Chapter 3

EXTENSIONS OF THE
MATRIX-VARIATE DIRICHLET
DISTRIBUTIONS

Matrix-variate functions appear in a wide variety of problems in Statistics, Economics, Physics and so on. When dealing with non-null distributions of likelihood ratio criteria in testing statistical hypotheses on the parameters of one or more multivariate normal or Gaussian distributions, functions of matrix argument come in naturally. An example may be seen from Conradie and Troskie (1984). Applications of matrix-variate functions in various areas may be seen from Mathai (1997) and the references therein. Applications in engineering problems may be seen from Biyari and Lindsey (1991), in statistical problems from Hayakawa (1972), and in generalized special functions from Mathai and Pederzoli (1997). Extensions of Dirichlet models on algebraic structures and group theoretic
considerations may be seen from Massam (1994). Liouville models are generalizations of the scalar Dirichlet models. Matrix-variate analogues of Liouville models may be seen from Gupta and Richards (1987).

### 3.1 A Generalized Real Inverted Dirichlet Model

The real matrix-variate type-2 or inverted Dirichlet density is given by

\[
 f(X_1, \ldots, X_k) = c |X_1|^{\alpha_1 - \frac{p+1}{2}} \cdots |X_k|^{\alpha_k - \frac{p+1}{2}} |I + X_1 + \cdots + X_k|^{-(\alpha_1 + \cdots + \alpha_k + 1)}
\]

for \( X_j = X_j' > 0, \ j = 1, \ldots, k, \ \Re(\alpha_j) > \frac{p-1}{2}, \ j = 1, \ldots, k + 1 \) and zero elsewhere. The normalizing constant \( c \) is given by

\[
 c = \frac{\Gamma_p(\alpha_1 + \cdots + \alpha_{k+1})}{\Gamma_p(\alpha_1) \cdots \Gamma_p(\alpha_{k+1})}.
\]

But when dealing with problems in survival analysis, reliability problems, Bayesian analysis and so on, apart from \( X_1, \ldots, X_k \) and \( X_1 + \cdots + X_k \) successive sums \( X_k, X_k + X_{k-1}, \ldots, X_k + \cdots + X_2 \) also enter into the picture. Hence as a generalization of the above model we consider

\[
 f(X_1, \ldots, X_k) = c_k |X_1|^{\alpha_1 - \frac{p+1}{2}} \cdots |X_k|^{\alpha_k - \frac{p+1}{2}} |I + X_2 + \cdots + X_k|^{\beta_1}
 \times |I + X_3 + \cdots + X_k|^{\beta_2} \cdots |I + X_k|^{\beta_{k-1}}
 \times |I + X_1 + \cdots + X_k|^{-(\alpha_1 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)}
\]

for \( X_j = X_j' > 0, \ j = 1, \ldots, k, \ \Re(\alpha_j) > \frac{p-1}{2}, \ \Re(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k) > \frac{p-1}{2} \) for \( j = 1, \ldots, k \), and \( f(X_1, \ldots, X_k) = 0 \) elsewhere, where \( X_1, \ldots, X_k \) are real symmetric positive definite \( p \times p \) matrix random
variables and $c_k$ is the normalizing constant so that the total integral is unity. Now $c_k$ can be evaluated by successive integration. For example, consider the integral over $X_1$ denoted by $I_{X_1}$

$$I_{X_1} = \int_{X_1>0} |X_1|^{\alpha_1 - \frac{p+1}{2}} \times |I + X_1 + \ldots + X_k|^{-(\alpha_1 + \ldots + \alpha_{k+1} + \beta_1 + \ldots + \beta_k)} dX_1$$

$$= |I + X_2 + \ldots + X_k|^{-(\alpha_1 + \ldots + \alpha_{k+1} + \beta_1 + \ldots + \beta_k)} \times \int_{X_1>0} |X_1|^{\alpha_1 - \frac{p+1}{2}} |I + (I + X_2 + \ldots + X_k)^{-\frac{1}{2}} X_1 \times (I + X_2 + \ldots + X_k)^{-\frac{1}{2}}|^{-(\alpha_1 + \ldots + \alpha_{k+1} + \beta_1 + \ldots + \beta_k)} dX_1$$

(3.2)

where $(I + X_2 + \ldots + X_k)^{-\frac{1}{2}}$ is the symmetric positive definite square root of the matrix $(I + X_2 + \ldots + X_k)^{-1}$. Note that for any two square non-singular matrices $A$ and $B$ where $AB$ is defined,

$$|I + AB| = |I + BA|$$

$$= |I + A^{\frac{1}{2}}BA^{\frac{1}{2}}| \text{ if } A = A' > 0, \quad A = \left(A^{\frac{1}{2}}\right)^2$$

$$= |I + B^{\frac{1}{2}}AB^{\frac{1}{2}}| \text{ if } B = B' > 0, \quad B = \left(B^{\frac{1}{2}}\right)^2$$

$$= |I + C'BC| \text{ if } A = A' > 0, \quad A = CC'.$$

Now, make the transformation

$$Y = (I + X_2 + \ldots + X_k)^{-\frac{1}{2}} X_1 (I + X_2 + \ldots + X_k)^{-\frac{1}{2}}$$

$$\Rightarrow dY = (I + X_2 + \ldots + X_k)^{-\frac{p+1}{2}} dX_1$$

for $X_2, \ldots, X_k$ fixed, by using the Jacobian in (1.7). Substituting in (3.2) we have

$$I_{X_1} = |I + X_2 + \ldots + X_k|^{-(\alpha_2 + \ldots + \alpha_{k+1} + \beta_1 + \ldots + \beta_k)}$$
\[ \int_{Y'=Y>0} |Y|^{\frac{-n_1}{2}} |I + Y|^{-(\alpha_1 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)} dY \]
\[ = |I + X_2 + \cdots + X_k|^{-(\alpha_2 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)} \times \frac{\Gamma_p(\alpha_1) \Gamma_p(\alpha_2 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)}{\Gamma_p(\alpha_1 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)} \]  
(3.3)

for \( \Re(\alpha_1) > e^{-1} + \Re(\alpha_2 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k) > e^{-1}/2 \). The \( Y \)-integral is evaluated with the help of type-2 beta integral in (1.16).

Proceeding in a similar manner and integrating out \( X_2, \ldots, X_k \) we have the following expression for \( c_k^{-1} \).

\[ c_k^{-1} = \prod_{j=1}^{k} \frac{\Gamma_p(\alpha_j) \Gamma_p(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k)}{\Gamma_p(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k)} \]  
(3.4)

where \( \Gamma_p(\cdot) \) is defined in (1.14) and (1.15).

We will derive some interesting results on (3.1) which will be stated as theorems.

**Theorem 3.1.1**

For arbitrary \( t_1, \ldots, t_k \), the joint product moment of the determinants of \( X_1, \ldots, X_k \), is given by the following:

\[ E \left[ |X_1|^{t_1} |X_2|^{t_2} \cdots |X_k|^{t_k} \right] = c_k \left\{ \prod_{j=1}^{k} \Gamma_p(\alpha_j + t_j) \right\} \]
\[ \times \prod_{j=1}^{k} \frac{\Gamma_p(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k - t_1 - \cdots - t_j)}{\Gamma_p(\alpha_j + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k - t_1 - \cdots - t_{j-1})} \]  
(3.5)

for \( \Re(\alpha_j + t_j) > e^{-1} + \Re(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k - t_1 - \cdots - t_j) > e^{-1}/2 \)

for \( j = 1, \ldots, k \), where \( E \) denotes the expected value.

This result is available from \( c_k^{-1} \) by replacing \( \alpha_j \) by \( \alpha_j + t_j \), for \( j = 1, \ldots, k \), rearranging parameters and then multiplying by the normalizing constant \( c_k \). The expected value of a real-valued scalar function
\( \psi(X_1, \ldots, X_k) \) of the \( p \times p \) real symmetric positive definite matrices \( X_1, \ldots, X_k \) is given by

\[
\int_{X_1 > 0} \cdots \int_{X_k > 0} \psi(X_1, \ldots, X_k) f(X_1, \ldots, X_k) \, dX_1 \wedge \cdots \wedge dX_k. \tag{3.6}
\]

From (3.5) we can get arbitrary moments of individual determinants such as \( E\{|X_j|^{t_j}\} \) by putting all other \( t_1, t_{j-1}, t_{j+1}, \ldots, t_k \), to zero. Let \( t_1 = 0 = \cdots = t_{j-1} = t_{j+1} = \cdots = t_k \). Then from (3.5) we get

\[
E[|X_j|^{t_j}] = \frac{\Gamma_p(\alpha_j + t_j) \Gamma_p(\alpha_j + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k - t_j)}{\Gamma_p(\alpha_j) \Gamma_p(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k)}
\]

\[
= \prod_{m=1}^{p} \frac{\Gamma(\alpha_j + t_j - \frac{m-1}{2})}{\Gamma(\alpha_j - \frac{m-1}{2})}
\]

\[
\times \prod_{m=1}^{p} \frac{\Gamma(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k - t_j - \frac{m-1}{2})}{\Gamma(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k - \frac{m-1}{2})}
\]

\[
= E(Z_{1j}^{t_j}) \cdots E(Z_{pj}^{t_j}) \tag{3.7}
\]

where \( Z_{1j}, \ldots, Z_{pj} \) are independently distributed type-2 beta random variables with the parameters \( (\alpha_j - \frac{m-1}{2}, \alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k - \frac{m-1}{2}) \), \( m = 1, 2, \ldots, p \).

**Corollary 3.1.1:** When \( X_1, \ldots, X_k \) have the joint density in (3.1) then for a specific \( j \), \(|X_j|\) is structurally a product of \( p \) independent real type-2 scalar beta random variables and the \( t_j^{th} \) moment of \(|X_j|\) is given in (3.7). One can obtain from (3.5) the generalized matrix transform or M-transform of \( f(X_1, \ldots, X_k) \). For a discussion of M-transforms see Mathai (1997). The M-transform also gives a clue to the various properties associated with the function \( f(X_1, \ldots, X_k) \). These can be guessed from the M-transforms, and they are stated as theorems and proved independently.
Theorem 3.1.2 Let $U_j, j = 1, \ldots, k$ be real symmetric positive definite matrices. Consider the transformation

$$
X_k = (I - U_k)^{-\frac{1}{2}} U_k (I - U_k)^{-\frac{1}{2}}
$$

$$
X_{k-1} = (I - U_k)^{-\frac{1}{2}} (I - U_{k-1})^{-\frac{1}{2}} U_{k-1} (I - U_{k-1})^{-\frac{1}{2}} (I - U_k)^{-\frac{1}{2}}
$$

$$
\vdots
$$

$$
X_1 = (I - U_k)^{-\frac{1}{2}} \cdots (I - U_1)^{-\frac{1}{2}} U_1 (I - U_1)^{-\frac{1}{2}} \cdots (I - U_k)^{-\frac{1}{2}}
$$

Then, when $(X_1, \ldots, X_k)$ has the joint density as $f(X_1, \ldots, X_k)$ of (3.1), $U_1, \ldots, U_k$ are statistically independently distributed, and further, $U_j$ is a real matrix-variate type-1 beta with the parameters $(\alpha_j, \alpha_{j+1} + \ldots, + \alpha_k + \beta_j + \ldots + \beta_k), j = 1, \ldots, k$.

**Proof.** From the transformation used in the theorem, we can have the following simplifications

$$
X_k = (I - U_k)^{-\frac{1}{2}} U_k (I - U_k)^{-\frac{1}{2}}.
$$

Writing $U_k = U_k^{\frac{1}{2}} U_k^{\frac{1}{2}}$ and then absorbing $U_k^{\frac{1}{2}}$ into each factor we have

$$
X_k = (U_k^{-1} - I)^{-\frac{1}{2}} (U_k^{-1} - I)^{-\frac{1}{2}} = (U_k^{-1} - I)^{-1}.
$$

Then $dX_k$ in terms of $dU_k$ is available from (1.8). That is,

$$
dX_k = |U_k^{-1} - I|^{-(p+1)}|U_k|^{-(p+1)}dU_k = |I - U_k|^{-(p+1)}dU_k.
$$

By using the same process in (3.9) we can write

$$
X_{k-1} = (I - U_k)^{-\frac{1}{2}} (U_{k-1}^{-1} - I)^{-1} (I - U_k)^{-\frac{1}{2}}.
$$
Then for fixed $U_k$ we can write $dX_{k-1}$ in terms of $dU_{k-1}$ by applying the results in (1.7) and (3.10) successively. That is, for fixed $U_k$,

$$dX_{k-1} = |I - U_k|^{-\frac{p+1}{2}} |I - U_{k-1}|^{-\frac{p+1}{2}} dU_{k-1}. \quad (3.12)$$

Proceeding like this we have finally,

$$X_1 = (I - U_k)^{-\frac{1}{2}} (I - U_{k-1})^{-\frac{1}{2}} \ldots \times (I - U_2)^{-\frac{1}{2}} (U_1^{-1} - I)^{-1} (I - U_2)^{-\frac{1}{2}} \ldots (I - U_k)^{-\frac{1}{2}}. \quad (3.13)$$

and for fixed $U_k, U_{k-1}, \ldots, U_2$,

$$dX_1 = |I - U_k|^{-\frac{p+1}{2}} \ldots |I - U_2|^{-\frac{p+1}{2}} |I - U_1|^{-\frac{p+1}{2}} dU_1. \quad (3.14)$$

From the transformation in (3.9) note that $X_k$ is free of $U_{k-1}, \ldots, U_1$ and $X_{k-1}$ is free of $U_{k-2}, \ldots, U_1$ and so on. Thus the Jacobian matrix is a triangular block matrix and the Jacobian is the product of the determinants of the diagonal blocks. This behaves like evaluating $dX_k$ in terms of $dU_k$ with $U_{k-1}, \ldots, U_1$ fixed, $dX_{k-1}$ in terms of $dU_{k-1}$ with all other $U_j$'s fixed and so on. Finally, the Jacobian is the following:

$$dX_k \wedge dX_{k-1} \wedge \ldots \wedge dX_1 = |I - U_k|^{-(k-1)\frac{p+1}{2}-(p+1)}$$

$$\times |I - U_{k-1}|^{-(k-2)\frac{p+1}{2}-(p+1)} \ldots$$

$$\times |I - U_1|^{-(p+1)} dU_k \wedge \cdots \wedge dU_1. \quad (3.15)$$

Now, let us evaluate other factors in (3.1). Consider

$$I + X_k = I + (I - U_k)^{-\frac{1}{2}} U_k (I - U_k)^{-\frac{1}{2}}$$

$$= (I - U_k)^{-\frac{1}{2}} (I - U_k + U_k) [I - U_k]^{-\frac{1}{2}}$$

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\[ I + X_k + X_{k-1} = (I - U_k)^{-1} + X_{k-1} \]
\[ = (I - U_k)^{-1} + (I - U_k)^{-\frac{1}{2}}(I - U_{k-1})^{-\frac{1}{2}}U_{k-1} \]
\[ \times (I - U_{k-1})^{-\frac{1}{2}}(I - U_k)^{-\frac{1}{2}} \]
\[ = (I - U_k)^{-\frac{1}{2}}[I + (I - U_{k-1})^{-\frac{1}{2}}U_{k-1}] \]
\[ \times (I - U_{k-1})^{-\frac{1}{2}}(I - U_k)^{-\frac{1}{2}} \]
\[ = (I - U_k)^{-\frac{1}{2}}(I - U_{k-1})^{-1}(I - U_k)^{-\frac{1}{2}} \]  \hspace{1cm} (3.17)

by using the same process as in (3.16). Proceeding like this we have

\[ I + X_k + \cdots + X_1 = (I - U_k)^{-\frac{1}{2}} \cdots (I - U_2)^{-\frac{1}{2}}(I - U_1)^{-1} \]
\[ \times (I - U_2)^{-\frac{1}{2}} \cdots (I - U_k)^{-\frac{1}{2}}. \]  \hspace{1cm} (3.18)

Now, let us substitute (3.9) to (3.18) in (3.1) to obtain the following:

\[ f(X_1, \ldots, X_k)dX_k \wedge \cdots \wedge dX_1 \]
\[ = c_k |X_1|^{\alpha_1 - \frac{p_1}{2}} \cdots |X_k|^{\alpha_k - \frac{p_k}{2}} |I + X_k + \cdots + X_2|^{\beta_1} \]
\[ \times |I + X_k + \cdots + X_3|^\epsilon_2 \cdots |I + X_k|^{\beta_{k-1}} \]
\[ \times |I + X_k + \cdots + X_1|^{-\alpha_1 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k}dX_k \wedge \cdots \wedge dX_1 \]
\[ = c_k \left\{ \prod_{j=1}^{k} |U_j|^{\alpha_j - \frac{p_j}{2}} |I - U_j|^{\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k - \frac{p_j}{2}} \right\} \]
\[ dU_k \wedge \cdots \wedge dU_1. \]  \hspace{1cm} (3.19)

But

\[ |U_j|^{\alpha_j - \frac{p_j}{2}} |I - U_j|^{\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k - \frac{p_j}{2}} \]
with the normalizing constant is a real matrix-variate type-1 beta density, as seen from (1.7), with the parameters \((\alpha_j, \alpha_{j+1} + \ldots, +\alpha_{k+1} + \beta_j + \ldots + \beta_k), \ j = 1, \ldots, k\). From (3.18) it is seen that the density of \((X_1, \ldots, X_k)\) factorizes into product of real matrix-variate type-1 beta densities and hence from the definition of independence, \(U_1, \ldots, U_k\) are independently distributed type-1 beta random variables. This establishes the theorem.

By retracing the steps we can prove the converse.

**Theorem 3.1.3** Let \(U_j = U_j' > 0, \ j = 1, \ldots, k\) be independently distributed real matrix-variate random variables with the parameters \((\alpha_j, \alpha_{j+1} + \ldots + \alpha_{k+1} + \beta_j + \ldots + \beta_k), \ j = 1, \ldots, k\). Consider the transformation in (3.9). Then \(X_1, \ldots, X_k\) have the joint distribution as given in (3.1).

Thus Theorems 3.1.2 and 3.1.3 also provide a characterization for the density in (3.1). It follows trivially from (1.7) that if the \(p \times p\) real symmetric positive definite matrix \(Y\) has a matrix-variate type-1 beta distribution with the parameters \((\alpha, \beta)\) then \(I - Y\) is type-1 beta distributed with the parameters \((\beta, \alpha)\). Hence if the \(U_j\)'s in (3.9) are independently distributed as type-1 betas with parameters \((\alpha_j, \alpha_{j+1} + \ldots + \alpha_{k+1} + \beta_j + \ldots + \beta_k), \ j = 1, \ldots, k\) then \(V_j = I - U_j\) is type-1 beta distributed with the parameters \((\alpha_{j+1} + \ldots + \alpha_{k+1} + \beta_j + \ldots + \beta_k, \alpha_j), \ j = 1, \ldots, k\). Hence we have the following theorem and its converse.

**Theorem 3.1.4** Let \((X_1, \ldots, X_k)\) have the joint density in (3.1). Con-
Consider the transformation

\[
X_k = V_k^{-\frac{1}{2}}(I - V_k)V_k^{-\frac{1}{2}}
\]
\[
X_{k-1} = V_k^{-\frac{1}{2}}V_{k-1}^{-\frac{1}{2}}(I - V_{k-1})V_{k-1}^{-\frac{1}{2}}V_k^{-\frac{1}{2}}
\]
\[
\vdots
\]
\[
X_1 = V_k^{-\frac{1}{2}}...V_1^{-\frac{1}{2}}(I - V_1)V_1^{-\frac{1}{2}}...V_k^{-\frac{1}{2}}
\]

(3.20)

for some \( p \times p \) symmetric positive definite matrices \((V_1, ..., V_k)\) such that

\( 0 < V_j < I, \ j = 1, ..., k. \)

Then \( V_1, ..., V_k \) are mutually independently distributed real matrix-variate type-1 beta random variables with the parameters \((\alpha_{j+1} + ... + \alpha_{k+1} + \beta_j + ... + \beta_k, \alpha_j), \ j = 1, ..., k. \)

**Theorem 3.1.5** Let \( 0 < V_j = V_j' < I, \ j = 1, ..., k \) be independently distributed real matrix-variate type-1 beta random variables with the parameters \((\alpha_{j+1} + ... + \alpha_{k+1} + \beta_j + ... + \beta_k, \alpha_j), \ j = 1, ..., k. \) Consider the transformation in (3.20). Then \( X_1, ..., X_k \) have the joint distribution as in (3.1).

The converse also holds here. This will be stated as a theorem also.

**Theorem 3.1.6** Let \( X_1, ..., X_k \) have the joint density as in (3.1). Let \( Z_1, ..., Z_k \) be real symmetric positive definite \( p \times p \) matrices. Consider the transformation

\[
X_k = Z_k^{-1}
\]
\[
X_{k-1} = (I + Z_k^{-1})^{\frac{1}{2}}Z_{k-1}^{-1}(I + Z_k^{-1})^{\frac{1}{2}}
\]
\[
\vdots
\]

(3.21)
Then $Z_1, \ldots, Z_k$ are mutually independently distributed real matrix-variate type-2 beta random variables with the parameters $(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k, \alpha_j)$, $j = 1, \ldots, k$.

**Proof:** Since the transformation is of a triangular format the Jacobian is easily seen to be the following:

$$
\begin{align*}
\frac{dX_k \wedge \cdots \wedge dX_1}{dX_k} &= \left|Z_k\right|^{-(k-1)\frac{p+1}{2}-(p+1)}|Z_{k-1}|^{-(k-2)\frac{p+1}{2}-(p+1)} \cdots \times |Z_1|^{-(p+1)} |I + Z_k|^{(k-1)\frac{p+1}{2}} \times \cdots |I + Z_2|^{\frac{p+1}{2}} \, dZ_k \wedge \cdots \wedge dZ_1.
\end{align*}
$$

Also note that

$$
I + X_k = I + Z_k^{-1}
$$

$$
I + X_k + X_{k-1} = (I + Z_k^{-1}) + (I + Z_k^{-1})^{1/2} (I + Z_k^{-1})^{1/2}
$$

$$
= (I + Z_k^{-1})^{1/2} (I + Z_k^{-1})^{1/2}
$$

$$
\vdots
$$

$$
I + X_k + \cdots + X_1 = (I + Z_k^{-1})^{1/2} \cdots (I + Z_2^{-1})^{1/2} |I + Z_1^{-1}|
$$

$$
\times (I + Z_2^{-1})^{1/2} \cdots (I + Z_k^{-1})^{1/2}.
$$

Now, substituting in terms of the $Z_j$'s and evaluating, we have

$$
f(X_1, \ldots, X_k) \frac{dX_k \wedge \cdots \wedge dX_1}{dX_k} = c_k |X_1|^{\alpha_1 - \frac{k+1}{2}} \cdots |X_k|^{\alpha_k - \frac{k+1}{2}} |I + X_k + \cdots + X_2|^{\beta_1}
$$

$$
\times |I + X_k + \cdots + X_3|^{\beta_2} \cdots |I + X_k|^{\beta_k-1}.
$$
which is a product of real matrix-variate type-2 beta densities, see (1.18), and hence the result. The converse is also true.

**Theorem 3.1.7** Let $Z_1, \ldots, Z_b$ be independently distributed real matrix-variate type-2 beta random variables with the parameters $\sum_{j=1}^{k} \alpha_{j+1} + \beta_1 + \cdots + \beta_k$. Consider the transformation in (3.21). Then $(X_1, \ldots, X_k)$ have the joint density as in (3.1). Thus Theorems 3.1.6 and 3.1.7 also provide a characterization for the density in (3.1). Many more such results can be obtained by exploiting the properties of real matrix-variate type-1 and type-2 beta random variables and the properties of Dirichlet integrals.

### 3.2 An Extension of the Dirichlet Model in the Complex Case

As an extension of the matrix-variate inverted Dirichlet density in the complex case, consider $k p \times p$ hermitian positive definite matrix random variables $\tilde{X}_1, \ldots, \tilde{X}_k$ having the joint density function

$$
\times |I + X_k + \cdots + X_1|^{-(\alpha_1 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)} dX_k \land \cdots \land dX_1
$$

$$
= c_k \left\{ \prod_{j=1}^{k} |Z_j|^{\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k - \frac{p+1}{2}} \times |I + Z_j|^{-(\alpha_{j+1} + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)} \right\} dZ_k \land \cdots \land dZ_1
$$

$$
(3.22)
$$

which is a product of real matrix-variate type-2 beta densities, see (1.18), and hence the result. The converse is also true.

**Theorem 3.1.7** Let $Z_1, \ldots, Z_b$ be independently distributed real matrix-variate type-2 beta random variables with the parameters $(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k, \alpha_j)$, $j = 1, \ldots, k$. Consider the transformation in (3.21). Then $(X_1, \ldots, X_k)$ have the joint density as in (3.1).

Thus Theorems 3.1.6 and 3.1.7 also provide a characterization for the density in (3.1). Many more such results can be obtained by exploiting the properties of real matrix-variate type-1 and type-2 beta random variables and the properties of Dirichlet integrals.
for $\Re(\alpha_j) > p - 1$, $\Re(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k) > p - 1$, $j = 1, \ldots, k$, and $f(\bar{X}_1, \ldots, \bar{X}_k) = 0$ elsewhere, where $c_k$ is the normalizing constant.

We derive here some properties of (3.23). Since

$$\int_{\bar{X}_1} \cdots \int_{\bar{X}_k} f(\bar{X}_1, \ldots, \bar{X}_k) d\bar{X}_1 \cdots d\bar{X}_k = 1$$

we can evaluate $c_k$ by successive integration starting with $\bar{X}_1$. For fixed $\bar{X}_2, \ldots, \bar{X}_k$ let

$$L_1 = \int_{\bar{X}_1 = \bar{X}_1 > 0} |\det(\bar{X}_1)|^{\alpha_1 - p} \times |\det(I + \bar{X}_1 + \cdots + \bar{X}_k)|^{-(\alpha_1 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)} d\bar{X}_1.$$

Note that,

$$|\det(I + \bar{X}_1 + \cdots + \bar{X}_k)| = |\det(I + \bar{X}_2 + \cdots + \bar{X}_k)| \times |\det[I + (I + \bar{X}_2 + \cdots + \bar{X}_k)^{-\frac{1}{2}}] \bar{X}_1 \times (I + \bar{X}_2 + \cdots + \bar{X}_k)^{-\frac{1}{2}}|.$$

Now, make the transformation

$$\bar{Y}_1 = (I + \bar{X}_2 + \cdots + \bar{X}_k)^{-\frac{1}{2}} \bar{X}_1 (I + \bar{X}_2 + \cdots + \bar{X}_k)^{-\frac{1}{2}}.$$

Then

$$d\bar{Y}_1 = |\det(I + \bar{X}_2 + \cdots + \bar{X}_k)^{-p}| d\bar{X}_1.$$

Substituting for $\bar{X}_1$ in terms of $\bar{Y}_1$ we have

$$L_1 = |\det(I + \bar{X}_2 + \cdots + \bar{X}_k)|^{-(\alpha_2 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)}$$
\[
\times \int_{Y_1=Y_1^*>0} |\det(\tilde{Y}_1)|^{\alpha_1-\rho} |\det(I + \tilde{Y}_1)|^{-(\alpha_1 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)} \, d\tilde{Y}_1
\]
\[
= |\det(I + \tilde{X}_2 + \cdots + \tilde{X}_k)|^{-(\alpha_2 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)}
\times \frac{\tilde{\Gamma}_p(\alpha_1)\tilde{\Gamma}_p(\alpha_2 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)}{\Gamma_p(\alpha_1 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k)}
\]

for \(\Re(\alpha_1) > p - 1\), \(\Re(\alpha_2 + \cdots + \alpha_{k+1} + \beta_1 + \cdots + \beta_k) > p - 1\). The \(\tilde{Y}_1\)-integral is evaluated by using the type-2 beta integral of (1.27). Successive integrations of \(\tilde{X}_2, \ldots, \tilde{X}_k\) will yield the result as follows:

\[
c_k^{-1} = \prod_{j=1}^{k} \frac{\tilde{\Gamma}_p(\alpha_j)\tilde{\Gamma}_p(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k)}{\Gamma_p(\alpha_j + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k)}
\]

for \(\Re(\alpha_j) > p - 1\), \(\Re(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k) > p - 1\), \(j = 1, \ldots, k\).

We can look into some interesting results and the corresponding matrix transformations. These will be stated as theorems. We need two more results in order to establish our main results. These will be listed as lemmas.

**Lemma 3.2.1.** Let \(\tilde{X}\) and \(A\) be \(p \times p\) hermitian positive definite matrices. Then \(\tilde{Y}_1 = A^{\frac{1}{2}} \tilde{X} A^{\frac{1}{2}}\) and \(\tilde{Y}_2 = \tilde{X}^{\frac{1}{2}} A \tilde{X}^{\frac{1}{2}}\) have the same eigenvalues.

This can be easily seen by looking at the determinantal equations for the eigenvalues.

\[
|A^{\frac{1}{2}} \tilde{X} A^{\frac{1}{2}} - \lambda I| = 0 \Rightarrow |\tilde{X} A - \lambda I| = 0 \Rightarrow |\tilde{X}^{\frac{1}{2}} A \tilde{X}^{\frac{1}{2}} - \lambda I| = 0.
\]

Thus \(\tilde{Y}_1\) and \(\tilde{Y}_2\) have the same equations giving rise to the same eigenvalues \(\lambda_1, \ldots, \lambda_p\), \(\lambda_j > 0\), \(j = 1, \ldots, p\).
Lemma 3.2.2. Let the common real eigenvalues of $\hat{Y}_1$ and $\hat{Y}_2$ of Lemma 3.2.1 be distinct. Then the wedge product $d\hat{Y}_1 = d\hat{Y}_2$ in the integrals.

Proof. Let $\hat{U}$ and $\hat{V}$ be unitary matrices with real diagonal elements such that

$$\hat{W}_1 = \hat{U}^*\hat{Y}_1\hat{U} = D = \text{diag}(\lambda_1, \ldots, \lambda_p) = \hat{V}^*\hat{Y}_2\hat{V} = \hat{W}_2.$$

Then from Theorem 1.4 of Mathai (1997)

$$d\hat{Y}_1 = d\hat{W}_1 = \left\{ \prod_{j>k} |\lambda_k - \lambda_j|^2 \right\} dD \wedge d\tilde{G}_1$$

and

$$d\hat{Y}_2 = d\hat{W}_2 = \left\{ \prod_{j>k} |\lambda_k - \lambda_j|^2 \right\} dD \wedge d\tilde{G}_2$$

where $d\tilde{G}_1$ and $d\tilde{G}_2$ are the following:

$$d\tilde{G}_1 = \wedge[\hat{U}(d\hat{U})] \text{ and } d\tilde{G}_2 = \wedge[\hat{V}(d\hat{V})].$$

Now from Corollary 4.3.1 of Mathai (1997),

$$\int_{O_1(p)} d\tilde{G}_1 = \int_{O_1(p)} d\tilde{G}_2 = \frac{\pi^{p(p-1)}}{\Gamma_p(p)}$$

where $O_1(p)$ is the orthogonal group of unitary matrices with real diagonal elements. $(d\hat{U})$ and $(d\hat{V})$ denote the matrices of differentials (entry-wise derivatives) in $\hat{U}$ and $\hat{V}$, respectively.

Theorem 3.2.1. Let $\bar{X}_1, \ldots, \bar{X}_k$ be matrix-variate random variables having the joint distribution as in (3.23). Consider the transformation

$$\hat{Y}_1 = (I + \bar{X}_1 + \ldots + \bar{X}_k)^{-\frac{1}{2}}\bar{X}_1(I + \bar{X}_1 + \ldots + \bar{X}_k)^{-\frac{1}{2}}$$

$$\hat{Y}_2 = (I + \bar{X}_2 + \ldots + \bar{X}_k)^{-\frac{1}{2}}\bar{X}_2(I + \bar{X}_2 + \ldots + \bar{X}_k)^{-\frac{1}{2}}$$

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\[ \tilde{Y}_k = (I + \tilde{X}_k)^{-\frac{1}{2}} \tilde{X}_k (I + \tilde{X}_k)^{-\frac{1}{2}}. \quad (3.25) \]

Then \( \tilde{Y}_1, \ldots, \tilde{Y}_k \) are independent, and further, \( \tilde{Y}_j \) has a type-1 beta density with the parameters \((\alpha_j, \alpha_{j+1} + \cdots + \alpha_k + \beta_j + \cdots + \beta_k), \ j = 1, \ldots, k. \)

**Proof:** From the transformation in (3.25), we have

\[
I - \tilde{Y}_1 = I - (I + \tilde{X}_1 + \cdots + \tilde{X}_k)^{-\frac{1}{2}} \tilde{X}_1 (I + \tilde{X}_1 + \cdots + \tilde{X}_k)^{-\frac{1}{2}}
\]

\[
= (I + \tilde{X}_1 + \cdots + \tilde{X}_k)^{-\frac{1}{2}} [(I + \tilde{X}_1 + \cdots + \tilde{X}_k) - \tilde{X}_1]
\times (I + \tilde{X}_1 + \cdots + \tilde{X}_k)^{-\frac{1}{2}}
\]

\[
= (I + \tilde{X}_1 + \cdots + \tilde{X}_k)^{-\frac{1}{2}} (I + \tilde{X}_2 + \cdots + \tilde{X}_k)(I + \tilde{X}_1 + \cdots + \tilde{X}_k)^{-\frac{1}{2}}
\]

\[
I - \tilde{Y}_2 = (I + \tilde{X}_2 + \cdots + \tilde{X}_k)^{-\frac{1}{2}} (I + \tilde{X}_3 + \cdots + \tilde{X}_k)(I + \tilde{X}_2 + \cdots + \tilde{X}_k)^{-\frac{1}{2}}
\]

\[
I - \tilde{Y}_{k-1} = (I + \tilde{X}_{k-1} + \tilde{X}_k)^{-\frac{1}{2}} (I + \tilde{X}_k)(I + \tilde{X}_{k-1} + \tilde{X}_k)^{-\frac{1}{2}}
\]

\[
I - \tilde{Y}_k = (I + \tilde{X}_k)^{-1}. \quad (3.26)
\]

From the above representations of \( I - \tilde{Y}_1, \ldots, I - \tilde{Y}_k \), from (1.10), (1.11) and from Lemma 3.2.2 we can evaluate the Jacobian of the transformation in (3.25) from (3.26).

\[
d\tilde{Y}_1 = d[(I + \tilde{X}_1 + \cdots + \tilde{X}_k)^{-\frac{1}{2}} (I + \tilde{X}_2 + \cdots + \tilde{X}_k)(I + \tilde{X}_1 + \cdots + \tilde{X}_k)^{-\frac{1}{2}}]
\]

\[
= d[(I + \tilde{X}_2 + \cdots + \tilde{X}_k)^{\frac{1}{2}} (I + \tilde{X}_1 + \cdots + \tilde{X}_k)^{-1} (I + \tilde{X}_2 + \cdots + \tilde{X}_k)^{\frac{1}{2}}]
\]

\[
= |\det(I + \tilde{X}_2 + \cdots + \tilde{X}_k)|^p |\det(I + \tilde{X}_1 + \cdots + \tilde{X}_k)|^{-2p} d\tilde{X}_1
\]

for fixed \( \tilde{X}_2, \ldots, \tilde{X}_k \). Similarly,

\[
d\tilde{Y}_2 = |\det(I + \tilde{X}_3 + \cdots + \tilde{X}_k)|^p |\det(I + \tilde{X}_2 + \cdots + \tilde{X}_k)|^{-2p} d\tilde{X}_2
\]

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and, finally,

$$d\tilde{Y}_k = |\det(I + \tilde{X}_k)|^{-2p} d\tilde{X}_k.$$  

Since the transformation in (3.25) is of a triangular nature, the Jacobian matrix will be a triangular block matrix with the Jacobian being the product of the determinants of the diagonal blocks and the Jacobian is given by, ignoring the sign,

$$d\tilde{Y}_1 \wedge \ldots \wedge d\tilde{Y}_k = |\det(I + \tilde{X}_1 + \ldots + \tilde{X}_k)|^{-2p} \left\{ |\det(I + \tilde{X}_2 + \ldots + \tilde{X}_k)|^{-p} \times \ldots |\det(I + \tilde{X}_k)|^{-p} \right\} d\tilde{X}_1 \wedge \ldots \wedge d\tilde{X}_k.$$ (3.27)

From (3.25), (3.26) and (3.27) we can compute the following product:

$$\left\{ \prod_{j=1}^{k} |\det(\tilde{Y}_j)|^{\alpha_j - p} |\det(I - \tilde{Y}_j)|^{\alpha_{j+1} + \ldots + \alpha_{k+1} + \beta_j + \ldots + \beta_k - p} \right\} d\tilde{Y}_1 \wedge \ldots \wedge d\tilde{Y}_k$$

$$= \left\{ \prod_{j=1}^{k} |\det(\tilde{X}_j)|^{\alpha_j - p} \right\} |\det(I + \tilde{X}_2 + \ldots + \tilde{X}_k)|^{\beta_1}$$

$$\times |\det(I + \tilde{X}_3 + \ldots + \tilde{X}_k)|^{\beta_2} \ldots |\det(I + \tilde{X}_k)|^{\beta_{k-1}}$$

$$\times |\det(I + \tilde{X}_1 + \ldots + \tilde{X}_k)|^{-(\alpha_1 + \ldots + \alpha_{k+1} + \beta_1 + \ldots + \beta_k)}$$

$$\times d\tilde{X}_1 \wedge \ldots \wedge d\tilde{X}_k.$$ (3.28)

Multiplying (3.28) on both sides by $c_k$ we have the result since the right side with $c_k$ is the density in (3.23) and the left side with $c_k$ is the product of complex matrix-variate type-1 beta densities.

It is easy to see that the converse also holds.

**Theorem 3.2.2.** Let the hermitian positive definite matrices $\tilde{Y}_1, \ldots, \tilde{Y}_k$ be independently distributed as complex matrix-variate type-1 beta random variables. Then, for any $p \geq 1$,

$$d\tilde{Y}_1 \wedge \ldots \wedge d\tilde{Y}_k = |\det(I + \tilde{X}_1 + \ldots + \tilde{X}_k)|^{-2p} \left\{ |\det(I + \tilde{X}_2 + \ldots + \tilde{X}_k)|^{-p} \times \ldots |\det(I + \tilde{X}_k)|^{-p} \right\} d\tilde{X}_1 \wedge \ldots \wedge d\tilde{X}_k.$$
variables where $\tilde{Y}_j$ has the parameters $(\alpha_j, \alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k)$ for $j = 1, \ldots, k$. Consider the transformation in (3.25) on the space of $k$-tuples of hermitian positive definite matrices $\tilde{X}_1, \ldots, \tilde{X}_k$. Then $\tilde{X}_1, \ldots, \tilde{X}_k$ have the joint density as given in (3.23).

Thus theorems 3.2.1 and 3.2.2 also provide a characterization of the density in (3.23). It is known that when $\tilde{Y}_j$ has a complex matrix-variate type-1 beta density then $I - \tilde{Y}_j$ again has a complex matrix-variate type-1 beta density. Thus, from theorems 3.2.1 and 3.2.2 we can get two more results as corollaries. One of them will be listed here as a theorem and it can be proved independently also by proceeding parallel to the proof in theorem 3.2.1.

**Theorem 3.2.3.** Let $\tilde{X}_1, \ldots, \tilde{X}_k$ have the joint density in (3.23). Consider the transformation

\[
\begin{align*}
\tilde{Z}_1 &= (I + \tilde{X}_1 + \cdots + \tilde{X}_k)^{-1}(I + \tilde{X}_2 + \cdots + \tilde{X}_k)(I + \tilde{X}_1 + \cdots + \tilde{X}_k)^{-1} \\
\tilde{Z}_2 &= (I + \tilde{X}_2 + \cdots + \tilde{X}_k)^{-1}(I + \tilde{X}_3 + \cdots + \tilde{X}_k)(I + \tilde{X}_2 + \cdots + \tilde{X}_k)^{-1} \\
\tilde{Z}_k &= (I + \tilde{X}_k)^{-1}
\end{align*}
\]

Then $\tilde{Z}_1, \ldots, \tilde{Z}_k$ are independent complex matrix-variate type-1 beta random variables with $\tilde{Z}_j$ having the parameters $(\alpha_{j+1} + \cdots + \alpha_{k+1} + \beta_j + \cdots + \beta_k, \alpha_j)$, for $j = 1, \ldots, k$. 

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Theorem 3.2.4. Let $X_1, ..., X_k$ have the joint density in (3.23). Consider the transformation

$$
\bar{U}_1 = \bar{X}_1^{-\frac{1}{2}}(I + \bar{X}_2 + ... + \bar{X}_k)\bar{X}_1^{-\frac{1}{2}}
$$

$$
\bar{U}_2 = \bar{X}_2^{-\frac{1}{2}}(I + \bar{X}_3 + ... + \bar{X}_k)\bar{X}_2^{-\frac{1}{2}}
$$

$$
\bar{U}_k = \bar{X}_k^{-1}.
$$

(3.30)

Then $\bar{U}_1, ..., \bar{U}_k$ are independent complex matrix-variate type-2 beta random variables with $\bar{U}_j$ having the parameters $(\alpha_{j+1} + ... + \alpha_{k+1} + \beta_j + ... + \beta_k, \alpha_j)$, for $j = 1, ..., k$.

Proof: From (3.30), (1.10), (1.11) and Lemma 3.2.2 we have the following:

$$
d\bar{U}_1 = |\det(I + \bar{X}_2 + ... + \bar{X}_k)|^p|\det(\bar{X}_1)|^{-2p}d\bar{X}_1
$$

for fixed $\bar{X}_2, ..., \bar{X}_k$.

$$
d\bar{U}_2 = |\det(I + \bar{X}_3 + ... + \bar{X}_k)|^p|\det(\bar{X}_2)|^{-2p}d\bar{X}_2
$$

and finally

$$
d\bar{U}_k = |\det(\bar{X}_k)|^{-2p}d\bar{X}_k.
$$

Since the transformation in (3.30) is of a triangular nature, we have the Jacobian given by

$$
d\bar{U}_1 \wedge ... \wedge d\bar{U}_k = |\det(\bar{X}_1)|^{-2p} ... |\det(\bar{X}_k)|^{-2p}|\det(I + \bar{X}_2 + ... + \bar{X}_k)|^p
$$

$$
\times |\det(I + \bar{X}_3 + ... + \bar{X}_k)|^p \ldots
$$

$$
\times |\det(I + \bar{X}_k)|^p d\bar{X}_1 \wedge ... \wedge d\bar{X}_k.
$$

(3.31)
From (3.30),

\[ I + \tilde{U}_1 = I + \tilde{X}_1^{-\frac{1}{2}}(I + \tilde{X}_2 + \ldots + \tilde{X}_k)\tilde{X}_1^{-\frac{1}{2}} \]
\[ = \tilde{X}_1^{-\frac{1}{2}}[I + (I + \tilde{X}_2 + \ldots + \tilde{X}_k)]\tilde{X}_1^{-\frac{1}{2}} \]
\[ = \tilde{X}_1^{-\frac{1}{2}}(I + \tilde{X}_1 + \ldots + \tilde{X}_k)\tilde{X}_1^{-\frac{1}{2}} \]
\[ I + \tilde{U}_2 = \tilde{X}_2^{-\frac{1}{2}}(I + \tilde{X}_2 + \ldots + \tilde{X}_k)\tilde{X}_2^{-\frac{1}{2}} \]
\[ I + \tilde{U}_k = \tilde{X}_k^{-\frac{1}{2}}(I + \tilde{X}_k)\tilde{X}_k^{-\frac{1}{2}}. \] (3.32)

Now from (3.30), (3.31) and (3.32) we have

\[
\left\{ \prod_{j=1}^{k} |\text{det}(\tilde{U}_j)|^{\alpha_{j+1} + \ldots + \alpha_{k+1} + \beta_j + \ldots + \beta_k - p} \right\}
\times |\text{det}(I + \tilde{U}_j)|^{{(\alpha_j + \ldots + \alpha_{k+1} + \beta_j + \ldots + \beta_k)}} \}
\times d\tilde{U}_1 \wedge \ldots \wedge d\tilde{U}_k
\]
\[
= \left\{ \prod_{j=1}^{k} |\text{det}(\tilde{X}_j)|^{\alpha_{j} - p} \right\}
\times |\text{det}(I + \tilde{X}_2 + \ldots + \tilde{X}_k)|^{\beta_k} \ldots |\text{det}(I + \tilde{X}_k)|^{\beta_{k-1}}
\times |\text{det}(I + \tilde{X}_1 + \ldots + \tilde{X}_k)|^{-(\alpha_1 + \ldots + \alpha_{k+1} + \beta_k + \ldots + \beta_k)}
\times d\tilde{X}_1 \wedge \ldots \wedge d\tilde{X}_k. \] (3.33)

Multiply both sides of (3.33) by \( \omega_k \) to see the result.

The converse also holds. Thus theorem 3.2.4 and its converse also provide a characterization of the density in (3.23). It is known that when \( \tilde{U}_j \) has a complex matrix-variate type-2 beta distribution then \( \tilde{U}_j^{-1} \) has a complex matrix-variate type-2 beta distribution with the parameters interchanged. This property also gives a couple of results. Instead of \( \tilde{U}_j^{-1} \), a slightly different transformation will be considered in the next theorem.
Theorem 3.2.5. Let $\bar{X}_1, \ldots, \bar{X}_k$ have the joint distribution as given in (3.23). Consider the transformation

\[
\begin{align*}
\tilde{V}_1 &= (I + \bar{X}_2 + \ldots + \bar{X}_k)^{-\frac{1}{2}} \bar{X}_1 (I + \bar{X}_2 + \ldots + \bar{X}_k)^{-\frac{1}{2}} \\
\tilde{V}_2 &= (I + \bar{X}_3 + \ldots + \bar{X}_k)^{-\frac{1}{2}} \bar{X}_2 (I + \bar{X}_3 + \ldots + \bar{X}_k)^{-\frac{1}{2}} \\
&\vdots \\
\tilde{V}_{k-1} &= (I + \bar{X}_k)^{-\frac{1}{2}} \bar{X}_{k-1} (I + \bar{X}_k)^{-\frac{1}{2}} \\
\tilde{V}_k &= \bar{X}_k.
\end{align*}
\]

Then $\tilde{V}_1, \ldots, \tilde{V}_k$ are independent complex matrix-variate type-2 beta random variables with $\tilde{V}_j$ having the parameters $(\alpha_j, \alpha_{j+1} + \ldots + \alpha_{k+1} + \beta_j + \ldots + \beta_k)$, for $j = 1, \ldots, k$.

The proof can be given by using the steps parallel to the ones in the proof of theorem 3.2.1. The converse of theorem 3.2.5 is also true. Further, theorem 3.2.5 and its converse also provide a characterization for the density in (3.23). The results given in this chapter are reported in Joy Jacob et al. (2004) and Joy Jacob et al. (2005).