Chapter 2

Level Density and Correlation Functions

In section (2.1), we review Dyson-Mehta theorems on quaternion determinants and correlation functions. The formal polynomial results for the correlation functions given in later Chapters follow from these theorems. In section (2.2), we summarize the asymptotic results for the correlation functions in the Gaussian ensembles. These are needed in the later Chapters for the proof of universality. In section (2.3), we give a non-polynomial method of determining the level density.

2.1 Dyson-Mehta Theorems on Correlation Functions

We start this section with definitions of quaternion matrices and quaternion determinants along with a useful theorem relating quaternion determinants and ordinary determinants (Mehta, 1989). This is followed by the theorem, required to calculate the n-point correlation function for the three ensembles corresponding to $\beta = 1, 2$ and 4. The theorems are due to Dyson and are generalizations of earlier Dyson-Mehta methods.

A quaternion is a linear combination of four units, $1$, $e_1$, $e_2$ and $e_3$ which can take the form

\[
1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\quad e_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\quad e_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},
\quad e_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.
\] (2.1)
and is given by
\[ q = q^{(0)} + \vec{q} \cdot \vec{e} = q^{(0)} + q^{(1)}e_1 + q^{(2)}e_2 + q^{(3)}e_3. \]  
(2.2)

Here \( q^{(0)}, q^{(1)}, q^{(2)}, q^{(3)} \) are complex numbers. The dual of a quaternion is given by
\[ q = q^{(0)} - \vec{q} \cdot \vec{e}. \]  
(2.3)

If a matrix \( Q \) contains quaternion units, i.e. \( Q = [q_{ij}] \), then the dual of the quaternion matrix is defined by \( \overline{Q} = [\overline{q}_{ji}] \). A quaternion matrix is self-dual if \( Q = \overline{Q} \). The quaternion determinant (denoted by \( Q \det \)) of a self-dual matrix \( Q \) is defined by
\[ Q \det Q = \sum_P \epsilon_P (q_{j_1}q_{j_2}q_{j_3} \ldots q_{j_r})^{(0)}(q_{k_1}q_{k_2}q_{k_3} \ldots q_{k_s})^{(0)} \ldots, \]
\[ = \sum_P (-1)^{n-1} \prod_{l=1}^{l} (q_{a_0}q_{b_0} \ldots q_{d_0})^{(0)}, \]  
(2.4)

where \( P \) is any permutation of indices \( (1, 2, \ldots, n) \) consisting of \( l \) exclusive cycles of the form
\[ (a \rightarrow b \rightarrow c \rightarrow \ldots \rightarrow d \rightarrow a) \]  
(2.5)

and \( \epsilon_P = (-1)^{n-1} \) is the parity of \( P \). Furthermore, the product of elements in a particular cycle depends on the index by which one begins the cycle, and the value of the whole product on the right hand side depends on the order in which the \( l \) cyclic factors are written. These ambiguities are removed by taking the scalar part of each cyclic factor and indicate this fact by a superscript zero on each cycle in (2.4).

**Theorem 1.**

If we consider an \( n \times n \) self-dual matrix \( Q \) and the corresponding \( 2n \times 2n \) matrix \( C(Q) \) (obtained by replacing the quaternion units by their \( 2 \times 2 \) matrix representations (2.1)) of twice the size, then
\[ \det C(Q) = (Q \det Q)^2. \]  
(2.6)

**Theorem 2.**
Consider $n$ real variables $x_1, x_2, \ldots, x_n$. Let the quaternion elements of the $n \times n$ matrix $Q_n$ satisfy the following properties:

1. The matrix $Q_n$ is self-dual.

2. The quaternion elements of the matrix satisfy

$$q_{j,k} = f(x_j, x_k) = \overline{q_{k,j}}$$

i.e. $q_{j,k}$ depends only on $x_j$ and $x_k$.

3. Moreover, let

$$\int f(x, x)d\mu(x) = c, \quad (2.8)$$
$$\int f(x, y)f(y, z)d\mu(x) = f(x, z) + g(x, z) \quad (2.9)$$

and

$$g(x, y) = \lambda f(x, y) - f(x, y)\lambda \quad (2.10)$$

where $d\mu$ is a suitable measure, $c$ a constant scalar (i.e. multiple of 1) and $\lambda$ a constant quaternion. Then the theorem states that

$$\int \det Q_n d\mu(x_n) = (c - n + 1) \det Q_{n-1} \quad (2.11)$$

where $\det Q_{n-1}$ is the $(n - 1) \times (n - 1)$ matrix obtained from $Q_n$ by removing the row and column containing the variable $x_n$. We can apply this theorem repeatedly to integrate $\det Q_n$ over as many variables as we like.

This theorem can be used to calculate the correlation function (1.4) provided we can express the joint-probability distribution of the eigenvalues in terms of determinants of self-dual quaternion matrices, with elements satisfying conditions (1)-(3). Dyson, (1972) has shown (see Appendix A) that the joint-probability distribution of eigenvalues for the three types of ensembles can be expressed in terms of such quaternion determinants and the correlation functions can thus be obtained.
2.2 Gaussian Ensemble Results for Correlation Functions

We summarize here the results for $R_\alpha$, first obtained for the Gaussian and circular ensembles (Dyson, 1970; Mehta, 1971; Dyson, 1972; Mehta, 1991). We have for the three types of ensembles

$$R_\alpha^{(\beta)}(r_1, \ldots, r_n) = Q \det [\sigma_{\beta}(r_j - r_k)]_{j,k=1,\ldots,n},$$

$$= \{\det [\sigma_{\beta}(r_j - r_k)]\}^{1/2}, \quad (2.12)$$

where

$$\sigma_2(r) = \begin{pmatrix} S(r) & 0 \\ 0 & S(r) \end{pmatrix}, \quad (2.13)$$

$$\sigma_1(r) = \begin{pmatrix} S(r) & D(r) \\ I(r) - \epsilon(r) & S(r) \end{pmatrix}, \quad (2.14)$$

$$\sigma_4(r) = \begin{pmatrix} S(2r) & D(2r) \\ I(2r) & S(2r) \end{pmatrix}, \quad (2.15)$$

$$S(r) = \sin(\pi r)/\pi r, \quad (2.16)$$

$$D(r) = dS(r)/dr, \quad (2.17)$$

$$I(r) = \int_0^r S(r')dr', \quad (2.18)$$

$$\epsilon(r) = r/2|r|. \quad (2.19)$$

The function $S(r)$, from which all other functions above derive, will be seen in the following Chapters as the asymptotic form of a kernel function involving orthogonal polynomials for $\beta = 2$ and skew-orthogonal polynomials for $\beta = 1, 4$. The universality of $R_\alpha$ will follow from that of the asymptotic kernel function $S(r)$. As an example of (2.12), we give below the $R_2$

$$R_2^{(1)} = 1 - [S(r)]^2 + J(r)D(r), \quad (2.20)$$

$$R_2^{(2)} = 1 - [S(r)]^2, \quad (2.21)$$

$$R_2^{(4)} = 1 - [S(2r)]^2 + I(2r)D(2r), \quad (2.22)$$
where $J(r) = I(r) - \epsilon(r)$.

The number variance $\sum^2(r)$ derives from the $R_2$ function.

### 2.3 Level Density of the Jacobi Ensemble

In this section we give a general procedure for deriving the level density for large $N$. Since

$$\frac{\partial P(x_1, \ldots, x_N)}{\partial x_1} = \left( \beta \sum_{j \neq 1} \frac{1}{x_1 - x_j} + \frac{w'(x_1)}{w(x_1)} \right) P(x_1, \ldots, x_N), \quad (2.23)$$

we find from (1.4) an exact hierarchic set of relations linking $R_n$ to $R_{n+1}$ (Pandey, 1995; French et al., 1988; Pandey and Shukla, 1991). For $n = 1$, this gives

$$\frac{\partial R_1(x)}{\partial x} = \beta \int \frac{R_2(x,y)}{x-y} dy + \frac{w'(x)}{w(x)} R_1(x). \quad (2.24)$$

For large $N$, the integral on the right side can be replaced by a principal-value integral involving $R_2(x,y) \approx R_1(x)R_1(y)$; moreover $\partial R_1/\partial x$ can be dropped. Both these approximations can be rigorously justified from the behavior of $R_1$ and $R_2$ for large $N$. We thus find (Dyson, 1972; Beenakker, 1997; Pandey, 1995)

$$\beta R_1(x) \int \frac{R_1(y)}{x-y} dy + \frac{w'(x)}{w(x)} R_1(x) = 0, \quad (2.25)$$

a result which could also be derived directly by maximizing $\log P$. Note that (2.25) is valid when $w \neq 0$. Moreover, $R_1$ is zero when $w = 0$ but it can also be zero elsewhere as, for example, in the Gaussian and Laguerre cases below. We solve below the integral equation (2.25), using the resolvent (Pandey, 1981)

$$G(z) = \int \frac{R_1(y)}{z-y} dy, \quad (2.26)$$

which satisfies

$$G(x+i0) = \int \frac{R_1(y)}{x-y} dy - i\pi R_1(x). \quad (2.27)$$
For the Jacobi weight function (1.24), one needs to consider carefully the singularities of $w'/w$ at $x = \pm 1$. Since $R_1(x) = 0$ for $|x| > 1$, and $O(N)$ for $|x| \leq 1$, we find from (2.25), after multiplication by $(1 - x^2)/(z - x)$ and integration over $x$, that

$$
\int_{-1}^{1} \frac{dx}{z - x} \int_{-1}^{1} \frac{dy}{x - y} R_1(x) R_1(y) = 0,
$$

(2.28)

terms ignored being of lower order in $N$. The expression on the left can be written as

$$
\frac{1}{2} \int \int_{-1}^{1} \frac{dx}{x} \frac{dy}{y} R_1(x) R_1(y) \left( \frac{1 - x^2}{z - x} - \frac{1 - y^2}{z - y} \right) = \frac{1}{2} \int \int_{-1}^{1} \frac{dx}{x} \frac{dy}{y} \frac{R_1(x) R_1(y)}{(z - x)(z - y)} \times \{1 - z(x + y) + xy\},
$$

$$
= \frac{1}{2} \int \int_{-1}^{1} \frac{dx}{x} \frac{dy}{y} \frac{R_1(x) R_1(y)}{(z - x)(z - y)} \times \{1 - z^2 + (z - x)(z - y)\},
$$

$$
= \frac{1}{2} (1 - z^2) G^2 + \frac{N^2}{2}.
$$

(2.29)

Thus, since $G(z) \approx N/z$ for large $|z|$, we get

$$
G(z) = \frac{N}{\sqrt{z^2 - 1}},
$$

(2.30)

and

$$
R_1(x) = \frac{N}{\pi \sqrt{1 - x^2}}, \quad |x| < 1,
$$

$$
= 0, \quad |x| > 1,
$$

(2.31)

the result being the same for all finite values of the parameters $a, b$. Note that the level density becomes indefinitely large at the end points. For the associated Laguerre weight function (1.25), $R_1(x) = 0$ for $x < 0$, and $w'/w$ has a singularity at $x = 0$. In this case we find from (2.25), after multiplication by $x/(z - x)$, integration over $x$, and neglecting the (lower-order) $aG(z)$ term, that

$$
- \int_{0}^{\infty} \frac{x R_1(x)}{z - x} dx + \beta \int_{0}^{\infty} \frac{dx}{z - x} \int_{0}^{\infty} \frac{dy}{x - y} R_1(y) = 0,
$$

(2.32)
so that
\[ \beta zG^2 - 2zG + 2N = 0, \quad (2.33) \]
giving
\[ G(z) = \frac{1}{\beta} - \frac{1}{\beta} \sqrt{\frac{z - 2\beta N}{z}}, \quad (2.34) \]
and
\[ R_1(x) = \frac{1}{\pi \beta} \sqrt{\frac{2\beta N - x}{x}}, \quad 2\beta N > x > 0, \]
\[ = 0, \quad x < 0, \text{ or } x > 2\beta N, \quad (2.35) \]
again independent of the parameter \( a \), but different from the Jacobi result (2.31). Finally, in the Gaussian case (Pandey, 1981), we get from (1.26, 2.25),
\[ - \int_{-\infty}^{\infty} \frac{xR_1(x)}{z - x} dx + \int_{-\infty}^{\infty} \frac{dx}{z - x} \int_{-\infty}^{\infty} dy \frac{R_1(y)}{x - y} = 0, \quad (2.36) \]
implies
\[ G^2 - 2zG + 2N = 0. \quad (2.37) \]
Thus
\[ G(z) = z - \sqrt{z^2 - 2N}, \quad (2.38) \]
and
\[ R_1(x) = \frac{1}{\pi} \sqrt{2N - x^2}, \quad |x| < \sqrt{2N}, \]
\[ = 0, \quad |x| > \sqrt{2N}, \quad (2.39) \]
the last result being the well-known semicircular density (1). One can similarly obtain results for \( G(z) \) and \( R_1(x) \) when \( u(x) \) is \( O(N) \) and a low order polynomial; this gives results different from above; see Chapter 8. The polynomial methods discussed in Chapters 3-6 will confirm the above density results.