4.1 Introduction

Server vacation models are very useful for the system in which the server wants to utilize the idle time for different purpose. Application of vacation models can be found in foundries, production assembly line systems, call centers with multi-task employees, designing of local area networks and data communication system etc. Various authors have analyzed the queueing problems, considering vacations with several combinations. Very few authors only have studied the comparable work on the bulk queueing models considering variant vacation policy. It is necessary to allow the server to

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do different types of secondary jobs with different threshold policies to optimize the overall cost.

Lee et al. (1991) considered a batch arrival queue with different vacations and showed that the waiting time distributions can be obtained by simple iterative procedure. Lee et al (1994) analyzed $M^{[X]}/G/1$ queueing system with N-policy and multiple vacations, using supplementary variable technique. Krishna Reddy and Anitha (1999) studied a $M/G(a,b)/1$ queue with different vacation polices and obtained Laplace transform of the joint distribution of the queue length and the remaining service time and the remaining vacation time depending on the state of the server.

Ke (2003) discussed the optimal control of a $M/G/1$ queueing system with server startup time and two types of vacation. Madan and Choudhury (2005) discussed a batch arrival queueing system, where the server provides two stages of heterogeneous service with a modified vacation model for a $M^{[X]}/G/1$ queueing systems. Ke (2007) used supplementary variable technique to study a $M^{[X]}/G/1$ queueing systems with balking under variant vacation.

A two phase queueing system with vacation have studied by Doshi (1991), Krishna Kumar et al. (2002a), Artalejo and Choudhury (2004), Choudhury and Paul (2005), Badamchi Zadeh and Shankar (2008), Arivudainambi and Godhandaraman (2012) have studied an $M/G/1$ queue with additional second stage service and optional re-service. $M^{[X]}/(G1,G2)/1$ queue with optional re-service have studied by Madan et al.(2004). Madan and Anabosi (2003) have studied a single server queue with two types of
service, Bernoulli schedule server vacation and a single vacation policy. Arumuganathan and Judeth Malliga (2006) analyzed a bulk queue with repair of service station and set up time.

In the area of optimal design and control of queues, the N-policy has received great attention. According to this policy, the server idle until a fixed number N of customers arrives in the queue that moment the server is switched on and servers exhaustively the queue until it empties. The server is then switched off and remains idle until N customers accumulate again in the queue. Given the costs of turning the server on and having customers a waiting in the queue, an optimal value of N can be determined that minimizes the expected cost of operating the queue. This model is found to be applicable in analyzing numerous real world queueing situations such as flexible manufacturing systems, service systems, computer and telecommunication system. In many production systems it is assumed that when all the jobs are served, the machine stays idle until next job arrives. If there is a cost associated with operating the machine. It is plausible that a rational way to operate the system is to shut down the machine when the queue length is zero and bring it up again as the queue length grows to a pre-determined level of, say $N(\geq 1)$ jobs. Such a control mechanism is usually good when the machine start-up and shut-down costs are high. Various authors have analysed queueing problems of server vacations with several combinations.
4.2 Model Description

In this chapter $M^{[X]} / G(a,b)/1$ queueing system with multiple vacation, set up time, N-policy and delayed service. After a service completion if the queue size is less than $a$ the server goes for multiple vacations also after serving $M$ batches continuously even if sufficient numbers are there in the queue the server goes for a vacation with probability $\alpha_j$ or resume service with probability $(1 - \alpha_j)$. When the server returns from a vacation and if the queue length is still less than $N$, he leaves for another vacation and so on until the server find $N (N > b)$ customers in the queue, if the server finds at least $N$ customers waiting for service, then he requires a setup time $R$ to start the service. After the setup he servers a batch of $b$ customers where $b \geq a$. The probability generating function of queue size at a random epoch is obtained. Some important performance measures such as expected queue size, expected busy period and idle period are derived. Along with the cost model, particular cases and some special cases are discussed. Numerical illustration for particular values of parameters are presented.

4.2.1 Notations:

- Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided service by general bulk service rule. Let $\lambda g_k dt$, $(k = 1, 2, 3,...)$ be the first order probability that a batch of $k$ customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq g_k \leq 1$, $\sum_{k=1}^{\infty} g_k = 1$ and $X(z)$ be its PGF. Let $\lambda > 0$ is the mean arrival rate of batches.
• The service time follows a general (arbitrary) distribution with cumulative distribution function $S(.)$ and density function $s(x)$. Let $\tilde{S}(\theta)$ denote the Laplace-Stieltjes transform of $S$ and $S^0(t)$ denote the remaining service time of a batch at an arbitrary time $t$.

• The server’s vacation time follows a general (arbitrary) distribution with cumulative distribution function $V(.)$ and density function $v(x)$. Let $\tilde{V}(\theta)$ denote the Laplace-Stieltjes transform of $V$ and $V^0(t)$ denote the remaining service time of a batch at an arbitrary time $t$.

• The setup time follows a general (arbitrary) distribution with cumulative distribution function $R(.)$ and density function $r(x)$. Let $\tilde{R}(\theta)$ denote the Laplace-Stieltjes transform of $R$ and $R^0(t)$ denote the remaining set up time of a batch at an arbitrary time $t$.

• $N_s(t)$ and $N_q(t)$ are the number of customers in the service and queue respectively.

The different states of the server at time $t$ are defined as follows:

$$Y(t) = \begin{cases} 0; & \text{if the server is on essential service} \\ 1; & \text{if the server is on vacation} \\ 2; & \text{if the server is on setup work} \end{cases}$$

$Z(t) = j$, if the server is on $j^{th}$ ($1 \leq j \leq M$) vacation. Let us define the following probabilities

$$P_{i,j}(x,t) dt = P \{ N_s(t) = i, N_q(t) = j, x \leq S^0(t) \leq x + dt, Y(t) = 0 \},$$
\[ a \leq x \leq b, \ j \geq 0 \]

\[ Q_{j,n}(x,t)dt = P \{ N_q(t) = n, x \leq V^0(t) \leq x + dt, Y(t) = 1, Z(t) = j \}, \]

\[ n \geq 0, \ 1 \leq j \leq M \]

\[ R_n(x,t)dt = P \{ N_q(t) = n, x \leq R^0(t) \leq x + dt, Y(t) = 2 \}, \ n \geq 0 \]

### 4.2.2 Steady state system equations

The model is then governed by the following set of differential-difference equations:

\[- P^1_{0,1}(x) = -\lambda P_{0,1}(x) + \sum_{i=a}^{b} R_i(0)s(x) \quad (4.1)\]

\[- P^1_{i,1}(x) = -\lambda P_{i,1}(x) + \sum_{k=1}^{i} P_{i-k}(x)\lambda g_k + R_{b+i}(0)s(x), \ i \geq 1 \quad (4.2)\]

\[- P^1_{0,j}(x) = -\lambda P_{0,j}(x) + \sum_{i=a}^{b} P_{i,j-1}(0)(1 - \alpha_{j-1})s(x), 2 \leq j \leq M \quad (4.3)\]

\[- P^1_{i,j}(x) = -\lambda P_{i,j}(x) + \sum_{i=a}^{i} P_{i-k,j}(x)\lambda g_k + (1 - \alpha_{j-1} P_{i+b,j-1}(0)s(x), \]

\[ 2 \leq i \leq M, i \geq 1 \quad (4.4)\]

\[- P^1_{b,j}(x) = -\lambda P_{b,j}(x) + \sum_{i=a}^{b} P_{i,b+j}(0)s(x) + \sum_{k=1}^{i} p_{i,j-1}(x)\lambda g_k, \]

\[ 1 \leq i \leq N - b - 1 \quad (4.5)\]

\[- P^1_{b,j}(x) = -\lambda P_{b,j}(x) + \sum_{i=a}^{b} P_{i,b+j}(0)s(x) + \sum_{k=1}^{i} p_{i,j-1}(x)\lambda g_k + R_{b+i}(0)s(x), \ i \geq N - b \quad (4.6)\]
\[ -Q_{0,1}(x) = -\lambda Q_{0,1}(x) + \sum_{j=1}^{M} P_{0,j}(0)v(x) \]  \hspace{1cm} (4.7)

\[ -Q_{i,1}(x) = -\lambda Q_{i,1}(x) + \sum_{k=1}^{i} Q_{i-k,1}(0)\lambda g_k + \sum_{j=l}^{M} P_{i,j}(0)v(x), \quad 1 \leq i \leq a - 1 \]  \hspace{1cm} (4.8)

\[ -Q_{i,1}(x) = -\lambda Q_{i,1}(x) + \sum_{k=1}^{i} Q_{i-k,1}(x)\lambda g_k + \sum_{j=l}^{M} P_{i,j}(0)\alpha_j v(x), \quad i \geq a \]  \hspace{1cm} (4.9)

\[ -Q_{0,n}(x) = -\lambda Q_{0,n}(x) + Q_{0,n-1}(0)v(x), \quad n \geq 2 \]  \hspace{1cm} (4.10)

\[ -Q_{i,n}(x) = -\lambda Q_{i,n}(x) + \sum_{k=1}^{i} Q_{i-k,n}(x)\lambda g_k + Q_{i-,n-1}(0)v(x), \quad n \geq 2, \quad 1 \leq i \leq a - 1, \]  \hspace{1cm} (4.11)

\[ -Q_{i,n}(x) = -\lambda Q_{i,n}(x) + \sum_{k=1}^{i} Q_{i-k,n}(x)\lambda g_k, \quad i \geq a, \quad n \geq N \]  \hspace{1cm} (4.12)

\[ -R_{1}(x) = -\lambda R_{1}(x) + \sum_{k=1}^{N-n} R_{i-k}(x)\lambda g_k \]  

\[ + \sum_{l=1}^{\infty} Q_{i,1}(0)r(x), \quad i \geq a, \quad n \geq N \]  \hspace{1cm} (4.13)

The Laplace transforms of \( P_{i,n}(x) \), \( Q_{j,n}(x) \) and \( R_{n}(x) \) are defined as:

\[ \tilde{P}_{i,n}(\theta) = \int_{0}^{\infty} e^{-\theta x} P_{i,n}(x)dx, \quad \tilde{Q}_{j,n}(\theta) = \int_{0}^{\infty} e^{-\theta x} Q_{j,n}(x)dx, \]

\[ \tilde{R}_{n}(\theta) = \int_{0}^{\infty} e^{-\theta x} R_{n}(x)dx, \]
Taking Laplace transform on both sides of the above equations, we get

\[
\theta \tilde{P}_{0,1}(\theta) - P_{0,1}(0) = \lambda P_{0,1}(\theta) - \sum_{i=a}^{b} R_i(0) \tilde{S}(\theta), \quad (4.14)
\]

\[
\theta \tilde{P}_{i,1}(\theta) - P_{i,1}(0) = \lambda \tilde{P}_{i,1}(\theta) - \lambda \sum_{k=1}^{i} \tilde{P}_{i-k,1}(\theta) g_k - R_{i+1}(0) \tilde{S}(\theta), \quad i \geq 1
\]

\[
\theta \tilde{P}_{0,j}(\theta) - P_{0,j}(0) = \lambda \tilde{P}_{0,j}(\theta) - \lambda \sum_{i=a}^{b} \tilde{P}_{i,j-1}(0)(1 - \alpha_{j-1}) \tilde{S}(\theta), \quad 2 \leq j \leq M
\]

\[
\theta \tilde{P}_{i,j}(\theta) - P_{i,j}(0) = \lambda \tilde{P}_{i,j}(\theta) - \lambda \sum_{k=1}^{i} \tilde{P}_{i-k,j}(\theta) g_k - \sum_{k=1}^{j} \tilde{P}_{i-j-k}(\theta) \lambda g_k
\]

\[
\theta \tilde{P}_{b,j}(\theta) - P_{b,j}(0) = \lambda \tilde{P}_{b,j}(\theta) - \sum_{m=a}^{b} P_{m,b+j}(0) \tilde{S}(\theta) - \sum_{k=1}^{j} \tilde{P}_{b-j-k}(\theta) \lambda g_k
\]

\[
\theta \tilde{P}_{b,j}(\theta) - P_{b,j}(0) = \lambda \tilde{P}_{b,j}(\theta) - \sum_{m=a}^{b} P_{m,b+j}(0) \tilde{S}(\theta) - \sum_{k=1}^{j} \tilde{P}_{i-j-k}(\theta) \lambda g_k + R_{b+j}(0) \tilde{S}(\theta), \quad j \geq N - b
\]

\[
\theta \tilde{Q}_{0,1}(\theta) - Q_{0,1}(0) = \lambda \tilde{Q}_{0,1}(\theta) - \sum_{j=1}^{M} p_{0,j}(0) \tilde{v}(\theta)
\]

\[
\theta \tilde{Q}_{i,1}(\theta) - Q_{i,1}(0) = \lambda \tilde{Q}_{i,1}(\theta) - \lambda \sum_{k=1}^{i} \tilde{Q}_{i-k,1}(\theta) g_k - \sum_{j=1}^{M} p_{i,j}(\theta) \tilde{v}(\theta)
\]

\[
\theta \tilde{Q}_{i,1}(\theta) - Q_{i,1}(0) = \lambda \tilde{Q}_{i,1}(\theta) - \lambda \sum_{k=1}^{i} \tilde{Q}_{i-k,1}(\theta) g_k - \sum_{j=1}^{M} p_{i,j}(\theta) \alpha_j \tilde{v}(\theta),
\]

\[
1 \leq i \leq a - 1
\]

\[
\theta \tilde{Q}_{i,1}(\theta) - Q_{i,1}(0) = \lambda \tilde{Q}_{i,1}(\theta) - \lambda \sum_{k=1}^{i} \tilde{Q}_{i-k,1}(\theta) g_k - \sum_{j=1}^{M} p_{i,j}(\theta) \alpha_j \tilde{v}(\theta),
\]

\[
1 \leq i \leq a - 1
\]
\[ \theta \tilde{Q}_{0,n}(\theta) - Q_{0,n}(0) = \lambda \tilde{Q}_{0,n}(\theta) - Q_{0,n-1}(0)\tilde{v}(\theta), n \geq 2 \quad (4.23) \]

\[ \theta \tilde{Q}_{i,n}(\theta) - Q_{i,n}(0) = \lambda \tilde{Q}_{i,n}(\theta) - \lambda \sum_{k=1}^{i} \tilde{Q}_{i-k,n}(\theta)g_k - Q_{i,n-1}(0)\tilde{v}(\theta), \]

\[ n \geq 2; 1 \leq i \leq a - 1 \quad (4.24) \]

\[ \theta \tilde{Q}_{i,n}(\theta) - Q_{i,n}(0) = \lambda \tilde{Q}_{i,n}(\theta) - \lambda \sum_{k=1}^{i} \tilde{Q}_{i-k,n}(\theta)g_k; n \geq N, i \geq a \quad (4.25) \]

\[ \theta \tilde{R}_i(\theta) - R_i(0) = \lambda \tilde{R}_i(\theta) - \sum_{j=1}^{\infty} Q_{i,1}(0)\tilde{r}(\theta) - \lambda \sum_{k=1}^{n-N} \tilde{R}_{i-k}(\theta)g_k, i \geq a \quad (4.26) \]

### 4.3 Probability Generating Function

To obtain Probability Generating Function (PGF) of the queue at an arbitrary time epoch, the following PGF are defined

\[ \tilde{P}_i(z, \theta) = \sum_{j=0}^{\infty} \tilde{P}_{i,j}(\theta)z^j, \quad P_i(z, 0) = \sum_{j=0}^{\infty} P_{i,j}(0)z^j, \quad a \leq i \leq b \]

\[ \tilde{Q}_i(z, \theta) = \sum_{l=0}^{\infty} \tilde{Q}_{i,l}(\theta)z^j, \quad Q_j(z, 0) = \sum_{l=0}^{\infty} Q_{i,j}(0)z^l, 1 \leq j \leq M \]

\[ \tilde{R}(z, \theta) = \sum_{n=0}^{\infty} \tilde{R}_n(\theta)z^n, \quad R(z, 0) = \sum_{n=0}^{\infty} R_n(0)z^n, \quad (4.27) \]

Multiply the equations (4.14) and (4.15) with suitable power of \( z^n \), summing over \( n \ (0 \leq n < \infty \) ) and using (4.27), we get

\[ z^b(\theta - \lambda + \lambda X(z)\tilde{P}_1(z, \theta)) = z^bP_1(z, 0) \]

\[ - \left[ R(z, 0) - \sum_{i=a}^{b-1} R_i(0)(z^b - z^i) \right] \tilde{S}(\theta), \quad (4.28) \]
Multiply the equations (4.16) and (4.17) with suitable power of $z^n$, summing over $n \ (0 \leq n < \infty )$ and using (4.27), we get

$$z^b(\theta - \lambda + \lambda X(z) \tilde{P}_j(z, \theta) = z^b P_j(z, 0)$$

$$- \left[ \sum_{i=a}^{b-1} P_{i,j-1}(0)(1 - \alpha_{j-1})(z^b - z^i) + P_{j-1}(z, 0) \right]$$

$$(1 - \alpha_{j-1}) \tilde{S}(\theta), \ 2 \leq j \leq M \quad (4.29)$$

Multiply the equations (4.18) and (4.19) with suitable power of $z^n$, summing over $n \ (0 \leq n < \infty )$ and using (4.27), we get

$$(\theta - \lambda + \lambda X(z) \tilde{P}_b(z, \theta) = P_b(z, 0) - \frac{\tilde{\mathcal{S}}(\theta)}{z^b}$$

$$\left[ \sum_{m=a}^{a-1} (P_m(z, 0) - \sum_{i=a}^{b-1} P_{ij-1}(0) z^i) + R(z, 0) \right] \quad (4.30)$$

Multiply the equations (4.20), (4.21) and (4.22) with suitable power of $z^n$, summing over $n \ (0 \leq n < \infty )$ and using (4.27), we get

$$(\theta - \lambda + \lambda X(z) \tilde{Q}_1(z, \theta) = \tilde{Q}_1(z, 0) - \tilde{V}(\theta) \left[ \sum_{i=a}^{a-1} \sum_{i=1}^{M} P_{ij}(0) z^i \right]$$

$$- \sum_{i=0}^{\infty} \sum_{i=1}^{M} P_{ij}(0) \alpha_j z^i \quad (4.31)$$
Multiply the equations (4.23), (4.24) and (4.25) with suitable power of $z^n$, summing over $n$ ($0 \leq n < \infty$) and using (4.27), we get

$$
(\theta - \lambda + \lambda X(z) \tilde{Q}_n(z, \theta) = Q_n(z, 0) \tilde{S}_n(z, 0) \left[ \sum_{i=0}^{a-1} \sum_{n=1}^{\infty} (Q_{i,n}(0) z^i) \right] \check{v}(\theta), n \geq 2
$$

(4.32)

Multiply the equations (4.26) with suitable power of $z^n$, summing over $n$ ($0 \leq n < \infty$) and using (4.27), we get

$$
(\theta - \lambda + \lambda X(z) \tilde{R}(z, \theta) = R(z, 0) - \left[ \sum_{l=1}^{\infty} Q_1(z, 0) - \sum_{i=0}^{N-1} Q_{i,j}(0) z^i \right] \check{R}(\theta)
$$

(4.33)

Let $c_i = \sum_{i=1}^{M} P_{i,j}(0)(1 - \alpha_j)$, $d_i = \sum_{i=1}^{\infty} Q_{i,n}(0)$

$$
\tilde{P}(z, 0) = \sum_{j=1}^{M} \tilde{P}_j(z, \theta), \quad P(z, 0) = \sum_{j=1}^{M} P_j(z, 0)
$$

$$
\tilde{Q}(z, \theta) = \sum_{n=1}^{\infty} \tilde{Q}_n(z, \theta), \quad Q(z, 0) = \sum_{n=1}^{\infty} Q_n(z, 0)
$$

By substituting $\theta = (\lambda - \lambda X(z))$ in equations (4.27) to (4.32)

$$
z^b P_1(z, 0) = \tilde{S}(\lambda - \lambda X(z)) \left[ R(z, 0) - \sum_{i=a}^{b-1} R_i(z, 0)(z^b - z^i) \right]
$$

(4.34)

$$
z^b P_j(z, 0) = \tilde{S}(\lambda - \lambda X(z)) \left[ \sum_{i=a}^{b-1} P_{ij-1}(0)(z^b - z^i) + P_{j-1}(z, 0)(1 - \alpha_{j-1}) \right]
$$

(4.35)
\[ z^b P_b(z, 0) = \tilde{S}(\lambda - \lambda X(z)) \left[ \sum_{i=a}^{b-1} P_i(z, 0) - \sum_{m=a}^{b-1} \sum_{j=0}^{b-1} P_{m,j}(0)z^j + R(z, 0) \right] \]  \hspace{1cm} (4.36)

\[ Q_1(z, 0) = \tilde{V}(\lambda - \lambda X(z)) \left[ \sum_{i=0}^{a-1} \sum_{j=1}^{M} P_{i,j}(0)z^i - \sum_{i=a}^{\infty} \sum_{j=1}^{M} P_{i,j}(0)\alpha_j z^i \right] \]  \hspace{1cm} (4.37)

\[ Q_n(z, 0) = \tilde{V}(\lambda - \lambda X(z)) \left[ \sum_{i=0}^{N-1} Q_{i,n-1}(0)z^i \right] \]  \hspace{1cm} (4.38)

\[ R(z, 0) = \tilde{R}(\lambda - \lambda X(z)) \left[ \sum_{i=1}^{\infty} Q_i(z, 0) - \sum_{i=0}^{N-1} Q_{i,i}(0)z^i \right] \]  \hspace{1cm} (4.39)

By using expressions of \( P_1(z, 0), P_j(z, 0), P_b(z, 0), Q_1(z, 0), Q_n(z, 0) \) and \( R(z, 0) \) from (4.34)to (4.39), we get

\[ z^b \tilde{P}_1(z, \theta) = \frac{\tilde{S}(\lambda - \lambda X(z)) - \tilde{S}(\theta) \left[ R(z, 0) - \sum_{i=a}^{b-1} R_i(z, 0)(z^b - z^i) \right]}{\theta - \lambda + \lambda X(z)} \]  \hspace{1cm} (4.40)

\[ z^b \tilde{P}_j(z, \theta) = \frac{\tilde{S}(\lambda - \lambda X(z)) - \tilde{S}(\theta)}{\theta - \lambda + \lambda X(z)} \times \left[ \sum_{i=a}^{b-1} \tilde{P}_{i,j-1}(0)(z^b - z^i) + P_{j-1}(z, 0)(1 - \alpha_{j-1}) \right] \]  \hspace{1cm} (4.41)

\[ z^b \tilde{P}_b(z, \theta) = \left[ \frac{\tilde{S}(\lambda - \lambda X(z)) - \tilde{S}(\theta)f(z)}{\theta - \lambda + \lambda X(z)} \right] \]  \hspace{1cm} (4.42)

where

\[ f(z) = \sum_{m=a}^{b-1} P_m(z, 0) - \sum_{m=a}^{b} \sum_{j=0}^{b-1} P_{m,j}(0)z^j + R(z, 0) \]
\[
\sum_{m=a}^{b-1} \sum_{j=0}^{M} P_{i,j}(0)z^j + \tilde{R}(\lambda - \lambda X(z)) \left[ \sum_{l=1}^{\infty} Q_l(z, 0) - \sum_{i=0}^{N-1} Q_{1,i}(0)z^i \right]
\]

Using equation (4.36) in equation (4.30), we get

\[
\tilde{Q}_1(z, \theta) = \frac{\tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(\theta)}{\theta - \lambda + \lambda X(z)}
\]

\[
\times \sum_{i=0}^{a-1} \sum_{j=1}^{M} P_{i,j}(0)z^j - \sum_{i=a}^{\infty} \sum_{j=1}^{M} P_{i,j}(0)\alpha_j z^i
\]

(4.43)

Using equation (4.37) in equation (4.31), we get

\[
\tilde{Q}_n(z, \theta) = \left[ \frac{\tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(\theta) \sum_{i=0}^{N-1} Q_{in-1}(0)z^i}{\theta - \lambda + \lambda X(z)} \right]
\]

(4.44)

Using equation (4.38) in equation (4.32), we get

\[
\tilde{R}(z, \theta) = \left[ \frac{\tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(\theta) \left[ \sum_{i=1}^{\infty} Q_l(z, 0) - \sum_{i=0}^{a-1} Q_{l,i}(0)z^i \right]}{\theta - \lambda + \lambda X(z)} \right]
\]

(4.45)

Let \( P(z) \) of the PGF of the queue length at arbitrary time epoch. Then,

\[
P(z) = \tilde{P}(z, 0) + \tilde{Q}(z, 0) + \tilde{R}(z, 0)
\]

(4.46)

Let \( p_i = \sum_{m=a}^{b} P_{m,i}(0), \quad q_i = \sum_{l=1}^{\infty} Q_{l,i}(0) \)

Substituting \( \theta=0 \) in equations (4.41) to (6.45), we get

\[
P(z) = \frac{Nr}{Dr}
\]

(4.47)
where

\[ Nr = (\hat{s}(\lambda - \lambda X(z) - 1)) \sum_{i=a}^{b-1} (z^b - z^i)(c_i - r_i) \]

\[ + (z^b - 1) \hat{R}(\lambda - \lambda X(z))(\hat{V}(\lambda - \lambda X(z)) - 1)\alpha_j \sum_{i=0}^{M} p_i z^i \]

\[ + (z^b - 1) \hat{R}(\lambda - \lambda X(z))(\hat{V}(\lambda - \lambda X(z)) - 1) \sum_{i=0}^{N-1} q_i z^i \]

\[ Dr = (z^b - \hat{S}(\lambda - \lambda X(z)))(-\lambda + \lambda X(z)) \]

This represents the probability generating function of the queue size at arbitrary time epoch. The equation (4.47) has \( N+b \) unknowns \( p_0, p_1, p_2, p_3, \ldots, p_{b-1}, q_0, q_1, \ldots, q_{N-1} \). We express \( q_i \) in terms of \( p_i, i = 0, 1, 2, \ldots, a - 1 \). Hence the equation (4.47) involving only \( b \) unknowns \( p_0, p_1, p_2, p_3, \ldots, p_{b-1} \). By Rouche’s theorem the expression, \( (z^b - 1)\hat{S}(\lambda - \lambda X(z)) \) has \( b-1 \) zeros inside and one of the unit circle \( |z| = 1 \). Since \( P(z) \) is analytic within and on the unit circle, the numerator must vanish at these points, which gives \( 'b' \) equations with \( 'b' \) unknowns. The probability generating function has to satisfy \( P(1) = 1 \). In order to satisfy this condition, applying L’Hopital’s rule and evaluating limit \( z \to 1 \) \( P(z) \), equating the expression to 1, we have

\[ E(S)(\sum_{a}^{b-1} (c_i - r_i)(b-i)) + bE(V) \sum_{i=0}^{a-1} (c_i + \sum_{j=1}^{M} p_j(z,0)\alpha_j) = b - \lambda E(X)E(s) \]

\[ (4.48) \]

Since left hand side of (6.47) are the probabilities of \( i \) customers being in the queue it follows that left hand side must be positive. Thus \( P(1)=1 \) is
satisfied iff \((b - \lambda E(X) E(S)) > 0\).

If \(\rho = \frac{\lambda E(X) E(S)}{b}\) then \(\rho < 1\) is the condition to be satisfied for the existence of steady state.

**Theorem 4.1.** The constants \(q_n\) involved in \(P(z)\) are expressed in terms of \(q_n\) as \(q_n = \sum_{i=0}^{n} k_i p_{n-i} \) for \(n = 0, 1, \ldots, a - 1\) where

\[
k_n = \frac{\alpha_n + \sum_{j=1}^{n} \alpha_j k_{n-j}}{1 - \alpha_0}; \quad n = 1, 2, \ldots, a - 1
\]

\(k_0 = \frac{\alpha_0}{1 - \alpha_0}\) and \(\alpha_n = \sum_{i=0}^{n} \alpha_n \beta_{n-i}\)

also \(\alpha_n\) and \(\beta_n\) are the probability that \(n\) customers arrive during a vacation time.

Using equations (4.37) and (4.38) \(\sum_{n=1}^{\infty} Q_n(z, 0)\) simplifies

\[
\sum_{n=0}^{\infty} q_n z^n = \bar{V} (\lambda - \lambda X(z)) \left[ \sum_{n=0}^{a-1} p_n z^n + \sum_{n=0}^{N-1} q_n z^n \right]
\]

\[
= \left[ \sum_{n=0}^{\infty} \alpha_n z^n \right] \left[ \sum_{n=0}^{\infty} \beta_j z^n + \sum_{n=0}^{N-1} q_n z^n \right]
\]

\[
= \sum_{n=0}^{a-1} \left[ \sum_{i=0}^{n} \alpha_{n-1} \alpha_i + q_i \right] z^n + \sum_{n=a}^{\infty} \left[ \sum_{i=0}^{a-1} \alpha_{n-1} q_i \right] z^n
\]

\[
+ \sum_{n=a}^{N-1} \left[ \sum_{i=a}^{n} \alpha_{n-1} q_i \right] z^n + \sum_{n=N}^{\infty} \left[ \sum_{i=a}^{N-1} \alpha_{n-1} q_i \right] z^n
\]

with \(\alpha_i = \sum_{i=0}^{n} \alpha_0 \beta_n - i, \quad n = 0, 1, \ldots, a - 1\) equating the coefficient of \(z^n\), \(n= 0, 1, 2, 3, \ldots, a - 1\) on both sides of above equation, we get
\[ q_n = \sum_{j=0}^{n} \left[ \sum_{i=0}^{n-j} \alpha_i \beta_j \right] p_j + \sum_{n=0}^{\infty} \alpha_{n-i} q_i \]

Solving for \( q_n \), we get

\[ q_n = \frac{\left[ \sum_{j=0}^{n} \left[ \sum_{i=0}^{n-j} \alpha_i \beta_j \right] p_j + \sum_{i=0}^{n-1} \alpha_{n-i} q_i \right]}{1 - \alpha_0} \]

Coefficient of \( p_n \) in \( q_n \) is

\[ K_0 = \frac{\alpha_0 \beta_0}{1 - \alpha_0} \]

and coefficient of \( p_n \) in \( q_{n-1} \) is \( (h_1 + \alpha_1 \) coefficient of \( p_{n-1} \) in

\[ \frac{Q_n}{1 - \alpha_0} = \frac{h_1 + \alpha_1 k_0}{1 - \alpha_0} = K_1 \]

Hence by induction, again by comparing the coefficient of \( z^n \) on both sides of (4.47) for \( n = a, a + 1, a + 2, .., N - 1 \), we obtained

\[ k_n = \frac{\alpha_n + \sum_{j=1}^{n} \alpha_j k_{n-j}}{1 - \alpha_0}; n = 1, 2, ..., a - 1 \]

Hence the theorem.
4.4 Performance measures

In this section, some useful performance measures of the proposed model such as, expected number of customers in the queue $E(Q)$, expected length of idle period $E(I)$, expected length of busy period $E(B)$ are derived which are useful to find the total average cost of the system.

4.4.1 Expected Queue Length

The expected queue length $E(Q)$ (i.e. mean number of customers waiting in the queue) is obtained by differentiating $P(z)$ at $z = 1$ is given by

$$
\lim_{z \to 1} P(z) = E(Q)
$$

$$
E(Q) = \frac{Nr}{2[\lambda E(X)(b - S_1)]^2} \quad (4.49)
$$

where

$$
Nr = f_1 \sum_{i=a}^{b-1} b(b - i) - i(i - 1)(c_i - r_i)
$$

$$
+ f_2 \sum_{i=a}^{b-1} b(b - 1)(c_i - r_i) + f_3 \sum_{i=0}^{a-1} (c_i - r_i)
$$

$$
+ f_4 \left( \sum_{i=0}^{a-1} \alpha_j p_j \sum_{i=0}^{N-1} q_i \right) + f_5 \sum_{i=0}^{a-1} ip_j \alpha_j + f_6 \left( \sum_{i=0}^{a-1} ip_j + \sum_{i=0}^{N-1} iq_i \right)
$$

$$
S_1 = \lambda E(X)E(S); \; V_1 = \lambda E(X)E(V);
$$
\[ R_1 = \lambda E(X)E(R); \]
\[ S_2 = 4\lambda E(S) + \lambda^2 E^2(X)E(S^2); \]
\[ V_2 = \lambda E''(X)E(V) + \lambda^2 E^2(X)E(V^2); \]
\[ R_2 = \lambda E''(X)E(R) + \lambda^2 E^2(X)E(R^2); \quad X_2 = X''(1) \]
\[ T = \lambda E(X)(b(b - 1) - S_2) + \lambda X_2(b - S_1); \]
\[ f_1 = \lambda E(X)[b - S_1]S_1; \]
\[ f_2 = [b - S_1]S_2 - TS_1 \]
\[ f_3 = \lambda E(X)[(b - S_1)[b(b - 1) - TbV_1R_1]; \]
\[ f_4 = \lambda E(X)(b - S_1)[2bV_1R_1 + b(b - 1)V_1 + bV_2] - TbV_1; \]
\[ f_5 = \lambda E(X)(b - S_1)[b + R_1] \text{ and } f_6 = \lambda E(X)(b - S_1)bV_1 \]

### 4.4.2 Expected Length of Idle Period

Let I be the idle period random variable, then the expected length of the idle period is given by.

\[ E(I) = E(I_1) + E(R) \]

where \( I_1 \) is the random variable denoting idle period due to multiple vacation process and \( E(R) \) is the expected length of setup time.

To find \( E(I_1) \), another random variable \( U_1 \) is defined as,

\[ U_1 = \begin{cases} 
0 : & \text{if the server finds at least } a \text{ customer after the first vacation} \\
1 : & \text{if the server finds less than } a \text{ customers after the first vacation}
\end{cases} \]

Now, the expected length of idle period due to multiple vacations \( E(I_1) \) is given by
\[ E(I_1) = E(I_1/U_1 = 0)P(U_1 = 0) + E(I_1/U_1 = 1)P(U_1 = 1) \]
\[ = E(V)P(U_1 = 0) + (E(V) + E(I_1))P(U_1 = 1). \]

solving for \( E(I_1) \) , we get

\[ E(I_1) = \frac{(E(V))}{P(U_1 = 0)} + E(R) \]
\[ = \frac{E(V)}{1 - \sum_{n=1}^{a-1} \sum_{i=0}^{n} \alpha_i c_{n-i}} + E(R) \]

where \( \alpha_i \) is the probability that \( i \) customers arrive during a vacation and \( c_i \) is the probability of \( i \) customers being in the queue at a departure epoch.

### 4.4.3 Expected Length of Busy Period

Let \( B \) be the busy period random variable. Another random variable \( J \) is defined as

\[ J = \begin{cases} 
0 : & \text{if the server finds less than } a \text{ customers after a first service} \\
1 : & \text{if the server finds at least } a \text{ customers after a first service} 
\end{cases} \]

Now, the expected length of busy period \( E(B) \) is given by

\[ E(B) = E(B/J = 0)P(J = 0) + E(B/J = 1)P(J = 1) \]
\[ = E(S)P(J = 0) + (E(S) + E(B))P(J = 1) \]

where \( E(S) \) is the expected service time.

solving for \( E(B) \), we get

\[ E(B) = \frac{E(S)}{P(J = 0)} \]
Thus, the expected length of busy period is obtained as

\[ E(B) = \frac{E(S)}{\sum_{n=0}^{a-1} p_i} \]

where \( E(S) \) is the expected service time.

### 4.5 Particular Case

When no setup time and probability of delayed service \( \alpha_j = 0 \) for \( j=1,3,\ldots,M \). Then equation (4.47) reduces to

\[ P(z) = \frac{N r}{(z^b - \tilde{S}(\lambda - \lambda X(z)))(-\lambda + \lambda X(z))} \]

(4.50)

where

\[ Nr = \tilde{s}(\lambda - \lambda X(z)) - 1 \sum_{i=a}^{b-1} (z^b - z^i)c_i \]

\[ + (z^b - 1)(V(\lambda - \lambda X(z)) - 1) \sum_{i=0}^{N-1} q_i z^i \]

Equation (4.47) gives the queue size distribution of \( M[X]/G(a,b)/1 \) queueing system with multiple vacations.
4.6 Special Cases

In this section, some special cases of the proposed model by specifying vacation time random variable and setup time random variable follows exponential distribution and service time random variable as hyper exponential distribution.

4.6.1 Case 1: $M[X]/G(a,b)/1$ queueing system with N-policy and exponential vacation time

If the vacation time is assumed as exponential with probability density function $v(x) = \gamma e^{-\gamma x}$, where $\gamma$ is the parameter, then, $
\tilde{V}(\lambda - \lambda X(z)) = \frac{\gamma}{\gamma + \lambda(1 - X(z))}$. Substituting this expression for $\tilde{V}(\beta(\lambda - \lambda X(z)))$ in (4.47) and after some algebra, the PGF of the queue size distribution of this special case of the queueing model is obtained as,

$$P(z) = \frac{ Nr }{ ((-\lambda + \lambda X(z))(z^b - \tilde{S}(\alpha(\lambda - \lambda X(z)))))} \quad (4.51)$$

where

$$Nr = (\tilde{S}((\lambda - \lambda(X))) - 1) \sum_{i=a}^{b-1} (z^b - z^i) (c_i - r_i) + (z^b - 1) \tilde{R}(\lambda - \lambda X(z)) \times \left( \frac{\gamma}{\gamma + \lambda(1 - X(z))} - 1 \right) \sum_{i=0}^{M} p_i z^i + (z^b - 1) \tilde{R}(\lambda - \lambda X(z)) \times \left( \frac{\gamma}{\gamma + \lambda\beta(1 - X(z))} - 1 \right) \sum_{i=0}^{N-1} q_i z^i$$

116
4.6.2 Case 2: $M^{[X]}/G(a, b)/1$ queueing system with multiple vacations and exponential setup time

In case of exponential setup time random variable with probability density function $u(x) = ue^{-ux}$, where 'u' is the parameter, then, $\tilde{R}(\lambda - \lambda X(z)) = \frac{u}{u + \lambda((1 - X(z)))}$. Substituting this expression for $\tilde{R}(\lambda - \lambda X(z))$ in (4.47) and after some algebra, the PGF of the queue size distribution of this special case of the queueing model is obtained as,

$$P(z) = \frac{Nr}{(z^b - \tilde{S}(\lambda - \lambda X(z))(-\lambda + \lambda X(z)))}$$

where

$$Nr = \tilde{S}(\lambda - \lambda X(z) - 1) \sum_{i=a}^{b-1} (z^b - z^i)(c_i - r_i) + (z^b - 1) \left( \frac{u}{u + \lambda((1 - X(z)))} \right)$$

$$\times (\tilde{V}(\lambda - \lambda X(z)) - 1)\alpha_j \sum_{i=0}^{M} p_i z^i + (z^b - 1) \left( \frac{u}{u + \lambda((1 - X(z)))} \right)$$

$$\times (\tilde{V}(\lambda - \lambda X(z)) - 1) \sum_{i=0}^{N-1} q_i z^i$$

4.6.3 Case 3: Service time follows hyper exponential distribution

In case of hyper exponential service time random variable with probability density function $s(x) = cu e^{-ux} + (1 - c)w e^{-wx}$, where u and w are the parameter, then $\tilde{S} = \frac{uc}{u + \lambda((1 - X(z)))} + \frac{w(1-c)}{w + \lambda((1 - X(z)))}$. Substituting this expression for $\tilde{S}(\lambda - \lambda X(z))$ in (4.47) and after some algebra, the PGF of the queue
size distribution of this special case of the queueing model is obtained as,

\[ P(z) = \frac{N_r}{(z^b - \frac{uc}{u+\lambda(1-X(z))} + \frac{w(1-c)}{w+\lambda(1-X(z)))})(-\lambda + \lambda x(z))} \]

where

\[ N_r = \left( \frac{uc}{u + \lambda(1 - X(z))} + \frac{w(1-c)}{w + \lambda(1 - X(z))} \right) - 1 \sum_{i=a}^{b-1} (z^b - z^i)(c_i - r_i) \]

\[ + (z^b - 1) \tilde{R}(\lambda - \lambda X(z)) \tilde{V}(\lambda - \lambda X(z)) - 1) \alpha_j \sum_{i=0}^{M} p_i z^i \]

\[ + (z^b - 1) \tilde{R}(\lambda - \lambda X(z)) (\tilde{V}(\lambda - \lambda X(z)) - 1) \sum_{i=0}^{N-1} q_i z^i \]

### 4.7 Cost Model

Cost analysis is the most important phenomenon in any practical situation at every stage. Cost involves startup cost, operating cost, holding cost, setup cost, service cost and reward cost. It is quite natural that the management of the system desires to minimize the total average cost and to optimize the cost. Addressing this, in this section, the cost model for the proposed queueing system is developed and the total average cost is obtained with the following assumptions:

- \( C_s \): Startup cost per cycle
- \( C_h \): Holding cost per customer per unit time
- \( C_o \): Operating cost per unit time
- \( C_r \): Reward cost per cycle due to vacation
$C_u$: Setup cost per cycle

Since the length of the cycle is the sum of the idle period and busy period, the expected length of cycle, $E(T_c)$ is obtained as

$$E(T_c) = E(\text{length of the Idle Period}) + E(\text{length of the Busy Period})$$

$$E(T_c) = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \left[ \sum_{i=0}^{a} \beta_i (p_{n-i} + r_{n-i}) \right]} + E(R)$$

$$= \frac{E(S)}{\sum_{n=0}^{a-1} (p_n + r_n)} + E(R)$$

Now, the total average cost (TAC) per unit time is obtained as

Total average cost (TAC) = Start-up cost per cycle + Holding cost of number of customers in the queue per unit time Operating + Operating cost per unit time $\times \rho$

- Reward due to vacation of type one per cycle

- Reward due to vacation of type two per cycle

$$TAC = \left[ C_s \frac{1}{E(T_c)} + C_h E(Q) + C_o \rho - \frac{1}{E(T_c)} \left[ C_r 1 \frac{E(V)}{P(U_1 = 0)} \right] + C_u E(U) \right]$$

where $\rho = \frac{\lambda E(X) E(S)}{b}$

To minimize the total average cost, we simple direct search method to find optimal policy for a threshold value $a^*$ to minimize the total average cost, is defined.

Step: 1. Fix the value of maximum batch size $b$
Step: 2. Select the value of $a$ which will satisfy the following relation 
\[ TAC(a^*) \leq TAC(a), \ 1 \leq a \leq b. \]

Step: 3. The value $a^*$ is optimum, since it gives minimum total average cost. Using above procedure, the optimal value of $a$ can be obtained, which minimize that total average cost function.

### 4.8 Numerical Results

The above queueing model is analysed numerically with the following assumptions:

(i) Service time distribution is Erlang-$k$ with $k = 2$

(ii) arrivals follows geometric distribution with mean 2

(iii) Vacation time and setup time are exponential with parameters $\nu = 10$, and $\eta = 8$ respectively

(iv) minimum Service capacity and Service capacity maximum $a = 2$, maximum $b = 5$ and $N=6$. The zero of the function $z^b - \tilde{S}(\lambda - X(z))$ and the simultaneous equations are solved by using MATLAB. Results are presented for the service rate $\mu = 5$ and the arrival rate ranging from 1.1 to 1.5 in the following data.
Table 4.1: Computed values of various queue characteristics

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\rho$</th>
<th>$E(Q)$</th>
<th>$E(I)$</th>
<th>$E(B)$</th>
<th>$E(W)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>0.1760</td>
<td>0.8870</td>
<td>0.4028</td>
<td>0.7652</td>
<td>0.4031</td>
</tr>
<tr>
<td>1.2</td>
<td>0.1920</td>
<td>2.2890</td>
<td>0.3889</td>
<td>1.2685</td>
<td>0.9537</td>
</tr>
<tr>
<td>1.3</td>
<td>0.2080</td>
<td>5.4286</td>
<td>0.3768</td>
<td>2.5911</td>
<td>2.0879</td>
</tr>
<tr>
<td>1.4</td>
<td>0.2240</td>
<td>7.1120</td>
<td>0.3668</td>
<td>4.4622</td>
<td>2.5400</td>
</tr>
<tr>
<td>1.5</td>
<td>0.2400</td>
<td>8.4142</td>
<td>0.3583</td>
<td>6.4004</td>
<td>2.8047</td>
</tr>
</tbody>
</table>

The Table 4.1 clearly shows as long as increasing the arrival rate $\lambda$, the proportion of expected length of idle period decreases while the utilization factor, the expected queue length, expected waiting time, expected length of busy period of our queueing model are all increases.
4.8.1 Graphical Study

Figure 4.1 shows that the effect of arrival rate $\lambda$ over the $E(Q)$ and $E(B)$, the expected number of customer in the queue and expected busy period increases as arrival rate $\lambda$ increase.