CHAPTER 1

INTRODUCTION, BASIC TOOLS AND TECNIQUES
1.1. General Theory, Motivation and Review of Literature

1.1.1. Oscillator and their importance

In everyday life we come across various things that move. The motion of physical systems can be classified into two broad categories: translational and vibrational motion. If the position of a body varies linearly with time its motion is translatory, e.g. a train moving on a straight track or a ball rolling on the ground. A motion that repeats itself in equal intervals of time is called periodic motion, e.g. the motion of the hands of a clock. If a body in periodic motion moves back and forth over the same path, its motion is called vibratory or oscillatory. Some examples of oscillatory motion are the oscillations of the arms of a walking person, the balance wheel of a flying mosquito etc [Bajaj (1984)]. Oscillations may be very complex such as those of a piano string or those of the earth during an earthquake. It may be remarked that mechanical systems are not the only ones that can oscillate. A tuned circuit in a radio can oscillate electro-magnetically. Radio waves, microwaves and visible light are just the oscillating electric and magnetic fields. Thus the study of oscillations is essential for the understanding of various systems- mechanical, acoustical, electrical and atomic.

Oscillators are a natural and expected part of the electronic scene. They occur in many applications and make possible circuits and subsystems that perform very useful functions. Oscillators occur sometimes even when we don’t want them, amplifiers can oscillate if stray feedback paths are present. Without oscillators we would probably live in a very dull world.

Oscillation occurs when an amplifier is furnished with a feedback path that satisfies two conditions:

1. Amplitude Condition- The cascaded gain and loss through the amplifier/feedback must be greater than unity!
2. Phase Condition- The frequency of oscillation will be at the point where loop phase shift totals 360 (or zero) degrees!
In most oscillator circuits, oscillation builds up from zero when power is first applied, under linear circuit operation. However, limiting amplifier saturation and other non-linear effects end up keeping the oscillator’s amplitude from building up indefinitely. Thus, oscillators are not the simplest devices in the world to accurately design, simulate or model. There is a real art to GOOD, STABLE oscillator design. As you learn more about oscillators you will certainly grow to appreciate them!

There are many types of oscillators and many different circuit configurations that produce oscillations. Some oscillators produce sinusoidal signals, others produce non sinusoidal signals. Non sinusoidal oscillators, such as pulse and ramp (or saw-tooth) oscillators, find use in timing and control applications. Pulse oscillators are commonly found in digital-systems clocks and ramp oscillators are found in the horizontal sweep circuit of oscilloscopes and television sets. There are number of applications of oscillators in various fields such as in consumer electronic equipment, radios, TVs, VCRs, test equipment, wireless systems. They are also used as network analyzers and signal generators.

**Free, Damped and Forced Oscillations**

Once started, the oscillations continue forever with constant amplitude and a constant frequency. Simple harmonic motions which persist indefinitely without loss of amplitude are called free or undamped. However, observation of the free oscillations of a real physical system reveals that the energy of the oscillator gradually decreases with time and the oscillator eventually comes to rest. For example, the amplitude of a pendulum oscillating in air decreases with time and it ultimately stops. The vibrations of a tuning fork die away with the passage of time. This happens because, in actual physical systems, the friction (or damping) is always present. Friction resists motion. The presence of resistance to motion implies that frictional or damping force acts on the system. The damping force acts against the motion, doing negative work on the system, leading to a dissipation of energy. When a body moves through a medium such as air, water etc. its energy is dissipated due to friction and appears as heat either in the body
itself or in the surrounding medium or both. There is another mechanism by which an oscillator loses energy. The energy of an oscillator may decrease not only due to friction in the system, but also due to radiation. The oscillating body imparts periodic motion to the particles of the medium in which it oscillates, thus producing waves. For example, a tuning fork produces sound waves in the medium by decreasing its energy. All sounding bodies are subject to dissipative forces, or otherwise, there would be no loss of energy by the body and consequently no emission of sound energy could occur. Thus sound waves are produced by radiation from mechanical oscillatory systems. The electromagnetic waves are produced by radiations from oscillating electric and magnetic fields. The effect of radiation by an oscillating system and of the friction present in the system is that the amplitude of oscillations gradually diminishes with time. The reduction in amplitude (or energy) of an oscillator is called damping and the oscillations are said to be damped.

Let us consider a system of unit mass particle, whose oscillations are described by an inhomogeneous, linear, second-order differential equation of the form

\[
\frac{d^2 x}{dt^2} + a \frac{dx}{dt} + \omega_0^2 x = F \sin \omega t
\]

When the damping and external forcing are absent, \( \alpha = 0, F = 0 \), the system is essentially a free linear harmonic oscillator. Its solution corresponding to the initial conditions \( x(0) = A, \dot{x}(0) = 0 \) is

\[ x(t) = A \cos \omega_0 t \]

When damping is present and the external forcing is absent, \( \alpha \neq 0, F = 0 \), the explicit solution of this linear differential equation can be readily written. It has the form

\[ x(t) = A_1 \exp (m_1 t) + A_2 \exp (m_2 t) \]

Where, \( m_{1,2} = \frac{1}{2} \left[ -\alpha \pm \sqrt{\alpha^2 - 4\omega_0^2} \right] \)
And $A_1, A_2$ are integration constants. We now have three possibilities:

Under damping: $0 < \alpha < 2\omega_o$

Critical damping: $\alpha = 2\omega_o$

Over damping: $\alpha > 2\omega_o$

When the damping and the external forcing both are present, $\alpha \neq 0, F \neq 0$, then the oscillations are said to be damped and forced oscillations. The solution will be

$$x(t) = (A_t\omega_o/C)e^{-\alpha t/2} \cos(Ct - \delta) + A_p \cos(\omega t - \gamma)$$

Where, $A_p = \frac{F}{(\omega_o^2 - \omega^2)^2 + \alpha^2\omega^2}, \gamma = \tan^{-1}\left(\frac{\omega^2 - \omega_o^2}{\alpha\omega}\right)$

1.1.2. Importance of Nonlinearity

Nonlinearity is abundant in nature. It is having an increasingly important impact on variety of applied subjects ranging from the study of turbulence and the behaviour of weather, through the investigation of electrical and mechanical oscillators in engineering systems to the analysis of various biological, ecological and economic phenomena. Nonlinearity in the oscillating system may exist in various forms e.g. in a mechanical system the nonlinearity may be due to the presence of nonlinear elastic / spring elements, nonlinear damping, system with fluid, nonlinear boundary conditions etc., in an electromagnetic system the nonlinear resistive, inductive, capacitive elements, hysteresis of ferromagnetic materials, nonlinear active elements like vacuum tube, transistor etc. may be responsible for nonlinear effects in the systems. The nature of such nonlinear systems is studied under the banner of nonlinear dynamics and its sub field chaos theory which particularly discusses the bounded aperiodic behaviour of certain nonlinear oscillators that are highly sensitive to the initial conditions.
Linear vs nonlinear: physical and mathematical implications

Mathematically, the essential difference between linear and nonlinear dynamical systems is clear. Linear combination of the solutions also form a valid solution, this is the superposition principle. In fact, a moment of serious thought allows one to recognize that superposition is responsible for the systematic methods used to solve, independent of other complexities, essentially any linear problem. Fourier and Laplace transform methods, for example, depend on being able to superpose solutions. Putting it naively, one breaks the problem into many small pieces, then adds the separate solutions to get the solution to the whole problem. In contrast, two solutions of a nonlinear equation cannot be added together to form another solution. Superposition fails. Thus, one must consider a nonlinear problem in toto; one cannot break the problem into small sub problems and add their solutions. It is therefore perhaps not surprising that no general analytic approach exists for solving typical nonlinear equations. In fact, certain nonlinear equations describing chaotic physical motions have no useful analytic solutions. Physically, the distinction between linear and nonlinear behavior is best abstracted from examples. For instance, when water flows through a pipe at low velocity, its motion is laminar and is characteristic of linear behavior: regular, predictable, and describable in simple analytic mathematical terms. However, when the velocity exceeds a critical value, the motion becomes turbulent, with localized eddies moving in a complicated, irregular, and erratic way that typifies nonlinear behavior. By reflecting on this and other examples, we can isolate at least three characteristics that distinguish linear and nonlinear physical phenomena.

First, the motion itself is qualitatively different. Linear systems typically show smooth, regular motion in space and time that can be described in terms of well behaved functions. Nonlinear systems, however, often show transitions from smooth motion to chaotic or erratic as we will see later, even apparently random behavior. The quantitative description of chaos is one of the triumphs of nonlinear science.
Second, the response of a linear system to small changes in its parameters or to external stimulation is usually smooth and in direct proportion to the stimulation. But for nonlinear systems, a small change in the parameters can produce an enormous qualitative difference in the motion. Further, the response to an external stimulation can be different from the stimulation itself: for example, a periodically driven nonlinear system may exhibit oscillations at, say, one-half, one-quarter, or twice the period of the stimulation.

Third, a localized “lump,” or pulse, in a linear system will normally decay by spreading out as time progresses. This phenomenon, known as dispersion, causes waves in linear systems to lose their identity and die out, such as when low-amplitude water waves disappear as they move away from the original disturbance. In contrast, nonlinear systems can have highly coherent, stable localized structures—such as the eddies in turbulent flow—that persist either for long times or, in some idealized mathematical models, for all time. The remarkable order reflected by these persistent coherent structures stands in sharp contrast to the irregular, erratic motion that they themselves can undergo.

**Nonlinearity and chaos:**

As described above that the linear systems follow superposition principle i.e. small changes in the initial state leads to proportional changes in the final state. However this is not necessarily true for the nonlinear systems. Some sudden dramatic changes in nonlinear systems may give rise to a complex behavior called chaos. Chaos is bounded aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions. The sensitive dependence on initial conditions means the two trajectories starting very close together rapidly diverge from each other and thereafter have totally different features. The occurrence of this dynamical phenomenon has important consequences for physics and other scientific disciplines. A set of differential equations (i.e. dynamical system) that models a physical system is often studied to make long-time predictions of the dynamical behaviour of this system. If, however the dynamical equations exhibit sensitive dependence on initial conditions, long-time
predictions becomes impossible: Suppose we measure the initial conditions of the physical system very accurately, there is always some error between the measured and the true initial state. With increasing time in our model equations, this discrepancy grows until it becomes as large as some measure of tolerance below which we may accept the prediction. But after this point prediction fails. As it turns out that the separation of nearby trajectories even occurs exponentially fast. Sensitive dependence on initial conditions seems to be intimately connected with the occurrence of aperiodic long-time dynamical behavior. The corresponding trajectories do not approach a fixed point, a periodic orbit or a quasiperiodic orbit for $t \to \infty$. Instead of this, they oscillate in an irregular manner for all time. However such irregular behaviour can not be compared to random or noisy input or parameters. It arises only from the nonlinearities of the deterministic dynamical system rather than from random driving forces. The above described ingredients are commonly accepted as what it is understood by the term chaos.

1.1.3. **Nonlinear Damping**

The mechanism of supply (source) and dissipation of energy in the oscillator collectively play a very important role in deciding the dynamical behaviour of the system. One of the very common and ubiquitous forms of dissipation in oscillating systems is **damping**. Forces that are functions of the velocity are called damping forces. When the damping force or simply the damping causes the amplitude of the unforced motion to decrease, it is called positive damping. When the damping causes the amplitude to increase, it is called negative damping. Damping may be linear or nonlinear depending on nature of the forces responsible for the dissipation of energy. There may be sometime simultaneous presence of linear and nonlinear dissipative forces. The damping may be present in various forms [Nayfeh and Mook (1995)]

*Coulomb Damping:* when contact surface between two solids is dry, the friction force opposing their relative motion is called Coulomb damping. When an external force is applied to move the block in Figure 1.1 (a) from rest, a friction
force which opposes the impending motion develops. The magnitude of the friction force $f$ increases until a critical value is reached and then the block moves. After the motion begins, the magnitude of $f$ decreases as long as $|\dot{x}| = |\dot{x}_m|$ and then increases when $|\dot{x}|$ becomes larger than $|\dot{x}_m|$ as shown schematically in Figure 1.1 (b). The critical value is usually expressed as $\mu_s N$, where $\mu_s$ is the so-called static coefficient of friction and $N$ is the normal force between the block and the surface, in this case it is $mg$.

![Figure 1.1. (a) Spring mass system. (b) Friction force as a function of velocity.](image)

In many applications, the Coulomb force is approximated by a constant. Thus referring to Figure 1.1 (a), one writes the equation of motion as

$$m\ddot{x} + F(x) = f = \begin{cases} 
\mu_d mg \text{ when } \dot{x} < 0 \\
-\mu_d mg \text{ when } \dot{x} > 0
\end{cases}$$

Where $\mu_d$ is the so-called dynamic or kinetic coefficient of friction and $F(x)$ is the negative of the restoring force of the spring.

**Linear Damping:** When the contact surface in Figure 1.1 (a) is covered with a thin liquid film so that the two surfaces do not touch, it is usual to assume that the friction force is proportional to the velocity gradient (i.e. $f \propto \dot{x}/h$, where $\dot{x}$ is the relative velocity and $h$ is the thickness of the film) and opposes the motion. Thus referring to Figure 1.1 (a), one writes the equation of motion in the form

$$m\ddot{x} + c\dot{x} + F(x) = 0$$
Where $c$ is a positive constant that is a function of the fluid properties and the condition of the surfaces. Mahalingam investigated the combined influence of Coulomb and linear damping on the response of vibratory systems [Mahalingam (1975)].

Another example of the drag force being proportional to the velocity occurs when an immersed body moves through a fluid at very low Reynolds numbers (Stokes flow).

**Nonlinear Damping:** When an immersed body moves through a fluid at high Reynolds numbers, the flow separates and the drag force is very nearly proportional to the square of the velocity. Thus one writes the equation of motion in the form

$$m\ddot{x} + F(x) = -c|\dot{x}|\dot{x}$$

Where $c$ is a positive constant that is a function of the body geometry and the fluid properties. For moderate Reynolds numbers, the damping force lies between the linear and quadratic forms. Consequently some researchers have represented the damping force as $-c\dot{x}^p\dot{x}$, where $0 < p < 1$. Since these models of damping are not analytic, other researchers have used damping forms such as $-cf(x)\dot{x}$ or $-cg(\dot{x})\dot{x}$, where $f(x)$ and $g(\dot{x})$ are even analytic functions of $x$ and $\dot{x}$ respectively. Hemp proposed a combination of Coulomb and quadratic damping for a runaway escapement mechanism [Hemp (1972)].

The viscous damping is usually small, the damping force exerted by the fluid in contact with the system is likely to be viscous. Viscous forces are generally much smaller than inertial and elastic forces in a system. However, damping devices called dampers are sometimes deliberately introduced in a system for vibration control. The damping force exerted by such devices may be comparable in magnitude to the inertial and elastic forces.

In real systems, it is likely that the moving part is in contact with an unlubricated surface, as in the case of horizontal oscillations of a body attached to a spring.
The oscillating body is always in contact with the horizontal surface. The resulting frictional force opposes the motion and can often be idealized as a force of constant magnitude. Such a force is usually referred to as a Coulomb friction force.

In a solid, some part of energy may be lost due to imperfect elasticity or internal friction of the material. It is very difficult to estimate this type of damping. Experiments suggest that a resistive force proportional to the amplitude and independent of the frequency may serve as a satisfactory approximation. This kind of damping in solids is referred to as structural damping.

Thus, the damping of a real system is a complex phenomenon involving several kinds of damping forces such as viscous damping, Coulomb friction and structural damping. Because it is generally very difficult to predict the magnitude of the damping forces, one usually has to rely on experience and experiment so as to make a reasonably good estimate. It is common practice to approximate the damping of a system by an equivalent viscous damping, for the simple reason that viscous damping is the most convenient to handle mathematically. Thus, according to this approximation, the magnitude of the viscous force to be used in a particular problem is chosen to be the one that would produce the same rate of energy dissipation as the actual damping forces. This usually provides a good estimate.

The inclusion of damping forces complicates the analysis considerably. Fortunately in actual systems, the damping forces are usually small and can often be ignored. In situations, where they are not very small, the viscous damping model is convenient mathematically. We shall use this model, under a simplifying assumption, that the velocity of the moving part of the system is small, so that the damping force is linear in velocity. If the velocity is not small, the damping force exerted on the system may be represented more closely by a force proportional to the square or higher power of the velocity.
1.1.4. Review of Literature and Motivation

Although dynamics is an interdisciplinary subject today, it was originally a branch of physics. The subject began in the mid-1600s, when Newton invented differential equations, discovered his laws of motion and universal gravitation and combined them to explain Kepler’s laws of planetary motion. Specifically, Newton solved the two-body problem - the problem of calculating the motion of the earth around the sun, given the inverse-square law of gravitational attraction between them. Subsequent generations of mathematicians and physicists tried to extend Newton’s analytical methods to the three-body problem (e.g. sun, earth, and moon) but curiously this problem turned out to be much more difficult to solve. After decades of effort, it was eventually realized that the three-body problem was essentially impossible to solve, in the sense of obtaining explicit formulas for the motions of the three bodies. At this point the situation seemed hopeless.

The breakthrough came with the work of Poincare in the late 1800s. He introduced a new point of view that emphasized qualitative rather than quantitative questions. For example, instead of asking for the exact positions of the planets at all times, he asked “Is solar system stable forever, or will some planets eventually fly off to infinity?” Poincare developed a powerful geometric approach to analyze such questions. That approach has flowered into the modern subject of dynamics, with applications reaching far beyond celestial mechanics. Poincare was also the first person to glimpse the possibility of chaos, in which a deterministic system exhibits aperiodic behavior that depends on the initial conditions, thereby rendering long-term prediction impossible.

But chaos remained in the background in the first half of this century; instead dynamics was largely concerned with nonlinear oscillators and their applications in physics and engineering. Nonlinear oscillators played a vital role in the development of such technologies as radio, radar, phase-locked loops, and lasers. On the theoretical side, nonlinear oscillators also stimulated the invention of new mathematical techniques- pioneers in this area include van der Pol, Andronov,
Little-wood, Cartwright, Levinson, and Smale. Meanwhile, in separate development, Poincare’s geometric methods were being extended to yield a much deeper understanding of classical mechanics, thanks to the work of Birkhoff and later Kolmogorov, Arnol’d and Moser [Birkhoff (1927)].

The invention of the high-speed computer in the 1950s was a watershed in the history of dynamics. The computer allowed one to experiment with equations in a way that was impossible before, and thereby to develop some intuition about nonlinear systems. Such experiments led to Lorenz’s discovery in 1963 of chaotic motion on a strange attractor [Lorenz (1963)]. He studied a simplified model of convection rolls in the atmosphere to gain insight into the notorious unpredictability of the weather. Lorenz found that the solutions to his equations never settled down to equilibrium or to a periodic state- instead they continued to oscillate in an irregular, aperiodic fashion. Moreover, if he started his simulations from two slightly different initial conditions, the resulting behaviors would soon become totally different. The implication was that the system was inherently unpredictable- tiny errors in measuring the current state of the atmosphere (or any other chaotic system) would be amplified rapidly, eventually leading to embarrassing forecasts. But Lorenz also showed that there was structure in the chaos- when plotted in three dimensions, the solutions to his equations fell onto a butterfly-shaped set of points (Figure 1.2). He argued that this set had to be “an infinite complex of surfaces”—today we would regard it as an example of a fractal.

![Figure 1.2. A solution in Lorentz Attractor.](image)

Lorenz’s work had little impact until the 1970s, the boom years for chaos. Here are some of the main developments of that glorious decade. In 1971 Ruelle and
Takens proposed a new theory for the onset of turbulence in fluids, based on abstract considerations about strange attractors [Ruelle and Takens (1971)]. A few years later, May found examples of chaos in iterated mappings arising in population biology, and wrote in influential review article that stressed the pedagogical importance of studying simple nonlinear systems, to counterbalance the often misleading linear intuition fostered by traditional education. Next came the most surprising discovery of all, due to the physicist Feigenbaum. He discovered that there are certain universal laws governing the transition from regular to chaotic behavior; roughly speaking, completely different systems can go chaotic in the same way. His work established a link between chaos and phase transitions, and enticed a generation of physicists to the study of dynamics. Finally, experimentalists such as Gollub, Libchaber, Argoul, Linsay, Moon, and Westervelt [Ouellette and Gollub (2008), Libchaber et. Al. (1983), Argoul et. Al. (1987), Linsay (1981), Moon (1991), Gwinn and Westervelt (1985)] tested the new ideas about chaos in experiments on fluids, chemical reactions, electronic circuits, mechanical oscillator, and semiconductors [Strogatz (1994)].

During last 50 years, after the discovery of Lorenz, a lot of work has been done on the bifurcation and chaos analysis, control and synchronization of chaos in ubiquitous physical oscillators like: driven Duffing, driven van der Pol, forced Pendulum, forced Duffing-Helmholtz, Duffing-van der Pol, parametrically driven Duffing, parametrically driven Duffing-van der Pol, Parametrically driven Duffing-Helmholtz oscillators etc. Here we give a brief review of the most relevant work:

Novak and Frehlich observed in the Duffing oscillator that a bifurcation from a solution composed of only odd harmonics to one composed of both even and odd harmonics precedes the period-doubling bifurcations [Novak and Frehlich (1982)]. Wiggins studied chaotic dynamics of the quasiperiodically forced Duffing oscillator. The results gave insight into the experimental results on the quasiperiodically forced Duffing oscillator obtained by Moon and Holmes [Wiggins (1987)]. Kao et al. studied the crises in a two-well forced oscillator of
Duffing type with an analog simulation. Its features are discussed with the aid of return maps and phase portraits. Two types of boundary crisis in a strange attractor following the Feigenbaum route to chaos are found [Kao Y. H. et al. (1987)]. Solari and Gilmore described the global organization of the periodic orbits in the periodically driven Duffing oscillator in terms of the relative rotation rates of pairs of orbits [Solari H. G. and Gilmore R. (1988)]. Nayfeh and Sanchez analyzed the response of a damped Duffing oscillator of the softening type to a harmonic excitation in a two-parameter space consisting of the frequency and amplitude of the excitation. They developed an approximate procedure is for the generation of the bifurcation diagram in the parameter space of interest [Nayfeh and Sanchez (1989)]. Ueda surveyed of regular and chaotic phenomena in the forced Duffing oscillator and gave a reasonably complete view of the behavior of this important and prototypical dynamical system [Ueda (1991)]. Englisch and Lauterborn investigated a special, seemingly infinite sequence of saddle-node bifurcations of the driven double-well Duffing oscillator. It occurs in resonances with even torsion number and shows period-adding behavior. The sequence of saddle-node bifurcations gives rise to a regular window structure of higher and higher period [Englisch V. and Lauterborn W. (1991)]. Lansbury et al. studied the control phase portraits encountered in the twin-well Duffing oscillator, concentrating on the loss of stability of attractors confined to a single well of the potential [Lansbury et al. (1992)]. Parthasarathy obtained the threshold condition for the occurrence of Smale-horseshoe chaos in the parametrically driven Duffing oscillator using the Melnikov-function approach. A detailed description of the homoclinic bifurcation sets in the five-dimensional parameter space involved in the system is provided [Parthasarathy S. (1992)]. Wang investigated the features of various bifurcations in forced Duffing oscillators by means of numerical simulation. Some common features in four types of potential well are found [Wang C. S. et al. (1992)]. Zhu et al presented the variance of the displacement, the probability densities of the displacement, velocity and amplitude and the joint probability density of the displacement and velocity of the stationary response of a Duffing oscillator to narrow-band Gaussian random excitation for a number of
combinations of the parameters of the oscillator and excitation obtained from digital simulation [Zhu et al. (1993)]. Murali and Lakshmanan investigated the phenomenon of chaos synchronization and efficient signal transmission in Van der Pol–Duffing oscillator. A criterion for synchronization based on asymptotic stability is discussed. By considering a cascaded synchronization system, they investigated the possibility of the secure communication of analog signals [Murali and Lakshmanan (1993)]. Gilmore and McCallum identified four levels of structure in the bifurcation diagram of the two-well periodically driven Duffing oscillator, plotted as a function of increasing control parameter $T$, the period of the driving term [Gilmore and McCallum (1995)]. Woafo et al. studied the dynamics of a system consisting of a van der Pol oscillator coupled to a Duffing oscillator. Chaotic behavior is observed using the Shilnikov theorem and from a direct numerical simulation of the coupled equations of motion [Woafo P. et al. (1996)]. Luo and Han presented an analytical approach for the quantitative prediction of stability and bifurcations of approximate periodic solutions of the generalized Duffing oscillator. This approach is based on an improved harmonic balance method [Luo and han (1997)]. Paul and Rajasekar studied the migration from one attractor to another coexisting attractor in two coupled Duffing oscillators by an open-plus-closed-loop control method and adaptive control algorithm. Suppression of chaos by these methods is also investigated [Paul S. and Rajasekar S. (1997)]. Liu et al. have done detailed studied the effect of bounded noise on the chaotic behavior of the Duffing oscillator under parametric excitation. The random Melnikov process is derived and a mean-square criterion is used to detect the chaotic dynamics in the system [Liu et al. (2001)]. Xu and Chung investigated the mechanism for the action of time delay in a non-autonomous system. The original mathematical model under consideration is a van der Pol–Duffing oscillator with excitation. A delayed system is obtained by adding both linear and nonlinear time delayed position feedbacks to the original system. They observed that time delay may be used as a simple but efficient “switch” to control motions of a system: either from order motion to chaos or from chaotic motion to order for different applications [Xu and Chung (2003)]. A
new signal detection and estimation method based on the intermittency transition between order and chaos is developed by Wang and He. The corresponding bifurcation process for the Duffing oscillator is discussed in detail, and the length of the laminar phase of the bifurcation is exploited to describe quantitatively the property of sensitive dependence on the initial condition of the chaotic Duffing system [Wang and He (2003)]. Batista et al. studied the hysteretic bistable response of Duffing oscillators and showed ways to control the switching between stable branches of this nonlinear response. They also showed that how memory effects in dissipation qualitatively and quantitatively alter the dynamics of Duffing oscillators [Batista A. A. et al. (2008)].

It is evident from the review of recent work on various ubiquitous nonlinear physical oscillators that the large amount of work has been done so far on these oscillators under the presence of viscous damping. However in real physical situations damping may be present in other forms also as discussed in Section 1.1.3. Hence the study of oscillating physical system under the presence of other forms of damping constitutes an important area of research. Specifically the study on nonlinear oscillating system under the nonlinear damping is relatively new and unexplored area of research. Hence the subject of the present thesis has been chosen centered around the study of dynamical behaviour of some of the very common nonlinear oscillators under the presence of nonlinear damping.

Here in this section we have only discussed the review of the recent research work on ubiquitous nonlinear physical oscillator under the linear i.e. viscous damping. The reviews of the very recent works on the specific oscillators (considered in this study) under the presence of nonlinear damping have been given respective chapters.
1.2. Basic Tools

1.2.1. Analytical methods

1.2.1.1. Linear Stability Analysis

To determine the stability of equilibrium points we infinitesimally disturb or perturb the system near the given equilibrium point linearly and analyse under what conditions the perturbation will lie down or grow exponentially fast [Lakshmanan and Rajasekar (2003)]. This will then give the required classification of equilibrium points based on the linear stability analysis. For this now let us consider a two dimensional dynamical system

\[ \dot{x} = P(x, y) \]  
\[ \dot{y} = Q(x, y) \]  

(1.1a)  
(1.1b)

Where \( P \) and \( Q \) are some well defined functions of \( x \) and \( y \). Let

\[ N = N^* = N_{u3} = \frac{u3}{v3} \]

Be the equilibrium point of Eq. (1.1), so that

\[ P(x_0, y_0) = Q(x_0, y_0) = 0 \]

In order to determine the stability of this equilibrium point we slightly disturb it.

\[ x = x_0 + \epsilon(t) \]
\[ y = y_0 + \varphi(t), \quad \epsilon, \varphi \ll 1 \]

For resultant motion, we make a Taylor expansion about the equilibrium point

\[ P(x, y) = P(x_0 + \epsilon, y_0 + \varphi) \]
\[ P(x, y) = P(x_0, y_0) + \frac{\partial P}{\partial x}|_{x_0, y_0} \cdot \epsilon + \frac{\partial P}{\partial y}|_{x_0, y_0} \cdot \varphi + \text{higher order term} \]
\[ Q(x, y) = Q(x_0 + \epsilon, y_0 + \varphi) \]
\[ Q(x, y) = Q(x_0, y_0) + \frac{\partial Q}{\partial x}|_{x_0, y_0} \cdot \epsilon + \frac{\partial Q}{\partial y}|_{x_0, y_0} \cdot \varphi + \text{higher order term} \]

Now using the fact that at the equilibrium point \( P(x_0, y_0) = Q(x_0, y_0) = 0 \) and that the higher order terms on the right hand sides can be neglected in most cases, then Eq. (1.1) can be rewritten as a system of two first-order linear coupled differential equations.
\[ \dot{e} = a\epsilon + b\varphi \quad (1.2a) \]
\[ \dot{\varphi} = c\epsilon + d\varphi \quad (1.2b) \]

Where,
\[ a = \frac{\partial P}{\partial x}|_{x_o,y_o}, \quad b = \frac{\partial P}{\partial y}|_{x_o,y_o} \]
\[ c = \frac{\partial Q}{\partial x}|_{x_o,y_o}, \quad d = \frac{\partial Q}{\partial y}|_{x_o,y_o} \]

Differentiating Eq. (1.2a) with respect to 't' and eliminating the \( \varphi \) dependence, we obtain a second order differential equation for the variable \( \epsilon \):

\[ \ddot{e} - (a + d)\dot{e} + (ad - bc)e = 0 \]

Its solution is given by,
\[ \epsilon(t) = Aexp(\lambda_1 t) + B(\lambda_2 t) \quad (1.3) \]

Where,
\[ \lambda_{1,2} = \frac{1}{2} \left[ (a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)} \right], \quad ad - bc \neq 0 \quad (1.4) \]

Substituting Eq. (1.3) into Eq. (1.2a), we get
\[ \varphi(t) = Cexp(\lambda_1 t) + Dexp(\lambda_2 t) \quad (1.5) \]

Where,
\[ C = A(\lambda_1 - a)/b, \quad D = B(\lambda_2 - a)/b \]

Eqs. (1.3) & (1.5) are the solutions of Eq. (1.2).

\( \lambda_1 \) & \( \lambda_2 \) are the eigenvalues of the matrix \( \mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), \( \det \mathbf{M} \neq 0 \).

An equilibrium point \( X_o = (x_o, y_o) \) of the system (1.1) is dynamically stable, if all the permissible perturbations \((\epsilon, \varphi)\) which solve (1.2) decay in time exponentially fast and that no admissible perturbation can grow in time. This means that the stability of the equilibrium point \((x_o, y_o)\) can be studied from Eqs. (1.3) & (1.5) using Eq. (1.4) and that is depends solely on the numerical value of eigenvalues \( \lambda_1 \) and \( \lambda_2 \). Thus for an equilibrium point to be stable, unstable or neutrally stable, the following conditions are to be satisfied:
### Stability nature of equilibrium point

<table>
<thead>
<tr>
<th></th>
<th>Conditions on the eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stable</td>
<td>Real parts of both the eigenvalues are negative</td>
</tr>
<tr>
<td>Unstable</td>
<td>Real part of even one of the two eigenvalues is positive</td>
</tr>
<tr>
<td>Neutral</td>
<td>Real parts of both the eigenvalues are zero</td>
</tr>
</tbody>
</table>

#### 1.2.1.2. Melnikov’s method for detecting homoclinic bifurcation

In this section we will explain Melnikov’s method for detecting homoclinic bifurcation [Jordan and Smith (1999)]. This is a global perturbation method applicable to systems which have a known homoclinic path in an underlying autonomous system. This system is then perturbed usually by damping and forcing terms, and conditions for which homoclinic manifolds intersect are determined to leading order. There are various versions of the theory of increasing generality, but here we shall consider systems of the form

\[
\dot{x} = y, \quad \dot{y} + f(x) = \varepsilon h(x, y, t)
\]  

(1.6)

where \( h(x, y, t) \) is \( T \)-periodic in \( t \), and \( |\varepsilon| \) is a small parameter. The unperturbed system is

\[
\dot{x} = y, \quad \dot{y} + f(x) = 0
\]

It is assumed that \( f(0) = 0 \), and that the origin is a simple saddle with a known homoclinic path \( x = x_0(t - t_0), y = y_0(t - t_0) \). Consider the loop from the origin lying in the half-plane \( x \geq 0 \). Since the system is autonomous we can include an arbitrary time translation \( t_0 \), which will be significant in Melnikov’s method.

It is assumed that \( f(x) \) and \( h(x, y, t) \) have continuous partial derivatives in each argument up to any required order, with a Taylor series in \( x \) and \( y \) in a neighbourhood of the origin, namely,

\[
f(x) = f'(0)x + \cdots
\]

\[
h(x, y, t) = h(0,0,t) + \left[ h_x(0,0,t)x + h_y(0,0,t)y \right] + \cdots
\]
with \( f'(0) < 0 \) (to guarantee a saddle) and \( h_x(0,0,t) \neq 0, h_y(0,0,t) \neq 0 \), except possibly for isolated values of \( t \).

As \( \varepsilon \) increases from zero, an unstable limit cycle emerges from the origin with solution \( x = x_\varepsilon(t), y = y_\varepsilon(t) \), say. We take as usual the Poincaré sequence starting with \( t = 0 \) and having period \( T \) and let the fixed point of the limit cycle be \( P_\varepsilon: (x_\varepsilon(0), y_\varepsilon(0)) \) (see Figure 1.3). Remember that any sequence can be used: if homoclinic tangency occurs for the sequence starting with \( t = 0 \), it will occur for any other starting time. Associated with \( P_\varepsilon \) there will be stable and unstable manifolds \( W^s_\varepsilon \) and \( W^u_\varepsilon \); if these manifolds intersect, then homoclinic bifurcation takes place. Melnikov’s method investigates the distance between the manifolds at a point on the unperturbed homoclinic path where \( t = 0 \), that is, at \( P_0(x_0(-t_0), y_0(-t_0)) \).

To approximate to the stable manifold \( W^s_\varepsilon \) we apply the regular perturbation to Eq. (1.6) for \( t \geq 0 \).

\[
\begin{align*}
x^s_\varepsilon(t, t_0) &= x_0(t - t_0) + \varepsilon x^s_1(t, t_0) + O(\varepsilon^2) \\
y^s_\varepsilon(t, t_0) &= y_0(t - t_0) + \varepsilon y^s_1(t, t_0) + O(\varepsilon^2)
\end{align*}
\]

To ensure that \( W^s_\varepsilon \) approaches \( P_\varepsilon \) we require

---

Figure 1.3. Distance function \( D(t_0) \) between the manifolds \( W^u_\varepsilon \) and \( W^s_\varepsilon \) at the point \( P_0 \) on the unperturbed autonomous homoclinic path.
\[ x^s_\varepsilon(t, t_0) - x^s_\varepsilon(t, t_0) \to 0, \quad y^s_\varepsilon(t, t_0) - y^s_\varepsilon(t) \to 0, \quad \text{as } t \to \infty \]

Substitute the expansions of \( x^s_\varepsilon(t, t_0) \) and \( y^s_\varepsilon(t, t_0) \) into Eq. (1.6). Then equate the coefficients of \( Q \) to zero, using also \[ @q_{x^s_\varepsilon(t, t_0)} = @q_{y^s_\varepsilon(t, t_0)} = Q @f_{x^s_\varepsilon(t, t_0)}, \]

It follows that \( x^s_\varepsilon(t, t_0) \) and \( y^s_\varepsilon(t, t_0) \) satisfy
\[ L_{x^s_\varepsilon(t, t_0)} + @f_{x^s_\varepsilon(t, t_0)} = h_{x^s_\varepsilon(t, t_0), y^s_\varepsilon(t, t_0), t} \]

In a similar manner, for the unstable manifold \( W^u_\varepsilon \) the perturbation series for \( x^u_\varepsilon \) is
\[ x^u_\varepsilon(t, t_0) = x_0(t - t_0) + \varepsilon x^u_1(t, t_0) + O(\varepsilon^2) \]
\[ y^u_\varepsilon(t, t_0) = y_0(t - t_0) + \varepsilon y^u_1(t, t_0) + O(\varepsilon^2) \]

For \( t \leq 0 \), where
\[ x^u_\varepsilon(t, t_0) - x^u_\varepsilon(t) \to 0, \quad y^u_\varepsilon(t, t_0) - y^u_\varepsilon(t) \to 0, \quad \text{as } t \to -\infty \]

Thus \( x^u_1(t, t_0) \) and \( y^u_1(t, t_0) \) satisfy
\[ y^u_1(t, t_0) = x^u_1(t, t_0) \]

Note that the stable manifold \( W^s_\varepsilon \) is defined by the set of points
\[ (x^s_\varepsilon(0, t_0), y^s_\varepsilon(0, t_0)) \] for all \( t_0 \), whilst the unstable manifold \( W^u_\varepsilon \) is defined by
\[ (x^u_\varepsilon(0, t_0), y^u_\varepsilon(0, t_0)). \]

If \( x^u_{\varepsilon, s}(t, t_0) = [x^u_{\varepsilon, s}(t, t_0), y^u_{\varepsilon, s}(t, t_0)]^T \)

Then the displacement vector at \( t = 0 \) can be defined as
\[ d(t_0) = x^u_1(0, t_0) - x^s_\varepsilon(0, t_0) \]
\[ d(t_0) = \varepsilon [x^u_1(0, t_0) - x^s_\varepsilon(0, t_0)] + O(\varepsilon^2) \]

Where
\[ x^u_{\varepsilon, s} = [x^u_{\varepsilon, s}, y^u_{\varepsilon, s}]^T \]

If \( n(0, t_0) \) is the unit outward normal vector at \( P_0 \) (see Figure 1.3), then the points \( x^u_{\varepsilon, s}(0, t_0), y^u_{\varepsilon, s}(0, t_0) \) will not lie exactly on the normal but will be displaced as indicated in Figure 1.3. We use a distance function \( D(t_0) \) which is obtained by projecting the displacement vector \( d(t_0) \) onto the unit normal \( n \). Thus
\[ D(t_0) = d(0, t_0) = n(0, t_0) + O(\varepsilon^2) \]

The tangent vector to the unperturbed homoclinic path at \( t = 0 \) is
\{\dot{x}_0(-t_0), \dot{y}_0(-t_0)\} = \{y_0(-t_0), f[x_0(-t_0)]\}

Therefore the unit outward normal vector is

\[ n(0, t_0) = \frac{\{f[x_0(-t_0)], y_0(-t_0)\}}{\sqrt{\{f[x_0(-t_0)]^2, y_0(-t_0)^2\}}} \]

Hence \( D(t_0) = d(t_0). n(t_0) \)

\[ D(t_0) = \frac{[x_1^s(t-t_0)-x_1^s(-t_0) f[x_0(t-t_0)] + y_1^s(t-t_0)y_0(t-t_0)]}{\sqrt{\{f[x_0(-t_0)]^2, y_0(-t_0)^2\}}} + O(\epsilon^2) \text{ (1.9)} \]

When \( D(t_0) = 0 \) homoclinic bifurcation must occur since the distance between the manifolds vanishes to \( O(\epsilon^2) \). However, \( D(t_0) \), as it stands, requires \( x_1^u \) and \( x_1^s \) but as we shall show, \( D(t_0) \) surprisingly does not need these solutions of Eqs. (1.7) and (1.8). Let

\[ \Delta^s(t-t_0) = x_1^s(t, t_0) f[x_0(t-t_0)] + y_1^s(t, t_0)y_0(t-t_0) \]

\[ D(t_0) = \frac{\epsilon(\Delta^u(t-t_0)-\Delta^s(t-t_0))}{\sqrt{\{f[x_0(0,t_0)]^2, y_0(0,t_0)^2\}}} + O(\epsilon^2) \text{ (1.10)} \]

We now show that \( \Delta^s(0, t_0) \) and \( \Delta^u(0, t_0) \) do not require \( x_1^u(t, t_0) \) and \( x_1^s(t, t_0) \).

Differentiate \( \Delta^s(t, t_0) \) with respect to \( t \):

\[ \frac{d\Delta^s(t, t_0)}{dt} = \frac{d}{dt}[x_1^s(t, t_0) f[x_0(t-t_0)] + y_1^s(t, t_0)y_0(t-t_0)] = \dot{x}_1^s(t, t_0) f[x_0(t-t_0)] + \dot{y}_1^s(t, t_0)y_0(t-t_0) \]

\[ \frac{d\Delta^s(t, t_0)}{dt} = \frac{d}{dt}[y_0(t-t_0) + \frac{h[x_0(t-t_0), y_0(t-t_0), t]}{f[x_0(t-t_0)]}] = \frac{d}{dt}[y_0(t-t_0) h[x_0(t-t_0), y_0(t-t_0), t]] \text{ [Using Eqs. (1.7) and (1.8)]} \text{ (1.10)} \]

Now integrate Eq. (1.10) between 0 and \( \infty \) with respect to \( t \), noting that, since \( y_0(t-t_0) \to 0 \) and \( f[x_0(t-t_0)] \to 0 \) as \( t \to \infty \), then \( \Delta^s(t, t_0) \to 0 \) also. Hence

\[ \Delta^s(0, t_0) = -\int_{t_0}^{\infty} y_0(t-t_0) h[x_0(t-t_0), y_0(t-t_0), t] dt \]

In a similar manner it can be shown that
\[\Delta^u(0, t_0) = \int_{-\infty}^{0} y_0(t - t_0) h[x_0(t - t_0), y_0(t - t_0), t] \, dt\]

The numerator of the coefficients of \( \varepsilon \) in Eq. (1.9) controls homoclinic bifurcation. Let
\[M(t_0) = \Delta^u(0, t_0) - \Delta^s(0, t_0)\]
\[M(t_0) = \int_{-\infty}^{\infty} y_0(t - t_0) h[x_0(t - t_0), y_0(t - t_0), t] \, dt\]

The function \( M(t_0) \) is known as the Melnikov function associated with the origin of this system. If \( M(t_0) = 0 \) has simple zeros for \( t_0 \), then there must exist, to order \( O(\varepsilon^2) \), transverse intersections of the manifolds for these particular values of \( t_0 \). The points (there will generally be two transverse intersections) \( P_0 \) on the unperturbed homoclinic path where these intersections occur have the approximate coordinates \( (x_0(-t_0), y_0(-t_0)) \).

More general versions of Melnikov’s method applicable to periodically perturbed systems of the form
\[\dot{x} = F(x) + \varepsilon H(x, t)\]
are given by [Guckenheimer and Holmes (1983)] and [Drazin (1992)].

### 1.2.2. Computational/ Graphical Tools

#### 1.2.2.1. Numerical analysis

The equation of motion of nonlinear oscillators is in general not solvable exactly. Qualitative and quantitative ideas on the types of oscillations and their stability for small strength of the nonlinearity parameter can be obtained by making use of the one of the perturbation methods. However, a more complete picture can be obtained essentially by straightforward and detailed numerical analysis using any of the standard numerical methods. Here we give a brief of the numerical methods used in the present thesis.

In earlier days, numerical methods were impractical because they required enormous amounts of tedious hand-calculation. But all that has changed due to
availability of fast computers which enable us to approximate the solutions to analytically intractable problems, and also to visualize those solutions. In this section we take our first look at dynamics on the computer, in the context of numerical integration of \( \dot{x} = f(x) \).

**Euler’s Method**

The problem can be posed this way: given the differential equation \( \dot{x} = f(x) \), subject to the condition \( x = x_o \) at \( t = t_o \), find a systematic way to approximate the solution \( x(t) \).

Suppose we use the vector field interpretation of \( \dot{x} = f(x) \). That is, we think of a fluid flowing steadily on the x-axis, with velocity \( f(x) \) at the location \( x \). Imagine we’re riding along with a phase point being carried downstream by the fluid. Initially we’re at \( x_o \), and the local velocity is \( f(x_o) \). If we flow for a short time \( \Delta t \), we’ll have moved a distance \( f(x_o) \Delta t \), because distance = rate \( \times \) time. Of course, that’s not quite right, because our velocity was changing a little bit throughout the step. But over a sufficient small step, the velocity will be nearly constant and our approximation should be reasonably good. Hence our new position \( x(t_o + \Delta t) \) is approximately \( x_o + f(x_o)\Delta t \). Let’s call this approximation \( x_1 \). Thus

\[
x(t_o + t) \approx x_1 = x_o + f(x_o)\Delta t
\]

Now we iterate. Our approximation has taken us to a new location \( x_1 \); our new velocity is \( f(x_1) \); we step forward to \( x_2 = x_1f(x_1)\Delta t \); and so on. In general, the update rule is

\[
x_{n+1} = x_n + f(x_n)\Delta t
\]

This is the simplest possible numerical integration scheme. It is known as Euler’s method.
Euler’s method can be visualized by plotting $x$ versus $t$ in above figure. The curve shows the exact solution $x(t)$, and the open dots show its values $x(t_n)$ at the discrete times $t_n = t_0 + n\Delta t$. The black dots show the approximate values given by the Euler method. As you can see, the approximation gets bad in a hurry unless $\Delta t$ is extremely small. Hence Euler’s method is not recommended in practice, but it contains the conceptual essence of the more accurate methods to be discussed next.

**Refinements**

One problem with the Euler method is that it estimates the derivative only at the left end of the time interval between $t_n$ and $t_{n+1}$. A more sensible approach would be to use the average derivative across this interval. This is the idea behind the improved Euler method. We first take a trial step across the interval, using the Euler method. This produces a trial value $\bar{x}_{n+1} = x_n + f(x_n)\Delta t$; the tilde above the $x$ indicates that this is a tentative step, used only as a probe. Now that we’ve estimated the derivative on both ends of the interval, we average $f(x_n)$ and $f(\bar{x}_{n+1})$, and use that to take the real step across the interval. Thus the improved Euler method is

$$\begin{align*}
\bar{x}_{n+1} &= x_n + f(x_n)\Delta t \quad \text{(the trial step)} \\
x_{n+1} &= x_n + \frac{1}{2} [f(x_n) + f(\bar{x}_{n+1})] \Delta t \quad \text{(the real step)}
\end{align*}$$
This method is more accurate than the Euler method, in the sense that it tends to make a smaller error $E = |x(t_n) - x_n|$ for a given step size $\Delta t$. In both cases, the error $E \to 0$ as $\Delta t \to 0$, but the error decreases faster for the improved Euler method. One can show that $E \propto \Delta t$ for the Euler method, but $E \propto \Delta t^2$ for the improved Euler method. In the jargon of numerical analysis, the Euler method is first order, whereas the improved Euler method is second order.

Methods of third, fourth and even higher orders have been concocted, but we should keep in mind that higher order methods are not necessarily superior. Higher order methods require more calculations and function evaluations, so there’s a computational cost associated with them.

In practice, a good balance is achieved by the fourth-order Runge-Kutta method. To find $x_{n+1}$ in terms of $x_n$, this method first requires us to calculate the following four numbers:

$$k_1 = f(x_n)\Delta t$$

$$k_2 = f\left(x_n + \frac{1}{2}k_1\right)\Delta t$$

$$k_3 = f\left(x_n + \frac{1}{2}k_2\right)\Delta t$$

$$k_4 = f(x_n + k_3)\Delta t$$

Then $x_{n+1}$ is given by

$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

This method generally gives accurate results without requiring an excessively small step size $\Delta t$. Of course, some problems are nastier, and may require small steps in certain time intervals, while permitting very large steps elsewhere. In such cases, you may want to use a Runge-Kutta routine with an automatic step size control [Press et al. (1986)]. Choosing very small step size $\Delta t$ not always increases accuracy, it sometimes carries a penalty in the form of round-off error.
Round-off error occurs during every calculation, and will begin to accumulate in a serious way if $\Delta t$ is too small [Hubbard and West (1991)].

**Practical Matters**

We have several options to solve differential equations on the computer. We may write our own numerical integration routines, and plot the results using whatever graphics facilities are available [Press et.al (1986)]. A second option is to use existing packages for numerical methods. The software libraries like IMSL and NAG have a wide variety of state-of-the-art numerical integrators. These libraries are well documented, reliable, and flexible, and can be found at most university computing centers or networks. The packages Matlab, Mathematica, and Maple are more interactive and also have programs for solving ordinary differential equations.

Another option is for the researchers who want to explore dynamics, not computing is the dynamical systems software where you have to type in the equations and specify the parameters and initial conditions; the program solves the equations numerically and plots the results. Some of the very popular such packages are Phaser [Kocak (1989)] or MacMath [Hubbard and West (1992)].

**1.2.2.2. Phase Portrait**

The general form of a vector field on the phase plane is

\[ \begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*} \]

Where $f_1$ and $f_2$ are given functions. This system can be written more compactly in vector notation as

\[ \dot{x} = f(x) \]

Where \( x = (x_1, x_2) \) and \( f(x) = (f_1(x), f_2(x)) \). Here \( x \) represents a point in the phase plane, and \( \dot{x} \) is the velocity vector at that point. By flowing along the vector
field, a phase point traces out a solution $x(t)$, corresponding to a trajectory winding through the phase plane [Strogatz (1994)].

Furthermore, the entire phase plane is filled with trajectories, since each point can play the role of an initial condition.

For nonlinear systems, there is typically no hope of finding the trajectories analytically. Even when explicit formulas are available, they are often too complicated to provide much insight. Instead we will try to determine the qualitative behaviour of the solutions. Our goal is to find the system’s phase portrait directly from the properties of $f(x)$. An enormous variety of phase portraits are possible, one example is shown in figure given below:

Some of the most silent features of any phase portrait are:

- Fixed points satisfy $f(x^*) = 0$ and correspond to steady states or equilibria of the system. For example: points A, B and C are the fixed points in above figure.
- Closed orbits correspond to periodic solutions, i.e. solutions for which $x(t + T) = x(t)$ for all $t$, for some $T > 0$. For example: closed orbit D in above figure.
• The arrangement of trajectories near the fixed points and closed orbits. For example, the flow pattern near A and C is similar and different from that near B.
• The stability or instability of the fixed points and closed orbits. Here, the fixed points A, B and C are unstable, because nearby trajectories tend to move away from them, whereas the closed orbit D is stable.

1.2.2.3. Poincare Map

The Poincare section of the state space dynamics simplifies the geometric description of the dynamics by removing one of the state space dimensions [Hilborn (2000)]. The key point is that this simplified geometry nevertheless contains the ‘essential’ information about the periodicity, quasiperiodicity, chaoticity and bifurcations of the system’s dynamics.

For a three dimensional state space, the Poincare section is a two dimensional plane chosen so that the trajectories intersect the plane transversely. Figure 1.4 shows such a Poincare plane, where we have set up a Cartesian coordinate system with coordinates $u$ and $v$.

The assumed uniqueness and determinism of the solutions of the differential equations describing the dynamics of the system imply the existence of a Poincare map function, which relates a trajectory intersection point to the next intersection point. Suppose a trajectory intersects the Poincare plane at the point $(u_1, v_1)$. Then after ‘circling’ around state space, the trajectory next crosses the plane at $(u_2, v_2)$. In essence, we assume that there exists a pair of Poincare map functions that relate $(u_2, v_2)$ to $(u_1, v_1)$:

$$u_2 = P_u(u_1, v_1)$$

$$v_2 = P_v(u_1, v_1)$$
From the pair \((u_2, v_2)\), we can find \((u_3, v_3)\) and so on. Hence, if the map functions \(P_u\) and \(P_v\) are known, we have essentially all the information we need to characterize the dynamics of the system. We want to emphasize that we need not restrict ourselves to the long term behaviour of the system, that is, we need not restrict ourselves to what we have called the attractor for dissipative systems. However, most of the applications of the Poincare section technique will focus on the attractors and how they evolve as parameters changed.

1.2.2.4. Bifurcation Diagram

There are many possible motions of a dynamical system such as fixed point or equilibrium point, periodic, quasi-periodic and chaotic motion. Out of them, the simplest motion is fixed point and fixed point may be stable or unstable and the stability or instability of fixed point depends on the various parameters used in the system. If it is stable then system remains in that state even if the system is disturbed by infinitesimally small disturbance. But the state of dynamical system changes with time due to changes in the parameters like in electronic system it
may be voltage, current etc, in chemical system it may be concentration of reactants, in mechanical systems it may be damping or applied external force or sometimes parameters also changes due to surrounding atmosphere. In general if the parameters of a dynamical system change the long term dynamical may also change from fixed point to periodic or chaotic motion or vice versa. This sudden change in dynamics with parameters of a dynamical system is called bifurcation [Blanchard et al. (2006)]. For example, in a simple pendulum for small disturbance or for small external amplitude, the motion is periodic but for higher values of external forcing amplitude its motion is no longer periodic. Its motion changes from periodic to chaotic and for higher values of external forcing amplitude periodicity returns. Thus, a bifurcation diagram gives the information of the possible motions of dynamical system for a particular range of parameter. The name “bifurcation” was first introduced by Henri Poincare in 1885 in the first paper in mathematics showing such a behaviour [Poincare (1885)].

Figure 1.5. Bifurcation diagram of simple Duffing oscillator for linear damping (i.e. \( p = 1 \)) at damping coefficient \( \alpha = 0.5 \).
Phase portrait and Poincare map gives the information about the dynamics on the particular set of parameters but bifurcation provides the summary of the dynamics of the dynamical system for a range of system parameter i.e. it gives the more global picture of the dynamics, which also allows us to compare the dynamical features of various systems. A typical bifurcation diagram for the forced Duffing oscillator with linear damping is shown in Figure 1.5. In this diagram we have depicted the dynamical behavior of forced Duffing oscillator with respect to change in the amplitude of external periodic force by keeping all other parameters fixed. For the plotting of bifurcation diagram, external forcing amplitude is taken on x-axis with the step size 0.001 and some thousand x-values (by skipping some hundred values) of the Poincare map have been plotted on the y-axis for each value of forcing amplitude. It is clear from the diagram that for lower values of forcing amplitude e.g. at \( F = 0.30 \), the system contains fixed point but if we increase the value of forcing amplitude its motion changes to periodic motion (e.g. at \( F = 0.35 \)) and here it is period doubling. On further increase of the value of forcing amplitude its motion becomes chaotic (e.g. at \( F = 0.40 \)) then again at higher values of forcing amplitude periodicity returns (e.g. at \( F = 0.45 \)) which are called periodic windows. Hence the use of bifurcation diagram conveys lot of information about the dynamical behavior of physical systems.

1.2.2.5. Lyapunov Exponent

Consider two points in a space \( X_o \) and \( X_o + \Delta x_o \) each of which will generate an orbit in state space using some equation or system of equations. These orbits can be thought of as parametric functions of a variable that is something like time. If we use one of the orbits as a reference orbit, then the separation between the two orbits will also be a function of time. Because sensitive dependence can arise only in some portions of a system (like the logistic equation), this separation is also a function of the location of the initial value and has the form \( \Delta x(X_o, t) \). In a system with attracting fixed points or attracting periodic points, \( \Delta x(X_o, t) \) diminishes asymptotically with time. If a system is unstable, like pins balanced on their points, then the orbits diverge exponentially for a while, but eventually settle.
down. For chaotic points, the function \( \Delta x(X_o, t) \) will behave erratically. It is thus useful to study the mean exponential rate of divergence of two initially close orbits using the formula

\[
\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{\Delta x(X_o, t)}{\Delta x_o} \right|
\]

The limit \( \Delta x_o \to 0 \) ensures the validity of the linear approximation at any time.

This number \( \lambda \) is called the Lyapunov exponent, is useful for distinguishing among the various types of orbits. It works for discrete as well as continuous systems.

If \( \lambda < 0 \), the orbit attracts to a stable fixed point or stable periodic orbit. Negative Lyapunov exponents are characteristic of dissipative or non-conservative systems (the damped harmonic oscillator for instance). Such systems exhibit asymptotic stability, the more negative the exponent, the greater the stability. Superstable fixed points and supersatble periodic points have a Lyapunov exponent of \( \lambda = -\infty \).

If \( \lambda = 0 \), the orbit is a neutral fixed point (or an eventually fixed point). A Lyapunov exponent of zero indicates that the system is in some sort of steady state mode. A physical system with this exponent is conservative. Such systems exhibit Lyapunov stability. Take the case of two identical simple harmonic

![Image of orbits with Lyapunov Exponents](image)

Figure 1.6. Some orbits with their Lyapunov Exponents.
oscillators with different amplitudes. Because the frequency is independent of the amplitude, a phase portrait of the two oscillators would be a pair of concentric circles. The orbits in this situation would maintain a constant separation, like two flecks of dust fixed in place on a rotating record.

If $\lambda > 0$, the orbit is unstable and chaotic. Nearby points, no matter how close, will diverge to any arbitrary separation. All neighborhoods in the phase space will eventually be visited. These points are said to be unstable. For a discrete system, the orbits will look like snow on a television set. This does not preclude any organization as a pattern may emerge. Thus the snow may be a bit lumpy. For a continuous system, the phase space would be a tangled sea of wavy lines like a pot of spaghetti. A physical example can be found in Brownian motion. Although the system is deterministic, there is no order to the orbit that ensues.

1.2.2.6. Basin of Attraction

Dynamical systems can have multiple attractors and which of these is approached depends on the initial condition of the particular orbit [Ott (1993)]. The closure of the set of initial conditions which approach a given attractor is the basin of attraction for that attractor. From this definition it is clear that the orbit through an initial condition inside a given basin must remain inside the basin. Thus, basins of attraction are invariant sets.

![Figure 1.7](image)

**Figure 1.7.** (a) Potential $V(x)$ for a point particle moving in one dimension. (b) the basins of attraction for the attractors at $x = x_0$ (crosshatched) and at $x = -x_0$ (uncrosshatched).
As an example, consider the case of a particle moving in one dimension under the action of friction and the two well potential \( V(x) \) illustrated in Figure 1.7 (a). Almost every initial condition comes to rest at one of the two stable equilibrium points \( x = x_o \) or \( x = -x_o \). Figure 1.7 (b) schematically shows the basins of attraction for these two attractors in the position velocity phase space of the system. Initial conditions starting in the crosshatched region are attracted to the attractor at \( x = +x_o, \ \frac{dx}{dt} = 0 \), while initial conditions starting in the uncrosshatched region are attracted to the attractor at \( x = -x_o, \ \frac{dx}{dt} = 0 \). The boundary separating these two regions (the basin boundary) is, in this case, a simple curve. This curve goes through the unstable fixed point \( x = 0 \). Initial conditions on the basin boundary generates orbits that eventually approach the unstable fixed point \( x = 0, \ \frac{dx}{dt} = 0 \). Thus, the basin boundary is the stable manifold of an unstable invariant set. In this case the unstable invariant set is particularly simple (it is the point \( x = 0, \ \frac{dx}{dt} = 0 \)). However, the above statement also holds when the unstable invariant set is chaotic.

1.3. **Subject of this Thesis**

Dynamical systems have grown from various roots into a field of great diversity that interacts with many branches of mathematics as well as with the sciences. Nonlinearity in the oscillating system may exist in various forms e.g. in a mechanical system the nonlinearity may be due to the presence of nonlinear elastic / spring elements, nonlinear damping, system with fluid, nonlinear boundary conditions etc., in an electromagnetic system the nonlinear resistive, inductive, capacitive elements, hysteresis of ferromagnetic materials, nonlinear active elements like vacuum tube, transistor etc. may be responsible for nonlinear effects in the systems. The mechanism of supply (source) and dissipation of energy in the oscillator combinedly plays a very important role in deciding the dynamical behaviour of the system. One of the very common and ubiquitous
forms of dissipation in oscillating systems is damping. Damping mainly reduces the amplitude of oscillations. The linear (viscous) damping is one of the most common forms of damping present in many physical systems. However the consideration of nonlinear damping is also necessary in many of the engineering/physical problems such as rolling in the ship dynamics [Thompson J. M. T. et al. (1990)], vibration isolators [Mallik A. K. (1990)], drag forces in flow induced vibrations [Pippard A. B. (1989)] etc. Hence the study of oscillating physical system under the presence of nonlinear damping is also an important area of active research and specifically the study on nonlinear oscillating system under the nonlinear damping is relatively new and unexplored area of research. Hence the subject of the present thesis has been chosen centered around the study of dynamical behaviour of some of the very common nonlinear oscillators under the presence of nonlinear damping.

The main objective of this thesis is to see the effect of nonlinear damping on the dynamical behaviour of some of the ubiquitous classical nonlinear oscillators. We mainly focus on the analytical and computational studies on the driven pendulum, forced Duffing, forced Helmholtz-Duffing, parametrically driven Duffing oscillators using the nonlinear damping term proportional to the $\nu \dot{v} [\nu v]^{p-1}$, where, ‘$p$’ is known as damping exponent and ‘$\nu$’ is the velocity. We particularly pay our attention on how the damping exponent affects the global dynamical behaviour of the oscillator. In particular, we obtain analytically the threshold condition for the occurrence of homoclinic bifurcation using Melnikov technique and compare the results with the computational results. We also identified the major route to chaos and the regions of the 2D parameter space (consists of external forcing amplitude and damping coefficient) corresponding to the various types of asymptotic dynamics under linear and nonlinear damping. We also attempt to analyze how the basin of attraction patterns change with the introduction of nonlinear damping. One of the prime objectives of the present study is to identify some general dynamical features of nonlinear oscillators under nonlinear damping.
Chapter 2 is devoted to the study of dynamical behaviour of forced Duffing oscillator under nonlinear damping. This chapter is divided into three parts. First part of this chapter includes review of literature and brief introduction of Duffing oscillator. In second part of this chapter we have applied the analytical techniques namely linear stability analysis to find out the equilibrium points and Melinkov technique to find out the critical value of forcing amplitude for which first homoclinic bifurcation occurs. In third part of this chapter we have discussed the computational results obtained through the extensive computation of bifurcation diagram, Lyapunov exponent, parameter space, basin of attraction etc. and their comparison with the analytical results wherever possible.

In the first part of Chapter 3 we describe the major research work done on forced simple pendulum so far and their relevance with the present study of forced pendulum under nonlinear damping. In the second part we discuss our analytical and computational results obtained through various tools described above in this chapter.

Chapter 4 is divided in three parts. In first part of this chapter, we review the various recent works related to parameterically driven Duffing oscillator and forced Helmholtz-Duffing oscillator. Second part of this chapter is devoted to the study of parametrically driven Duffing oscillators under nonlinear damping and the third part contains our results for the forced Helmholtz Duffing oscillator under nonlinear damping.

Finally Chapter 5 summarizes all the major findings of the thesis work and presents future perspectives.