Chapter 4

GENERALIZED LAPLACIAN DISTRIBUTION AND SOME APPLICATIONS

4.1 The Generalized Laplacian Distribution

Let $X$ and $Y$ be two independent gamma random variables with parameters $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ where $\alpha_1 > 0$, $\beta_1 > 0$, $j = 1, 2$. Now $U = X - Y$ has the characteristic function

$$
\phi_u(t) = (1 + i\beta_1 t)^{-\alpha_1} (1 - i\beta_2 t)^{-\alpha_2}.
$$

(4.1)

when $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, we get

$$
\phi_u(t) = (1 + \beta^2 t^2)^{-\alpha}
$$

(4.2)

The distribution corresponding to (4.2) is called generalized Laplacian, denoted by $GL(\alpha, \beta)$ and was introduced by Mathai (1993b). When $\alpha$ is
a positive integer, the probability density function corresponding to (4.2) is given by

\[ g(y) = \frac{1}{(2\beta)^\alpha} |y|^{\alpha-1} e^{-\frac{|y|}{\beta}} \sum_{r=0}^{\alpha-1} \frac{(\alpha)_r}{r!(\alpha - r - 1)!} \left[ \frac{2|y|}{\beta} \right]^{-r} \]  

(4.3)

where \((\alpha)_n\) is the Pochammer's symbol defined by

\[ (\alpha)_n = \begin{cases} 1, & n = 0 \\ \alpha(\alpha+1)...(\alpha+n-1), & n \in \mathbb{N}, \ \alpha \neq 0 \end{cases} \]

and \(-\infty < y < \infty\). When \(\alpha = 1\) the above distribution reduces to the Laplace distribution. This distribution belongs to the class of self-decomposable distributions.

The generalized Laplacian density finds application in a wide range of contexts such as the production of a chemical called melatonin in human body, solar neutrino fluxes in cosmos, growth decay mechanisms like formation of sand dunes in deserts, input-output situations in econometric contexts and industrial productions etc. Details can be seen from Mathai (1993a, 1993b, 1994, 2000). It is also surprising to observe that the growth-decay model of Mathai (1993c) also covers the distribution of sample covariance structures. We will discuss an autoregressive process with generalized Laplacian as marginals.

Consider the usual linear additive autoregressive equation

\[ X_n = aX_{n-1} + \epsilon_n \quad n = 0, \pm 1, \ldots \]  

(4.4)

where \(|a| < 1\) and the innovations \(\{\epsilon_n\}\) are independent identically distributed random variables and \(\epsilon_n\) is independent of \(\{X_n, X_{n-1}\}\).
Lawrance (1978) derived gamma and Laplace solutions of the equation (4.4). Dewald and Lewis (1985) developed an autoregressive process of order 2 in Laplace variables. Gibson (1986) used the first order autoregressive Laplace processes in image source modeling in data compression task. An autoregressive model with Laplace innovations developed by Damsleth and El-Shaarawi (1989) was found more suitable for modeling sulphate concentration data when compared with the Gaussian model.

Recently many authors have shown interest in the field of financial modeling. Anderson and Arnold (1993) developed Linnik processes to model time series data on stock price returns. Kozubowski and Panorska (1999) describe a class of multivariate geometric stable laws that can be used in modeling multivariate financial portfolios of securities. Klebanov et al. (2000) use geometric compounding models for modeling data from financial contexts. We construct an autoregressive process having marginal as the generalized Laplacian distribution described in (4.3).

4.2 The Generalized Laplacian AR(1) Process

The Generalized Laplacian autoregressive process of order 1 [GLAR(1)] is defined by \( \{X_n, n \geq 1\} \), where \( X_n \) satisfies the equation (4.4). It may be noted that this model is defined only if the marginal distribution belongs to the class of self-decomposable distributions. Now we have the following theorem.
**Theorem 4.2.1** The first order autoregressive process

\[ X_n = aX_{n-1} + \epsilon_n, \quad |a| < 1 \]

is strictly stationary Markovian with generalized Laplacian marginal distribution as in (4.3) if and only if the \( \{\epsilon_n\} \) are distributed independently and identically as the \( \alpha \)-fold convolution of random variables \( \{V_n\} \) where

\[ V_n = \begin{cases} 
0 & \text{with probability } a^2 \\
L_n & \text{with probability } 1 - a^2 
\end{cases} \]

where \( \{L_n\} \) are independently and identically distributed Laplace random variables provided \( X_0 \overset{d}{=} GL(\alpha, \beta) \) and independent of \( \epsilon_1 \).

**Proof:** Consider the equation

\[ X_n = aX_{n-1} + \epsilon_n, \quad |a| < 1 \quad (4.5) \]

where \( \{X_n\} \) are assumed to be distributed as generalized Laplacian with parameters \( \alpha \) and \( \beta \). Writing (4.5) in terms of characteristic function we get

\[ \phi_{X_n}(t) = \phi_{\epsilon_n}(t) \phi_{X_{n-1}}(at). \quad (4.6) \]

Assuming stationarity we have

\[
\phi_{\epsilon}(t) = \frac{\phi_X(t)}{\phi_X(at)} = \left[ \frac{1 + \beta^2 t^2} {1 + \beta^2 a^2 t^2} \right]^{-\alpha}
\]

\[
= \left[ \frac{1 + \beta^2 a^2 t^2} {1 + \beta^2 t^2} \right]^\alpha
\]

\[
= \left[ a^2 + (1 - a)^2 \frac{1} {1 + \beta^2 t^2} \right]^\alpha.
\]
If $X_0 \overset{d}{=} GL(\alpha, \beta)$, then the process is strictly stationary. For this it suffices to verify that $X_n \overset{d}{=} GL(\alpha, \beta)$ for every $n$. An inductive argument can be presented as follows. Suppose $X_{n-1} = GL(\alpha, \beta)$. Then from (4.6) we have

$$
\phi_{X_n}(t) = \left[ \frac{1 + \beta^2 a^2 t^2}{1 + \beta^2 t^2} \right]^\alpha \left( 1 + \beta^2 a^2 t^2 \right)^{-\alpha}
= \left( 1 + \beta^2 t^2 \right)^{-\alpha}.
$$

Hence the process is strictly stationary.

**Corollary 4.1.1** If $X_0$ is distributed arbitrary, then also the process is asymptotically Markovian with generalized Laplacian marginal distribution.

**Proof:**

$$
X_n = aX_{n-1} + \epsilon_n
= a^n X_0 + \sum_{k=0}^{n-1} a^k \epsilon_{n-k}
$$

Writing in terms of characteristic function

$$
\phi_{X_n}(t) = \phi_{X_0}(a^n t) \prod_{k=0}^{n-1} \phi_{\epsilon}(a^k t)
= \phi_{X_0}(a^n t) \prod_{k=0}^{n-1} \left[ \frac{1 + a^2 \beta^2 a^2 k^2 t^2}{1 + \beta^2 a^2 k^2 t^2} \right]^\alpha
= \phi_{X_0}(a^n t) \left[ \frac{1 + \beta^2 a^2 n^2 t^2}{1 + \beta^2 t^2} \right]^\alpha \to (1 + \beta^2 t^2)^{-\alpha} \text{ as } n \to \infty.
$$

It can be seen that the model generalizes the models of Gibson (1986) and Dewald and Lewis (1985).
Remark 4.1.2 The model is defined for all values of \( |a| < 1 \). The autocorrelation is given by

\[
\rho_r = \text{Cor}(X_n, X_{n-r}) = a^{|r|}; \quad r = 0, \pm 1, \pm 2, ...
\]

and it can assume both positive and negative values.

4.2.1 Distribution of sums and bivariate distribution of \((X_n, X_{n+1})\)

We have

\[
X_{n+j} = a^j X_n + a^{j-1} \epsilon_{n+1} + a^{j-2} \epsilon_{n+2} + \ldots + \epsilon_{n+j} \quad j = 0, 1, 2, ...
\]

Hence

\[
T_r = X_n + X_{n+1} + \ldots + X_{n+r-1} = \sum_{j=0}^{r-1} \left[ a^j X_n + a^{j-1} \epsilon_{n+1} + \ldots + \epsilon_{n+j} \right] = X_n \left[ \frac{1 - a^r}{1 - a} \right] + \sum_{j=1}^{r-1} \epsilon_{n+j} \left[ \frac{1 - a^{r-j}}{1 - a} \right].
\]

Therefore the distribution of the sum \( T_r \) is uniquely determined by the characteristic function

\[
\phi_{T_r}(t) = \phi_{X_n} \left[ t \frac{1 - a^r}{1 - a} \right] \left\{ \prod_{j=1}^{r-1} \phi_{\epsilon} \left[ t \frac{1 - a^{r-j}}{1 - a} \right] \left[ 1 + \beta^2 t^2 \left[ \frac{1 - a^{r-j}}{1 - a} \right]^{-2} \right]^{-a} \right\} \times \left\{ \prod_{j=0}^{r-1} \left[ a^2 + (1 - a^2) \left[ 1 + \beta^2 t^2 \left[ \frac{1 - a^{r-j}}{1 - a} \right]^{-2} \right]^{-1} \right]^{a} \right\}.
\]

The distribution of \( T_r \) can be obtained by inverting the above expression. 

Next, the joint distribution of contiguous observations \((X_n, X_{n+1})\) can be
given in terms of bivariate characteristic function as,

\[ \phi_{X_n, X_{n+1}}(t_1, t_2) = E[\exp(-it_1 X_n - it_2 X_{n+1})] \]
\[ = E[\exp(-it_1 X_n - it_2(aX_n + \epsilon_n))] \]
\[ = E[\exp(-i(t_1 + at_2)X_n - it_2\epsilon_n)] \]
\[ = \phi_{\epsilon_n}(t_2)\phi_{X_n}(t_1 + at_2) \]
\[ = \left[1 + \beta^2(t_1 + at_2)^2\right]^{-\alpha}\left[a^2 + (1 - a^2)\frac{1}{1 + \beta^2t_2^2}\right]^\alpha. \]

Since this expression is not symmetric in \( t_1 \) and \( t_2 \), it follows that the GLAR(1) process is not time reversible. However when \( \alpha \) is large enough, the process tends to be Gaussian and in this case it becomes time reversible. Therefore the process can give rise to a wide variety of sample paths. It can be suitably applied as an alternative for the different Laplacian and Gaussian models for modeling time series data consisting of positive and negative observations. These results are reported in Joy Jacob and Jose, K.K. (2000).

4.2.2 Simulation Studies

Simulation studies of GLAR(1) process were carried out for various values of the parameters. The sample paths of GLAR(1) processes with \( \alpha = 2, 3 \) \( \beta = 1 \), and for various values of \( \alpha \) are given below.
Sample Path of GLAR(2,1) With $a=5$

Sample Path of GLAR(2,1) With $a=8$
GLAR(3,1) with $a=.5$

GLAR(3,1) with $a=.8$
4.3 Probabilistic Design of Mechanical Systems Using Generalized Laplacian Distribution

The classical approach to the design of mechanical systems is based on the use of many factors such as safety factor, margin of safety, cycles of operation, efficiency factor etc., neglecting the probabilistic nature of the design parameters. Such an approach usually leads to over-design, reflecting in excess weight. In the designing of rockets and spacecrafts, we have to consider the randomness of various environmental factors such as atmospheric pressure, temperature, wind velocity etc. Hence the probabilistic approach to design of mechanical systems is of great significance. The problem can be tackled using the stress strength interference (SSI) theory, wherein we take into account the statistical distributions of the environmental stress acting on the component and the component's ability (strength) to withstand that stress. With probabilistic considerations incorporated into data analysis, considerable improvement in design can be achieved. The recognition that the design parameters generally are characterized by a spectra of values, which are statistical in nature, points to the probabilistic methods as a logical design. Also conventional methods can be regarded as special cases of the more general probabilistic methods, in which the variability of all parameters is zero, and such situations are very rare. In the classical design methodology, the concept of safety factor is used. Safety factor is the ratio of ultimate strength to working
stress, both considered as fixed known values and here no consideration is given to the variability of stress or strength at all. This thought provided the deterministic design of a unit. This design is not a realistic one, since a component can fail even if designed with a high safety factor. This is primarily due to the fact that no consideration is given to the variability of stress and strength of the system.

The stress on a system is the total sum of internal and external stresses, created by the operational use and imposed by the environmental conditions prevailing. The stress and strength are not fixed quantities and they do vary from unit to unit, even if best quality control procedures are used. Therefore it is proper to consider the stress and strength as random variables and to outline a fundamental basis of computing the reliability from the knowledge of distributions associated with these random variables, since variability can be best described through a probability distribution.

Rajagopalan (2004) describes a model assuming that stress and strength are normally distributed. But stress and strength can never be negative. So here we introduce a model assuming that stress and strength are independent random variables distributed as gamma random variables. This is more realistic since a gamma variable can also be a sum of independent exponential variables. In a study on modeling growth-decay mechanism, Mathai (1993a,b) introduced a generalized Laplacian distribution (GLD) and obtained closed form solution for its density in terms of Whittaker functions, and this can be computed exactly. The GLD is
found to be useful in modelling solar neutrinos in cosmological studies and also for modeling the amount of melatonin produced in the human body. Mathai (1995) developed a non-symmetric Laplacian model for studying the destruction-dampening mechanism for nuclear reactions characterized by slow rise and rapid fall. The non-symmetric Laplacian can be approximated by a simple non-symmetric triangular distribution also. Details can be seen from Mathai and Haubold (1995).

4.3.1 Stress Strength Interference Theory with Gamma Distributed Stress and Strength

We assume that the stress \( (S_1) \) and strength \( (S_2) \) are independently distributed gamma random variables with \( S_j \) having the density

\[
    f_{S_j}(x) = \begin{cases} 
        \frac{e^{-\frac{x}{\beta_j}} x^{\alpha_j-1}}{\beta_j^{\alpha_j} \Gamma(\alpha_j)} & \text{if } x > 0, \alpha_j > 0, \beta_j > 0 \\
        0 & \text{elsewhere.}
    \end{cases}
\] (4.7)

The moment generating function (m.g.f) of \( S_j \) is given by

\[
    M_{S_j}(t) = E(e^{tS_j}) = (1 - \beta_j t)^{-\alpha_j}
\] (4.8)

for \( j = 1, 2 \) and \(|\beta_j t| < 1\).

If \( X = S_1 - S_2 \), the m.g.f of \( X \) can be obtained as

\[
    M_X(t) = M_{S_1}(t)M_{S_2}(-t) = (1 - \beta_1 t)^{-\alpha_1}(1 + \beta_2 t)^{-\alpha_2}
\] (4.9)

for \(|\beta_j t| < 1\), \( j = 1, 2 \). The m.g.f in (4.9) can be inverted for general parameter values. Mathai has given a series of papers on the inversion of
(4.9) and various types of generalized version. Since our interest is only on a special case we will deal with that situation. When $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$, the expression becomes

$$M_X(t) = (1 - \beta^2 t^2)^{-\alpha}. \quad (4.10)$$

When $\alpha = 1$, (4.10) corresponds to the m.g.f of the Laplace distribution. Hence the distribution corresponding to (4.10) is called a generalized Laplacian distribution. For details see SeethaLekshmi, Joy and Jose (2003).

4.3.2 Density for a Gamma Difference

Using Mathai(1993a), the density of $X$ corresponding to (4.9) can be obtained as

$$g(x) = \begin{cases} \frac{c e^{\frac{x}{2}}}{\beta_0} \int_x^\infty y^{\alpha_1-1}(y-x)^{\alpha_2-1}e^{-y} \, dy; & x \geq 0 \\ \frac{c e^{\frac{x}{2}}}{\beta_0} \int_0^x y^{\alpha_1-1}(y-x)^{\alpha_2-1}e^{-y} \, dy; & x \leq 0 \end{cases} \quad (4.11)$$

where $c = (\Gamma(\alpha_1)\Gamma(\alpha_2)\beta_1^{-\alpha_1}\beta_2^{-\alpha_2})^{-1}$, and $\beta_0 = \frac{1}{\beta_1} + \frac{1}{\beta_2}$. Using Gradshteyn and Ryshik (1980), one can evaluate the density as

$$f(x) = \begin{cases} c_1 x^{\frac{\alpha_1+\alpha_2}{2}} e^{\frac{x}{2}} \left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) \, W_{\frac{\alpha_1-\alpha_2}{2}, \frac{1-\alpha_1-\alpha_2}{2}}(\beta_0 x) & \text{for } x > 0 \\ c_2 x^{\frac{\alpha_1+\alpha_2}{2}} e^{\frac{x}{2}} \left(\frac{1}{\beta_2} - \frac{1}{\beta_1}\right) \, W_{\frac{\alpha_2-\alpha_1}{2}, \frac{1-\alpha_2-\alpha_1}{2}}(-\beta_0 x) & \text{for } x \leq 0 \end{cases} \quad (4.12)$$

where

$$\beta_0 = \frac{1}{\beta_1} + \frac{1}{\beta_2},$$

$$c_1^{-1} = \Gamma(\alpha_1)\beta_1^{\frac{\alpha_1-\alpha_2}{2}}\beta_2^{\frac{\alpha_2}{2}}(\beta_1 + \beta_2)^{\frac{\alpha_1+\alpha_2}{2}},$$

$$c_2^{-1} = \Gamma(\alpha_2)\beta_1^{\frac{\alpha_1-\alpha_2}{2}}\beta_2^{\frac{\alpha_2}{2}}(\beta_1 + \beta_2)^{\frac{\alpha_1+\alpha_2}{2}}.$$
and $W(\cdot)$ denotes the Whittaker function.

When $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ it can be seen that (4.13) reduces to the following form

$$f(x) = \frac{|x|^\alpha^{-1}}{\Gamma(\alpha)(2\beta)^\alpha} W_{\alpha,\frac{1}{2}-\alpha}(\frac{2}{\beta}|x|), \quad -\infty < x < \infty$$

and this corresponds to the m.g.f in (4.10). In general the distribution corresponding to (4.13) is called the distribution of a gamma difference. In any input-output situation with gamma type input and output, the residual effect is a gamma difference.

A non-central version of (4.13) is also defined by Mathai (1993b). This corresponds to the m.g.f given by

$$M_U(t) = (1 - \beta^2 t^2)^{-\alpha} \exp\{-2\lambda + 2\lambda(1 - \beta^2 t^2)^{-1}\}$$

where $U$ denotes the noncentral random variable, $\lambda > 0$, $\beta > 0$, $\alpha > 0$, $|\beta t| < 1$. Here $U$ will be called a noncentral generalized Laplacian (NGL) with the parameters $(c, \beta, \lambda)$. When $\alpha_1 = \frac{m}{2}$, $\alpha_2 = \frac{n}{2}$, $\beta_1 = \beta_2 = 2$, the distribution corresponding to (4.9), is the distribution of the difference of two independent chi-squares with $m$ and $n$ degrees of freedom, where $m, n = 1, 2, \ldots$. But a non-central chi-square with $m$ degrees of freedom and non-centrality parameter $\lambda$ has the m.g.f

$$M_X(t) = (1 - 2t)^{-\frac{m}{2}} e^{-\lambda + \lambda(1 - 2t)^{-1}}$$

for $|2t| < 1$. For details see Mathai (2000). Hence if $S_i$ has a non-central chi-square with $m_i$ degrees of freedom and noncentrality parameter $\lambda_i$,
for $i = 1, 2$ then if $S_1$ and $S_2$ are independent, $X = S_1 - S_2$ has the m.g.f
given by

$$M_X(t) = (1 - 2t)^{\frac{-m_1}{2}}(1 + 2t)^{\frac{-m_2}{2}}e^{-(\lambda_1 + \lambda_2) + \lambda_1(1 - 2t)^{-1} + \lambda_2(1 + 2t)^{-1}}.$$  

Now consider a random variable $U$ having the m.g.f

$$M_U(t) = (1 - \beta_1 t)^{-\alpha_1}(1 + \beta_2 t)^{-\alpha_2}e^{-(\lambda_1 + \lambda_2) + \lambda_1(1 - \beta_1 t)^{-1} + \lambda_2(1 + \beta_2 t)^{-1}}$$  

then $U$ can be regarded as a non-central gamma difference, where $\beta_i > 0$, $\alpha_i > 0$, $|\beta_i t| < 1$; $\lambda_i > 0$, $i = 1, 2$. When $\lambda_1 = \lambda_2 = \lambda$, $\beta_1 = \beta_2 = \beta$, $\alpha_1 = \alpha_2 = \alpha$, we get

$$M_U(t) = (1 - \beta^2 t^2)^{-\alpha}e^{-2\lambda + 2\lambda(1 - \beta^2 t^2)^{-1}},$$  

and then $U$ will be called a non-central generalized Laplace variable (NGL) with parameters $(\alpha, \beta, \lambda)$. It follows that when $\beta = 2$ and $\alpha = \frac{k}{2}$; $k = 1, 2, \ldots$ $U$ represents the difference of two independently and identically distributed non-central chi-square stress variables with $k$ degrees of freedom and non-centrality parameter $\lambda$.

### 4.3.3 Reliability of Mechanical Systems

The reliability of mechanical systems like rockets can be measured by the probability that strength $(S_2)$ exceeds stress $(S_1)$. When stress and strength are independently distributed as gamma variates the reliability namely $P[S_2 > S_1] = P[S_2 - S_1 > 0]$ can be computed using the density of $S_2 - S_1$. Explicit computational forms are given in (4.11) to (4.13) and
integration is not too difficult. We have also carried out simulation studies to estimate the probability of $S_2 - S_1$. The reliability figures assuming various values of the parameters are computed by simulating 10000 observations with the help of SAS programming. These figures are presented in the table below.

Table 4.1–Reliabilities for various stress and strength

<table>
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<th>$G(1, 1)$</th>
<th>$G(2, 1)$</th>
<th>$G(3, 2)$</th>
<th>$G(4, 2)$</th>
<th>$G(6, 2)$</th>
<th>$G(8, 3)$</th>
<th>$G(10, 3)$</th>
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<td>$G(1, 1)$</td>
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<td>.7479</td>
<td>.9636</td>
<td>.9886</td>
<td>.9989</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G(1, 1)$</td>
<td>.4997</td>
<td>.7479</td>
<td>.9636</td>
<td>.9886</td>
<td>.9989</td>
<td>1</td>
<td>1</td>
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<tr>
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<td>.8915</td>
<td>.9569</td>
<td>.9943</td>
<td>.9997</td>
<td>1</td>
</tr>
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<td>.9971</td>
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