CHAPTER-3
CHAPTER III

ON SOME COMMON RANDOM FIXED POINT THEOREMS WITH PPF DEPENDENCE IN BANACH SPACES

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Summary

In this chapter, some results concerning the existence of common random fixed points and random coincidence points with PPF dependence are proved for the pairs of random operators in Banach spaces satisfying a generalized contractive condition. The novelty of the present work lies in the fact that the domain and the range spaces of the random operators in question are not same and all the results are obtained via constructive methods. Our results generalize and extend the random fixed point theorems with PPF dependence of Dhage [39] and Dhage et al. [46] under more general contractive conditions.

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3.1 Introduction

In a recent paper [39], the author proved some random fixed point theorems for nonlinear random operators in Banach spaces, where the domain and range of the operators are not same. The random fixed point theorems of this type are called PPF (past, present and future) dependent random fixed point theorems or the random fixed point theorems with PPF dependence. Some basic random fixed point theorems along this line such as those established in Dhage [39] are useful for proving the random solutions of nonlinear functional random differential and integral equations which may depend upon the past history, present data and future consideration. The properties of a special Razumikhin class of functions are employed in the development of random fixed point theory with PPF dependence in abstract spaces.

The topic of common fixed point theorem for pairs or families of mappings in metric and abstract spaces is of great interest and has already been studied in the literature since long time. It seems that theory of common fixed point theorems has reached its culmination point and there are a good number of common fixed point theorems available for commuting as well as noncommuting mappings in metric spaces satisfying different contractive conditions. However, to the best of our knowledge there is no any result proved so far in the literature concerning the common random fixed point theorems for the mappings in abstract spaces with different domain and range spaces. In the present chapter,
some common random fixed point theorems with PPF dependence are proved for pairs of random operators in Banach spaces satisfying generalized contractive conditions. We claim that our results of this chapter are new and generalize some known basic results those proved in Bernfeld et al. [13] and Dhage [39] under more general contractive conditions.

3.2 Preliminaries

Throughout this chapter, let $(\Omega, \mathcal{A})$ be a measurable space and let $E$ be a separable Banach space with norm $\| \cdot \|_E$. We equip the Banach space $E$ with a $\sigma$-algebra $\beta_E$ of Borel subsets of $E$ so that $(E, \beta_E)$ becomes a measurable space. A mapping $x : \Omega \to E$ is called measurable if

$$x^{-1}(B) = \{ \omega \in \Omega \mid x(\omega) \in B \} \in \mathcal{A} \quad (3.2.1)$$

for all Borel sets $B \in \beta_E$.

Given two Banach spaces $E_1$ and $E_2$, a mapping $Q : \Omega \times E_1 \to E_2$ is called a random operator if $Q(\omega, x)$ is measurable in $\omega$ for all $x \in E_1$. We also denote a random operator $Q$ on $E_1$ by $Q(\omega)x = Q(\omega, x)$. A random operator $Q(\omega)$ is called continuous on $E$ if $Q(\omega, x)$ is continuous in $x$ for each $\omega \in \Omega$. Similarly, $Q$ is called compact on $\Omega \times E_1$ if $Q(\Omega \times E_1)$ is a relatively compact subset of $E_2$. Finally, $Q(\omega)$ is called compact on $E_1$ if $Q(\omega, E_1)$ is a relatively compact subset of $E_2$ for each $\omega \in \Omega$.

The following theorem is often used in the theory of random
fixed point theory and nonlinear discontinuous random differential equations. We also need this result in the subsequent part of this chapter.

**Theorem 3.2.1 (Carathéodory).** Let $Q : \Omega \times E_1 \to E_2$ be a mapping such that $Q(\omega, x)$ is measurable in $\omega$ for each $x \in E_1$ and $Q(\omega, x)$ is continuous in $x$ for each $\omega \in \Omega$. Then the map $(\omega, x) \mapsto Q(\omega, x)$ is jointly measurable.

Given a closed and bounded interval $I = [a, b]$ in $\mathbb{R}$, the set of real numbers, for some $a, b \in \mathbb{R}$, $a < b$, and given a Banach space $E$, let $E_0 = C(I, E)$ denote the Banach space of continuous $E$-valued functions defined on $I$ equipped with the supremum norm $\| \cdot \|_{E_0}$ defined by

$$\|x\|_{E_0} = \sup_{t \in I} \|x(t)\|_E.$$  \hspace{1cm} (3.2.2)

Let $Q : \Omega \times E_0 \to E$ be a random operator. A measurable function $\xi^* : \Omega \to E_0$ is called a PPF dependent random fixed point of the random operator $Q(\omega)$ if

$$Q(\omega, \xi^*(\omega)) = \xi^*(c, \omega)$$

for some $c \in I$. Any mathematical statement that guarantees the existence of PPF dependent random fixed point of the random operator $Q(\omega)$ is a random fixed point theorem with PPF dependence or a PPF dependent random fixed point theorem.

For a fixed $c \in I$, a **Razumikhin class** or **minimal class** of functions in $E_0$ (cf. [13]) is defined as

$$\mathcal{R}_c = \{ \phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E \}.$$ \hspace{1cm} (3.2.3)
The class $R_c$ is algebraically closed with respect to difference if $\phi - \xi \in R_c$ whenever $\phi, \xi \in R_c$. Similarly, $R_c$ is topologically closed if it is closed w.r.t. the topology on $E_0$ generated by the norm $\| \cdot \|_{E_0}$.

It is known that Razumikhin class of functions plays a significant role in proving the existence of PPF dependence fixed points with different domain and range of the operators in question. See Bernfeld et al. [13], Dhage [39] and the references therein. Below we give different classes of contractive random operators for having common random fixed point theorems with PPF dependence in Banach spaces.

### 3.3 PPF Random Contractions and Fixed Points

In this section we state a basic PPF dependent random fixed point theorem proved earlier in Dhage [39] for PPF random contraction operators in Banach spaces. Throughout this chapter, unless otherwise mentioned let $E$ denote a Banach space and let $E_0 = C(I, E)$, where $I$ is a closed and bounded interval in $\mathbb{R}$. The following definitions have been introduced in Dhage [39] and Dhage et. al. [46].

**Definition 3.3.1.** A random operator $Q : \Omega \times E_0 \to E$ is called a PPF random contraction if for each $\omega \in \Omega$,

$$\|Q(\omega, \xi) - Q(\omega, \eta)\|_E \leq \lambda(\omega)\|\xi - \eta\|_{E_0} \quad (3.3.1)$$
for all $\xi, \eta \in E_0$, where $\lambda : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \leq \lambda(\omega) < 1$ for all $\omega \in \Omega$.

**Definition 3.3.2.** A random operator $Q : \Omega \times E_0 \to E$ is called a strong PPF random contraction if for a given $c \in I$ and for each $\omega \in \Omega$,

$$
\|Q(\omega, \xi) - Q(\omega, \eta)\|_E \leq \lambda(\omega) \|\xi(c, \omega) - \eta(c, \omega)\|_E \quad (3.3.2)
$$

for all $\xi, \eta \in E_0$, where $\lambda : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \leq \lambda(\omega) < 1$ for all $\omega \in \Omega$.

**Definition 3.3.3.** A random operator $Q : \Omega \times E_0 \to E$ is called a Kannan type strong PPF random contraction if for a given $c \in I$ and for each $\omega \in \Omega$,

$$
\|Q(\omega, \xi) - Q(\omega, \eta)\|_E \leq \alpha(\omega) \left( \|\xi(c, \omega) - Q(\omega, \xi(\omega))\|_E + \|\eta(c, \omega) - Q(\omega, \eta(\omega))\|_E \right) \quad (3.3.3)
$$

for all $\xi, \eta \in E_0$, where $\alpha : \Omega \to \mathbb{R}_+$ is measurable function satisfying $\alpha(\omega) < 1/2$ for all $\omega \in \Omega$.

**Definition 3.3.4.** A random operator $Q : \Omega \times E_0 \to E$ is called a strong PPF random contraction of Riech type if for a given $c \in I$ and for each $\omega \in \Omega$,

$$
\|Q(\omega, \xi) - Q(\omega, \eta)\|_E \\
\leq \alpha(\omega) \|\xi(c) - Q(\omega, \xi(\omega))\|_E \\
+ \beta(\omega) \|\eta(c) - Q(\omega, \eta(\omega))\|_E + \gamma(\omega) \|\xi(c) - \eta(c)\|_E \quad (3.3.4)
$$
for all $\xi, \eta \in E_0$, where $\alpha, \beta, \gamma : \Omega \to \mathbb{R}_+$ are measurable functions satisfying $\alpha(\omega) + \beta(\omega) + \gamma(\omega) < 1$ for all $\omega \in \Omega$.

**Definition 3.3.5.** A random operator $Q : \Omega \times E_0 \to E$ is called a PPF random contraction of Reich type if for a given $c \in I$ and for each $\omega \in \Omega$,

$$
\|Q(\omega, \xi) - Q(\omega, \eta)\|_E \\
\leq \alpha(\omega) \|\xi(c) - Q(\omega, \xi(\omega))\|_E \\
+ \beta(\omega) \|\eta(c) - Q(\omega, \eta(\omega))\|_E + \gamma(\omega)\|\xi - \eta\|_{E_0}
$$

(3.3.5)

for all $\xi, \eta \in E_0$, where $\alpha, \beta, \gamma : \Omega \to \mathbb{R}_+$ are measurable functions satisfying $\alpha(\omega) + \beta(\omega) + \gamma(\omega) < 1$ for all $\omega \in \Omega$.

**Remark 3.3.1.** Notice that every strong PPF random contraction is PPF random contraction, but the converse may not be true. Further Reich type random operators include the classes of PPF random contraction as well as Kannan type random operators in Banach spaces.

The following basic random fixed point theorem with PPF dependence is proved in Dhage et al. [46].

**Theorem 3.3.1.** Let $(\Omega, A)$ be a measurable space and let $E$ be a separable Banach space. If the random operator $Q : \Omega \times E_0 \to E$ is continuous and PPF random contraction, then the following statements hold in $E$.

(a) If $R_c$ is algebraically closed with respect to difference, then for a given $\xi_0 \in E_0$ and for a given $c \in I$, every sequence
\{\xi_n(\omega)\} of measurable functions satisfying

\[
\begin{aligned}
Q(\omega, \xi_n(\omega)) &= \xi_{n+1}(c, \omega) \\
\|\xi_n(\omega) - \xi_{n+1}(\omega)\|_{E_0} &= \|\xi_n(c, \omega) - \xi_{n+1}(c, \omega)\|_{E}
\end{aligned}
\] (3.3.6)

converges to a PPF dependent random fixed point of the random operator $Q(\omega)$, i.e. there is a measurable function $\xi^* : \Omega \to E_0$ such that for each $\omega \in \Omega$,

\[
Q(\omega, \xi^*(\omega)) = Q(\omega)(\xi^*(\omega)) = \xi^*(c, \omega).
\]

(b) Given the measurable functions $\xi_0, \eta_0 : \Omega \to E_0$, let $\{\xi_n(\omega)\}$ and $\{\eta_n(\omega)\}$ be the sequences of iterates of measurable functions corresponding to $\xi_0(\omega)$ and $\eta_0(\omega)$ constructed as in (a). Then,

\[
\|\xi_n(\omega) - \eta_n(\omega)\|_{E_0} \leq \frac{1}{1 - \lambda(\omega)} \left[\|\xi_0(\omega) - \xi_1(\omega)\|_{E_0} + \|\xi_0(\omega) - \xi_1(\omega)\|_{E_0}\right] + \|\xi_0(\omega) - \eta_0(\omega)\|_{E_0}.
\]

If, in particular $\xi_0(\omega) = \eta_0(\omega)$ and $\{\xi_n(\omega)\} \neq \{\eta_n(\omega)\}$ for each $\omega \in \Omega$, then

\[
\|\xi_n(\omega) - \eta_n(\omega)\|_{E_0} \leq \frac{2}{1 - \lambda(\omega)} \|\xi_0(\omega) - \xi_1(\omega)\|_{E_0}.
\]

(c) Finally, if $R_c$ is topologically closed, then for a given measurable function $\xi_0 : \Omega \to E_0$, every sequence $\{\xi_n(\omega)\}$ of iterates of $Q(\omega)$ constructed as in (a), converges to a unique PPF dependent random fixed point $\xi^*(\omega)$ of $Q(\omega)$, i.e. there is a unique measurable function $\xi^* : \Omega \to E_0$ such that $Q(\omega, \xi^*(\omega)) = \xi^*(c, \omega)$ for all $\omega \in \Omega$. 

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The generalizations of Theorem 3.3.1 can be obtained in two ways. Firstly, it can be obtained by generalizing the class of PPF random contractions and secondly, it can be obtained by extending it to two or more PPF random contraction operators for existence of the common PPF dependent random fixed points. In this chapter, we unify both the above approaches and establish some interesting common random fixed point theorems with PPF dependence for a pair of random operators in separable Banach spaces. In the following section we prove our main PPF dependent common random fixed point theorems for pairs of random operators satisfying generalized PPF contractive conditions in separable Banach spaces.

3.4 PPF Dependent Common Fixed Point Theory

Let \( S, T : \Omega \times E_0 \to E \) be two random operators. A measurable function \( \xi^* : \Omega \to E_0 \) is called a PPF dependent common random fixed point of \( S \) and \( T \) if \( S(\omega, \xi^*(\omega)) = \xi^*(\omega, c) = T(\omega, \xi^*(\omega)) \) for some \( c \in I \) and any statement that guarantees existence of the PPF dependent common random fixed points of the random operators \( S \) and \( T \) is called a PPF dependent common random fixed point theorem for the random operators in Banach spaces.

Example 3.4.1. Given a measurable space \((\Omega, \mathcal{A})\) and given a
closed and bounded interval $I = [0, 1]$ in $\mathbb{R}$, the real line, define $E_0 = C(I, \mathbb{R})$. Consider two random operators $S, T : \Omega \times E_0 \to \mathbb{R}$ defined by

$$S(\omega, \phi) = \|\phi\| \quad \text{and} \quad T(\omega, \phi) = \|\phi\|^2$$

for all $\omega \in \Omega$.

Then the random operator $S$ and $T$ have a PPF dependent common fixed point $\phi^*(t) = t^2$, $t \in [0, 1]$. Note that

$$S(\omega, \phi) = \|\phi\|_C = \sup_{t \in [0,1]} |\phi(t)| = \sup_{t \in [0,1]} t^2 = 1$$

and

$$T(\omega, \phi) = \|\phi\|^2_C = 1 \quad \text{and} \quad \phi(\omega, 1) = 1.$$

Hence, $\phi(\omega, t) = t^2$ is a PPF dependent common random fixed point of $S(\omega)$ and $T(\omega)$, since $S(\omega, \phi) = \phi(\omega, 1) = T(\omega, \phi)$ for all $\omega \in \Omega$.

We remark that the two random operators defined above may have more than one PPF dependent random fixed points. Therefore, the topic of uniqueness of a PPF dependent common fixed point is of great interest and in the present section we prove some common fixed point theorems for a pair of random operators with PPF dependence. We need the following definitions in what follows.

**Definition 3.4.1.** Two operators $S, T : \Omega \times E_0 \to E$ is said to satisfy a condition of strong generalized PPF random contraction.
if there exists a measurable function \( \lambda : \Omega \to \mathbb{R}_+ \), \( 0 < \lambda(\omega) < 1 \) satisfying for each \( \omega \in \Omega \),

\[
\|S(\omega, \phi) - T(\omega, \xi)\|_E \\
\leq \lambda(\omega) \max \left\{ \|\phi(\omega, c) - \xi(\omega, c)\|_E, \|\phi(c) - S(\omega, \phi)\|_E, \|\xi(c) - T(\omega, \xi)\|_E, \frac{1}{2} \left[ \|\phi(\omega, c) - T(\omega, \xi)\|_E + \|\xi(\omega, c) - S(\omega, \phi)\|_E \right] \right\}
\]

(3.4.1)

for all measurable functions \( \phi, \xi : \Omega \to E_0 \) and for some \( c \in [a, b] \).

**Definition 3.4.2.** Two operators \( S, T : \Omega \times E_0 \to E \) is said to satisfy a condition of generalized PPF random contraction if there exists a measurable function \( \lambda : \Omega \to \mathbb{R}_+ \), \( 0 < \lambda(\omega) < 1 \) satisfying for each \( \omega \in \Omega \),

\[
\|S(\omega, \phi) - T(\omega, \xi)\|_E \leq \lambda(\omega) \max \left\{ \|\phi(\omega) - \xi(\omega)\|_{E_0}, \|\phi(\omega, c) - S(\omega, \phi)\|_E, \|\xi(c) - T(\omega, \xi)\|_E, \frac{1}{2} \left[ \|\phi(\omega, c) - T(\omega, \xi)\|_E + \|\xi(\omega, c) - S(\omega, \phi)\|_E \right] \right\}
\]

(3.4.2)

for all measurable functions \( \phi, \xi : \Omega \to E_0 \) and for some \( c \in [a, b] \).

It is known that every strong generalized PPF random contraction is generalized PPF random contraction, however the converse is not necessarily true. Again, the class of generalized contraction operators include all other classes of random operators mentioned in Dhage [39] and Dhage et al. [46] as special cases.
Theorem 3.4.1. Suppose that $S, T : \Omega \times E_0 \to E$ are two continuous random operators and satisfy the condition of generalized PPF random contraction. Then the following statements hold.

(a) If $\mathcal{R}_c$ is algebraically closed w.r.t the difference, then for a given measurable function $\phi_0 : \Omega \to E_0$ and $c \in [a, b]$, every sequence $\{\phi_n(\omega)\}$ of iterates of $S$ and $T$ defined by

$$
S(\omega, \phi_{2n}(\omega)) = \phi_{2n+1}(\omega, c), \quad T(\omega, \phi_{2n+1}(\omega)) = \phi_{2n+2}(\omega, c);
$$

$$
\|\phi_n(\omega) - \phi_{n+1}(\omega)\|_{E_0} = \|\phi_n(\omega, c) - \phi_{n+1}(\omega, c)\|_{E_0}
$$

(3.4.3)

for $n = 0, 1, 2, \ldots$, converges to a PPF dependent common random fixed point of $S$ and $T$.

(b) If $\phi_0, \xi_0 : \omega \to E_0$ are measurable and $\{\phi_n(\omega)\}$, $\{\xi_n(\omega)\}$ are sequences of measurable functions defined by (3.4.3). Then, for each $\omega \in \Omega$,

$$
\|\phi_n(\omega) - \xi_n(\omega)\|_{E_0} \\
\leq \left(\frac{1}{1 - \lambda(\omega)}\right) \left[\|\phi_0(\omega) - \phi_1(\omega)\|_{E_0} + \|\xi_0(\omega) - \xi_1(\omega)\|_{E_0}\right]
$$

$$
+ \|\phi_0(\omega) - \xi_0(\omega)\|_{E_0}.
$$

If, in particular $\phi_0 \equiv \xi_0$ on $\Omega$ and $\{\phi_n(\omega)\} \neq \{\xi_n(\omega)\}$, then

$$
\|\phi_n(\omega) - \xi_n(\omega)\|_{E_0} \leq \left[\frac{2}{1 - \lambda(\omega)}\right] \|\phi_0(\omega) - \xi_0(\omega)\|_{E_0},
$$

for each $\omega \in \Omega$.

(c) If $\mathcal{R}_c$ is topologically closed, then $S(\omega)$ and $T(\omega)$ have a unique PPF dependent random fixed point in $\mathcal{R}_c$. 

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**Proof.** Let \( \phi_0 : \Omega \rightarrow E_0 \) be an arbitrary measurable function and define a sequence \( \{\phi_n(\omega)\} \) in \( E_0 \) as follows. By hypothesis, \( S(\omega, \phi_0(\omega)) \in E \). Suppose that \( S(\omega, \phi_0(\omega)) = x_1(\omega) \). Choose \( \phi_1 : \Omega \rightarrow E_0 \) such that

\[
x_1(\omega) = \phi_1(c, \omega) \quad \text{and} \quad \|\phi_1(\omega) - \phi_0(\omega)\|_{E_0} = \|\phi_1(c, \omega) - \phi_0(c, \omega)\|_E.
\]

Again, by hypothesis, \( T(\omega, \phi_1) \in E \). Suppose that \( T(\omega, \phi_1) = x_2(\omega) \). Choose \( \phi_2 : \Omega \rightarrow E_0 \) such that

\[
x_2(\omega) = \phi_2(c, \omega) \quad \text{and} \quad \|\phi_2(\omega) - \phi_1(\omega)\|_{E_0} = \|\phi_2(c, \omega) - \phi_1(c, \omega)\|_E.
\]

Proceeding in this way, by induction, we obtain

\[
S(\omega, \phi_{2n}(\omega)) = \phi_{2n+1}(c, \omega); \quad T(\omega, \phi_{2n+1}(\omega)) = \phi_{2n+2}(c, \omega)
\]

and

\[
\|\phi_n(\omega) - \phi_{n+1}(\omega)\|_{E_0} = \|\phi_n(c, \omega) - \phi_{n+1}(c, \omega)\|_E
\]

for all \( n = 0, 1, \ldots \).

We claim that \( \{\phi_n(\omega)\} \) is a Cauchy sequence in \( E_0 \). Now for \( n = 0 \), we have the following estimate:

\[
\|\phi_1(\omega) - \phi_2(\omega)\|_{E_0} \\
= \|\phi_1(c, \omega) - \phi_2(c, \omega)\|_E \\
= \|S(\omega, \phi_0(\omega)) - T(\omega, \phi_1(\omega))\|_E \\
\leq \lambda \max \left\{ \|\phi_0(\omega) - \phi_1(\omega)\|_{E_0}, \|\phi_0(c, \omega) - S\phi_0\|_E, \|\phi_1(c, \omega) - T(\omega, \phi_1(\omega))\|_E, \right. \\
\left. \frac{1}{2} \left[ \|\phi_0(c, \omega) - T(\omega, \phi_1(\omega))\|_E \right] \right\}
\]

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\[
\begin{align*}
&+ \| \phi_1(c, \omega) - S(\omega, \phi_0(\omega)) \|_E \bigg]\bigg) \\
\leq & \lambda \max \left\{ \| \phi_0(\omega) - \phi_1(\omega) \|_{E_0}, \right. \\
&\left. \| \phi_0(c, \omega) - \phi_1(c, \omega) \|_E, \| \phi_1(c, \omega) - \phi_2(c, \omega) \|_E, \right. \\
&\left. \frac{1}{2} \left[ \| \phi_0(c, \omega) - \phi_2(c, \omega) \|_E + \| \phi_1(c, \omega) - \phi_1(c, \omega) \|_E \right] \right\} \\
\leq & \lambda \max \left\{ \| \phi_0(\omega) - \phi_1(\omega) \|_{E_0}, \right. \\
&\left. \| \phi_0(\omega) - \phi_1(\omega) \|_{E_0}, \| \phi_1(\omega) - \phi_2(\omega) \|_{E_0}, \right. \\
&\left. \frac{1}{2} \left[ \| \phi_0 - \phi_2 \|_{E_0} + \| \phi_1(\omega) - \phi_1(\omega) \|_{E_0} \right] \right\} \\
\leq & \lambda \max \left\{ \| \phi_0(\omega) - \phi_1(\omega) \|_{E_0}, \right. \\
&\left. \| \phi_0(\omega) - \phi_1(\omega) \|_{E_0}, \right. \\
&\left. \frac{1}{2} \| \phi_0(\omega) - \phi_2(\omega) \|_{E_0} \right\} \\
\leq & \lambda \max \left\{ \| \phi_0(\omega) - \phi_1(\omega) \|_{E_0}, \right. \\
&\left. \frac{1}{2} \left[ \| \phi_0(\omega) - \phi_1(\omega) \|_{E_0} \\
&+ \| \phi_1(\omega) - \phi_2(\omega) \|_{E_0} \right] \right\} \\
\leq & \lambda(\omega) \| \phi_0(\omega) - \phi_1(\omega) \|_{E_0}.
\end{align*}
\]

Similarly,

\[
\begin{align*}
\| \phi_2(\omega) - \phi_3(\omega) \|_{E_0} \\
= & \| \phi_2(c, \omega) - \phi_3(c, \omega) \|_E \\
= & \| S(\omega, \phi_2(\omega)) - T(\omega, \phi_1(\omega)) \|_E \\
\leq & \lambda \max \left\{ \| \phi_2(\omega) - \phi_1(\omega) \|_{E_0}, \right. \\
&\left. \| \phi_2(c, \omega) - S(\omega, \phi_2(\omega)) \|_E, \| \phi_1(c, \omega) - T(\omega, \phi_1(\omega)) \|_E, \right. \\
&\left. \frac{1}{2} \left[ \| \phi_2(c, \omega) - T(\omega, \phi_1(\omega)) \|_E \\
&+ \| \phi_1(c, \omega) - S(\omega, \phi_2(\omega)) \|_E \right] \right\} \\
\end{align*}
\]
\[ \leq \lambda \max \left\{ \| \phi_2(\omega) - \phi_1(\omega) \|_{E_0}, \right. \\
\left. \| \phi_2(c, \omega) - \phi_3(c, \omega) \|_E, \| \phi_1(c) - \phi_2(c) \|_E, \right. \\
\left. \frac{1}{2} \left[ \| \phi_2(c) - \phi_2(c) \|_E \\
+ \| \phi_1(c, \omega) - \phi_3(c, \omega) \|_E \right] \right\} \\
\leq \lambda \max \left\{ \| \phi_1(\omega) - \phi_2(\omega) \|_{E_0}, \right. \\
\left. \| \phi_2(\omega) - \phi_3(\omega) \|_{E_0}, \| \phi_1(\omega) - \phi_2(\omega) \|_{E_0}, \right. \\
\left. \frac{1}{2} \left[ \| \phi_2(\omega) - \phi_2(\omega) \|_{E_0} \\
+ \| \phi_1(\omega) - \phi_3(\omega) \|_{E_0} \right] \right\} \\
\leq \lambda \max \left\{ \| \phi_1(\omega) - \phi_2(\omega) \|_{E_0}, \right. \\
\left. \| \phi_2(\omega) - \phi_3(\omega) \|_{E_0}, \frac{1}{2} \| \phi_1(\omega) - \phi_3(\omega) \|_{E_0} \right\} \\
\leq \lambda \max \left\{ \| \phi_1(\omega) - \phi_2(\omega) \|_{E_0}, \right. \\
\left. \frac{1}{2} \left[ \| \phi_1(\omega) - \phi_2(\omega) \|_{E_0} + \| \phi_2(\omega) - \phi_3(\omega) \|_{E_0} \right] \right\} \\
\leq \lambda(\omega)\| \phi_1(\omega) - \phi_2(\omega) \|_{E_0}. \]

Proceeding in this way, by induction, we obtain

\[ \| \phi_n(\omega) - \phi_{n+1}(\omega) \|_{E_0} \leq \lambda(\omega)\| \phi_{n-1}(\omega) - \phi_n(\omega) \|_{E_0} \quad (3.4.4) \]

for all \( n = 1, 2, \ldots \).

Hence, by repeated application of the above inequality yields

\[ \| \phi_n(\omega) - \phi_{n+1}(\omega) \|_{E_0} \leq \lambda^n(\omega)\| \phi_0(\omega) - \phi_1(\omega) \|_{E_0} \]

for all \( n = 1, 2, \ldots \).
If \( m > n \), by triangle inequality, we obtain

\[
\|\phi_m(\omega) - \phi_n(\omega)\|_{E_0} \\
\leq \|\phi_n(\omega) - \phi_{n+1}(\omega)\|_{E_0} + \cdots + \|\phi_{m-1}(\omega) - \phi_m(\omega)\|_{E_0} \\
\leq \lambda^n(\omega)\|\phi_0(\omega) - \phi_1(\omega)\|_{E_0} + \cdots + \\
\quad + \lambda^{m-1}(\omega)\|\phi_0(\omega) - \phi_1(\omega)\|_{E_0} \\
\leq (\lambda^n(\omega) + \cdots + \lambda^{m-1}(\omega))\|\phi_0(\omega) - \phi_1(\omega)\|_{E_0} \\
\leq \frac{\lambda^n(\omega)}{1 - \lambda(\omega)}\|\phi_0(\omega) - \phi_1(\omega)\|_{E_0}.
\] (3.4.5)

Hence,

\[
\lim_{m \to n \to \infty} \|\phi_m(\omega) - \phi_n(\omega)\|_{E_0} = 0.
\]

As a result, the sequence \( \{\phi_n(\omega)\} \) is a Cauchy sequence of measurable functions on \( \Omega \) into \( E_0 \). Since \( E_0 \) is complete, \( \{\phi_n(\omega)\} \) and every subsequence of it converges to a limit point \( \phi^*(\omega) \) in \( E_0 \), that is, \( \lim_{n \to \infty} \phi_{2n+2}(\omega) = \phi^*(\omega) \) and that \( \lim_{n \to \infty} \phi_{2n+1}(\omega) = \phi^*(\omega) = \lim_{n \to \infty} \phi_{2n+2}(\omega) \). Since \( E \) is separable, \( E_0 \) is also separable and \( \phi^* \) is also a measurable function on \( \Omega \) into \( E_0 \). Now from the continuity of the random operators \( S \) and \( T \) it follows that

\[
S(\omega, \phi^*(\omega)) = \lim_{n \to \infty} S(\omega, \phi_{2n+2}(\omega)) = \lim_{n \to \infty} \phi_{2n+2}(c, \omega) = \phi^*(c, \omega)
\]

and

\[
T(\omega, \phi^*(\omega)) = \lim_{n \to \infty} T(\omega, \phi_{2n+2}(\omega)) = \lim_{n \to \infty} \phi_{2n+2}(c, \omega) = \phi^*(c, \omega).
\]

Hence, \( \phi^*(\omega) \) is a PPF dependent common random fixed point of \( S(\omega) \) and \( T(\omega) \).
Let $\phi_0, \xi_0 : \Omega \to E_0$ be two given measurable functions and let $\{\phi_n(\omega)\}$ and $\{\xi_n(\omega)\}$ be two sequences of measurable functions of iterations of $S$ and $T$ defined by (3.4.3). Then,

\[ \|\phi_n(\omega) - \xi_n(\omega)\|_{E_0} \]

\[ \leq \|\phi_n(\omega) - \phi_{n-1}(\omega)\|_{E_0} + \|\phi_{n-1}(\omega) - \xi_{n-1}(\omega)\|_{E_0} \]

\[ + \|\phi_{n-1}(\omega) - \xi_n(\omega)\|_{E_0} \]

\[ \leq \lambda^n(\omega)\|\phi_0(\omega) - \phi_1(\omega)\|_{E_0} + \|\phi_{n-1}(\omega) - \xi_{n-1}(\omega)\|_{E_0} \]

\[ + \lambda^n(\omega)\|\xi_0(\omega) - \xi_1(\omega)\|_{E_0} \]

\[ \leq \lambda^n(\omega)\left[\|\phi_0(\omega) - \phi_1(\omega)\|_{E_0} + \|\xi_0(\omega) - \xi_1(\omega)\|_{E_0}\right] \]

\[ + \|\phi_{n-1}(\omega) - \xi_{n-1}(\omega)\|_{E_0} \]

\[ \leq (\lambda^n(\omega) + \cdots + 1)\times \]

\[ \times \left[\|\phi_0(\omega) - \phi_1(\omega)\|_{E_0} + \|\xi_0(\omega) - \xi_1(\omega)\|_{E_0}\right] \]

\[ + \|\phi_{n-1}(\omega) - \xi_{n-1}(\omega)\|_{E_0} \]

\[ \leq \frac{1}{1 - \lambda(\omega)}\left[\|\phi_0(\omega) - \phi_1(\omega)\|_{E_0} + \|\xi_0(\omega) - \xi_1(\omega)\|_{E_0}\right] \]

\[ + \|\phi_{n-1}(\omega) - \xi_{n-1}(\omega)\|_{E_0}. \quad (3.4.7) \]

In particular, if $\phi_0(\omega) = \xi_0(\omega)$, then $\phi_0(c, \omega) = \xi_0(c, \omega)$ so that $S\phi_0 = S\xi_0$ and $\phi_1(\omega) = \xi_1(\omega)$. Hence, from inequality (3.4.7) it follows that

\[ \|\phi_n(\omega) - \xi_n(\omega)\|_{E_0} \leq \frac{2}{1 - \lambda(\omega)}\|\phi_0(\omega) - \phi_1(\omega)\|_{E_0}. \]

(c) To prove uniqueness of common random fixed point in $\mathcal{R}_c$, let $\phi^*$ and $\xi^*$ be two common random fixed points of $S(\omega)$ and
$T(\omega)$, then for a fixed $\omega \in \Omega$,

$$\|\phi^*(\omega) - \xi^*(\omega)\|_{E_0}$$

$$= \|\phi^*(c, \omega) - \xi^*(c, \omega)\|_E$$

$$\leq \|S(\omega, \phi^*(\omega)) - T(\omega, \xi^*(\omega))\|_E$$

$$\leq \lambda(\omega) \max \left\{ \|\phi^*(\omega) - \xi^*(\omega)\|_{E_0}, \right. \right.$$

$$\left. \|\phi^*(c, \omega) - S(\omega, \phi^*(\omega))\|_E, \|\xi^*(\omega) - T(\omega, \xi^*(\omega))\|_E, \right. \right.$$ 

$$\frac{1}{2} \left[ \|\phi^*(c, \omega) - T(\omega, \xi^*(\omega))\|_E$$

$$+ \|\xi^*(c, \omega) - S(\omega, \phi^*(\omega))\|_E \right] \left. \right\}$$

$$\leq \lambda(\omega) \max \left\{ \|\phi^*(\omega) - \xi^*(\omega)\|_{E_0}, 0, 0, \right. \right.$$ 

$$\frac{1}{2} \left[ \|\phi^*(c, \omega) - \xi^*(c, \omega)\|_E + \|\xi^*(c, \omega) - \phi^*(c, \omega)\|_E \right] \left. \right\}$$

$$\leq \lambda(\omega) \max \{\|\phi^*(\omega) - \xi^*(\omega)\|_{E_0}, 0, 0, \|\phi^*(\omega) - \xi^*(\omega)\|_{E_0}\}$$

which yields $\phi^*(\omega) = \xi^*(\omega)$ since $\lambda(\omega) < 1$ for all $\omega \in \Omega$. This completes the proof. \hfill \Box

As a special case of Theorem 3.4.1 we obtain the following corollary which is again new to the literature.

**Corollary 3.4.1.** Suppose that $S, T : \Omega \times E_0 \to E$ are two continuous random operators satisfying for a given $c \in I$ and for each $\omega \in \Omega$,

$$\|S(\omega, \xi) - T(\omega, \eta)\|_E$$

$$\leq \alpha(\omega) \|\xi(c) - S(\omega, \xi(\omega))\|_E$$

$$+ \beta(\omega) \|\eta(c) - T(\omega, \eta(\omega))\|_E + \gamma(\omega)\|\xi(c) - \eta(c)\|_E$$

(3.4.8)
for all \( \xi, \eta \in E_0 \), where \( \alpha, \beta, \gamma : \Omega \to \mathbb{R}_+ \) are measurable functions satisfying \( \alpha(\omega) + \beta(\omega) + \gamma(\omega) < 1 \) for all \( \omega \in \Omega \). Then the statements (a) through (c) of Theorem 3.4.1 hold.

**Proof.** Since inequality 3.4.8 implies condition (3.4.2) the conclusion follows by an application of Theorem 3.4.1. □

On taking \( S = T \) in Corollary 3.4.1, we obtain Theorem 3.3.1 proved in Dhage et. al. [46] as a special case. Again, if \( S = T \), PPF random contraction condition (3.4.1) reduces to the following PPF random contraction condition.

**Definition 3.4.3.** An operator \( T : \Omega \times E_0 \to E \) is called a generalized PPF random contraction if there exists a measurable function \( \lambda : \Omega \to \mathbb{R}, 0 < \lambda(\omega) < 1 \) satisfying for each \( \omega \in \Omega \),

\[
\|T(\omega, \phi) - T(\omega, \xi)\|_E \\
\leq \lambda(\omega) \max \left\{ \|\phi(\omega) - \xi(\omega)\|_{E_0}, \right. \\
\left. \|\phi(c, \omega) - T(\omega, \phi)\|_E, \|\xi(c, \omega) - T(\omega, \xi)\|_E, \right. \\
\frac{1}{2} \left[ \|\phi(c, \omega) - T(\omega, \xi)\|_E + \|\xi(c, \omega) - T(\omega, \phi)\|_E \right] \right\}
\tag{3.4.10}
\]

for all \( \phi, \xi \in E_0 \) and for some \( c \in [a, b] \).

**Remark 3.4.1.** It is clear that contractions and strong Kannan type PPF random contractions are generalized PPF random contractions, but the converse may not be true. The class of generalized PPF random contraction operators is supposed to be the most
general one and includes several classes of PPF random contraction operators in metric spaces.

As a special case of Theorem 3.4.1 we obtain the following corollary.

**Corollary 3.4.2.** Suppose that $T : \Omega \times E_0 \to E$ is a continuous generalized PPF random contraction. Then the following statements hold in $E_0$.

(a) If $\mathcal{R}_c$ is closed with respect to difference, then for a given measurable function $\phi_0 : \Omega \to E_0$, every sequence $\{\phi_n(\omega)\}$ of measurable functions of iterates of $T(\omega)$ defined by (3.3.6) converges to a PPF dependent random fixed point of $T(\omega)$.

(b) If $\mathcal{R}_c$ is algebraically and topologically closed, then for a given measurable function $\phi_0 : \Omega \to E_0$, every sequence $\{\phi_n(\omega)\}$ of measurable functions of iterates of $T(\omega)$ defined by (3.3.6) converges to a PPF dependent random fixed point of $T(\omega)$.

**Proof.** The proof is similar to Theorem 3.4.1 and hence we omit the details. □

**Remark 3.4.2.** Note that Corollary 3.4.2 includes Theorem 3.3.1 as special case in view of Remark 3.4.1.

### 3.5 Random Coincidence Points with PPF Dependence

We need the following definitions in what follows.
Definition 3.5.1. Let $A : \Omega \times E_0 \to E$ and $S : \Omega \times E_0 \to E_0$ be two random operators. A point $\phi^* \in E_0$ is called a PPF dependent random coincidence point of $A(\omega)$ and $S(\omega)$ if $A(\omega, \phi^*(\omega)) = S(\omega, \phi^*(\omega))(c)$ for some $c \in I$ and any mathematical statement that guarantees the existence of such a coincidence point is called a common coincident point theorem with PPF dependence.

Definition 3.5.2. Two operators $A : \Omega \times E_0 \to E$ and $S : \Omega \times E_0 \to E_0$ are said to satisfy a condition of strong generalized PPF random contraction of type (C) if there exists a measurable function $\lambda : \Omega \to \mathbb{R}^+$, $0 < \lambda(\omega) < 1$ satisfying for each $\omega \in \Omega$, 

$$
\|A(\omega, \phi) - A(\omega, \xi)\|_E \\
\leq \lambda(\omega) \max \left\{ \|S(\omega, \phi(\omega))(c) - S(\omega, \xi(\omega))(c)\|_E, \\
\|S(\omega, \phi(\omega))(c) - A(\omega, \phi(\omega))\|_E, \\
\|S(\omega, \xi(\omega))(c) - A(\omega, \xi(\omega))\|_E, \\
\frac{1}{2}\left[\|S(\omega, \phi(\omega))(c) - A(\omega, \xi(\omega))\|_E \\
+ \|S(\omega, \xi(\omega))(c) - A(\omega, \phi(\omega))\|_E \right]\right\}
$$

(3.5.1)

for all measurable functions $\phi, \xi : \Omega \to E_0$ and for some $c \in [a, b]$.

Definition 3.5.3. Two operators $A : \Omega \times E_0 \to E$ and $S : \Omega \times E_0 \to E_0$ are said to satisfy a condition of generalized PPF random contraction of type (C) if there exists a measurable func-
tion $\lambda : \Omega \to \mathbb{R}_+, 0 < \lambda(\omega) < 1$ satisfying for each $\omega \in \Omega$,

$$
\|A(\omega, \phi) - A(\omega, \xi)\|_E \\
\leq \lambda(\omega) \max \left\{ \|S(\omega, \phi(\omega)) - S(\omega, \xi(\omega))\|_E, \\
\|S(\omega, \phi(\omega))(c) - A(\omega, \phi(\omega))\|_E, \\
\|S(\omega, \xi(\omega))(c) - A(\omega, \xi(\omega))\|_E, \\
\frac{1}{2} \left[ \|S(\omega, \phi(\omega))(c) - A(\omega, \phi(\omega))\|_E \\
+ \|S(\omega, \xi(\omega))(c) - A(\omega, \phi(\omega))\|_E \right] \right\} \quad (3.5.2)
$$

for all measurable functions $\phi, \xi : \Omega \to E_0$ and for some $c \in [a, b]$.

Our main coincident random point theorem with PPF dependence is the following.

**Theorem 3.5.1.** Let $T : \Omega \times E_0 \to E$ and $S : \Omega \times E_0 \to E_0$ be two continuous random operators satisfying a generalized contraction of type (C). Further suppose that

(a) $T(\Omega \times E_0) \subset S(\Omega \times E_0)(c)$, and

(b) $S(\Omega, E_0)$ is complete.

If $\mathcal{R}_c$ is algebraically closed w.r.t the difference, then $T(\omega)$ and $S(\omega)$ have a PPF dependent random coincidence point.

**Proof.** Let $\phi_0 : \Omega \to E_0$ be arbitrary measurable function and define a sequence $\{\xi_n(\omega)\}$ of measurable functions in $E_0$ as follows. By hypothesis, $A(\omega, \phi_0(\omega)) \in E$. Suppose that $A(\omega, \phi_0(\omega)) = \ldots$
$x_1(\omega)$. Since $A(\Omega, E_0) \subset S(\Omega, E_0)(c)$, choose a measurable function $\phi_1 : \Omega \rightarrow E_0$ such that

$$x_1(c, \omega) = S(\omega, \phi_1(\omega)(c)) = \xi_1(c, \omega)$$

and

$$||\xi_1(\omega) - \xi_0(\omega)||_{E_0} = ||\xi_1(c, \omega) - \xi_0(c, \omega)||_E.$$

Again, by hypothesis, $A(\omega, \phi_1(\omega)) \in E$. Let $A(\omega, \phi_1(\omega)) = x_2(\omega)$. Since $A(\Omega, E_0) \subset S(\Omega, E_0)(c)$, choose measurable function $\phi_2 : \Omega \rightarrow E_0$ such that

$$x_2(\omega) = S(\omega, \phi_2(\omega))(c) = \xi_2(c, \omega)$$

and

$$||\xi_2(\omega) - \xi_1(\omega)||_{E_0} = ||\xi_2(c, \omega) - \xi_1(c, \omega)||_E.$$

Proceeding in this way, by induction, we obtain

$$A(\omega, \phi_n(\omega)) = S(\omega, \phi_{n+1}(\omega))(c); \quad S(\omega, \phi_{n+1}(\omega))) = \xi_{n+1}(\omega)$$

$$||\xi_n(\omega) - \xi_{n+1}(\omega)||_{E_0} = ||\xi_n(c, \omega) - \xi_{n+1}(c, \omega)||_E$$

for all $n = 0, 1, \ldots$.

We claim that $\{\xi_n(\omega)\}$ is a Cauchy sequence in $E_0$. Now for $n = 0$, we have the following estimate:

$$||\xi_1(\omega) - \xi_2(\omega)||_{E_0}$$

$$= ||\xi_1(c, \omega) - \xi_2(c, \omega)||_E$$

$$= ||A(\omega, \phi_0(\omega)) - A(\omega, \phi_1(\omega))||_E$$

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\[ \leq \lambda(\omega) \max \left\{ \left\| \mathbf{S}(\omega, \phi_0(\omega)) - \mathbf{S}(\omega, \phi_1(\omega)) \right\|_{E_0} , \right. \\
\left. \quad \left\| \mathbf{S}(\omega, \phi_0(\omega))(c) - A(\omega, \phi_0(\omega)) \right\|_{E} , \right. \\
\left. \quad \left\| \mathbf{S}(\omega, \phi_1(\omega))(c) - A(\omega, \phi_1(\omega)) \right\|_{E} , \right. \\
\left. \quad \frac{1}{2} \left[ \left\| \mathbf{S}(\omega, \phi_0(\omega))(c) - A(\omega, \phi_1(\omega)) \right\|_{E} \\
\quad + \left\| \mathbf{S}(\omega, \phi_1(\omega))(c) - A(\omega, \phi_0(\omega)) \right\|_{E} \right] \right\} \]

\[ \leq \lambda(\omega) \max \left\{ \left\| \xi_0(\omega) - \xi_1(\omega) \right\|_{E_0} , \right. \\
\left. \left\| \xi_0(c, \omega) - \xi_1(c, \omega) \right\|_{E} , \left\| \phi_1(c, \omega) - \phi_2(c, \omega) \right\|_{E} , \right. \\
\left. \frac{1}{2} \left[ \left\| \xi_0(c, \omega) - \xi_2(c, \omega) \right\|_{E} + \left\| \xi_1(c, \omega) - \xi_1(c, \omega) \right\|_{E} \right] \right\} \]

\[ \leq \lambda(\omega) \max \left\{ \left\| \xi_0(\omega) - \xi_1(\omega) \right\|_{E_0} , \right. \\
\left. \left\| \xi_0(\omega) - \xi_1(\omega) \right\|_{E_0} , \left\| \xi_1(\omega) - \xi_2(\omega) \right\|_{E_0} , \right. \\
\left. \frac{1}{2} \left[ \left\| \xi_0(\omega) - \xi_2(\omega) \right\|_{E_0} + \left\| \xi_1(\omega) - \xi_1(\omega) \right\|_{E_0} \right] \right\} \]

\[ \leq \lambda(\omega) \max \left\{ \left\| \xi_0(\omega) - \xi_1(\omega) \right\|_{E_0} , \right. \\
\left. \left\| \xi_1(\omega) - \xi_2(\omega) \right\|_{E_0} , \frac{1}{2} \left\| \xi_0(\omega) - \xi_2(\omega) \right\|_{E_0} \right\} \]

\[ \leq \lambda(\omega) \max \left\{ \left\| \xi_0(\omega) - \xi_1(\omega) \right\|_{E_0} , \right. \\
\left. \left\| \xi_0(\omega) - \xi_1(\omega) \right\|_{E_0} , \frac{1}{2} \left[ \left\| \xi_0(\omega) - \xi_1(\omega) \right\|_{E_0} + \left\| \xi_1(\omega) - \xi_2(\omega) \right\|_{E_0} \right] \right\} \]

\[ \leq \lambda(\omega) \|\xi_0(\omega) - \xi_1(\omega)\|_{E_0}. \]

Similarly,

\[ \|\xi_2(\omega) - \xi_3(\omega)\|_{E_0} \]

\[ = \|\mathbf{S}(\omega, \phi_2(\omega))(c) - \mathbf{S}(\omega, \phi_3(\omega))(c)\|_{E} \]

\[ = \|\mathbf{A}(\omega, \phi_2(\omega)) - \mathbf{A}(\omega, \phi_1(\omega))\|_{E} \]

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\[
\leq \lambda(\omega) \max \left\{ \|S(\omega, \phi_2(\omega)) - S(\omega, \phi_1(\omega))\|_{E_0}, \right.
\]
\[
\left. \|S(\omega, \phi_2(\omega))(c) - A(\omega, \phi_2(\omega))\|_E, \right.
\]
\[
\left. \|S(\omega, \phi_1(\omega))(c) - A(\omega, \phi_1(\omega))\|_E, \right.
\]
\[
\left. \frac{1}{2} \left[ \|S(\omega, \phi_2(\omega))(c) - A(\omega, \phi_1(\omega))\|_E 
\right. 
\left. + \|S(\omega, \phi_1(\omega))(c) - A(\omega, \phi_2(\omega))\|_E \right] \right\} 
\]
\[
\leq \lambda(\omega) \max \left\{ \|\xi_2(\omega) - \xi_1(\omega)\|_{E_0}, \right. 
\]
\[
\left. \|\xi_2(c,\omega) - \xi_3(c,\omega)\|_E, \|\xi_1(c,\omega) - \xi_2(c,\omega)\|_E, \right. 
\]
\[
\left. \frac{1}{2} \left[ \|\xi_2(c,\omega) - \xi_2(c,\omega)\|_E + \|\xi_1(c,\omega) - \xi_3(c,\omega)\|_E \right] \right\} 
\]
\[
\leq \lambda(\omega) \max \left\{ \|\xi_1(\omega) - \xi_2(\omega)\|_{E_0}, \right. 
\]
\[
\left. \|\xi_2(\omega) - \xi_3(\omega)\|_{E_0}, \|\xi_1(\omega) - \xi_2(\omega)\|_{E_0}, \right. 
\]
\[
\left. \frac{1}{2} \left[ \|\xi_2(\omega) - \xi_2(\omega)\|_{E_0} + \|\xi_1(\omega) - \xi_3(\omega)\|_{E_0} \right] \right\} 
\]
\[
\leq \lambda(\omega) \max \left\{ \|\xi_1(\omega) - \xi_2(\omega)\|_{E_0}, \right. 
\]
\[
\left. \|\xi_2(\omega) - \xi_3(\omega)\|_{E_0}, \frac{1}{2} \|\xi_1(\omega) - \xi_3(\omega)\|_{E_0} \right\} 
\]
\[
\leq \lambda(\omega) \max \left\{ \|\xi_1(\omega) - \xi_2(\omega)\|_{E_0}, \right. 
\]
\[
\left. \|\xi_1(\omega) - \xi_2(\omega)\|_{E_0}, \frac{1}{2} \left[ \|\xi_1(\omega) - \xi_2(\omega)\|_{E_0} + \|\xi_2(\omega) - \xi_3(\omega)\|_{E_0} \right] \right\} 
\]
\[
\leq \lambda(\omega) \|\xi_1(\omega) - \xi_2(\omega)\|_{E_0}. 
\]

Proceeding in this way, by induction, we obtain

\[
\|\xi_n(\omega) - \xi_{n+1}(\omega)\|_{E_0} \leq \lambda(\omega) \|\xi_{n-1}(\omega) - \xi_n(\omega)\|_{E_0}
\]

for all \( n = 1, 2, \ldots \).
Hence, by repeated application of the above inequality yields
\[ \| \xi_n(\omega) - \xi_{n+1}(\omega) \|_{E_0} \leq \lambda^n(\omega) \| \xi_0(\omega) - \xi_1(\omega) \|_{E_0} \]
for all \( n = 1, 2, \ldots \).

If \( m > n \), by triangle inequality, we obtain
\[
\begin{align*}
\| \xi_m(\omega) - \xi_n(\omega) \|_{E_0} & \leq \| \xi_n(\omega) - \xi_{n+1}(\omega) \|_{E_0} + \cdots + \| \xi_{m-1}(\omega) - \xi_m(\omega) \|_{E_0} \\
& \leq \lambda^n(\omega) \| \xi_0(\omega) - \xi_1(\omega) \|_{E_0} + \cdots + \lambda^{m-1}(\omega) \| \xi_0(\omega) - \xi_1(\omega) \|_{E_0} \\
& \leq \left( \lambda^n(\omega) + \cdots + \lambda^{m-1}(\omega) \right) \| \xi_0(\omega) - \xi_1(\omega) \|_{E_0} \\
& \leq \frac{\lambda^n(\omega)}{1 - \lambda(\omega)} \| \xi_0(\omega) - \xi_1(\omega) \|_{E_0}.
\end{align*}
\]
Hence,
\[ \lim_{m>n \to \infty} \| \xi_m(\omega) - \xi_n(\omega) \|_{E_0} = 0. \]

As a result, the sequence \( \{ \xi_n(\omega) \} \) is Cauchy sequence of measurable functions in \( E_0 \). Since \( E_0 \) is complete, \( \{ \xi_n(\omega) \} \) and every subsequence of it converges to a limit point \( \xi^*(\omega) \) in \( E_0 \), that is,
\[ \lim_{n \to \infty} \xi_n(\omega) = \lim_{n \to \infty} S(\omega, \phi_n(\omega)) = \xi^*(\omega) \]
and
\[ \lim_{n \to \infty} \xi_n(c, \omega) = \lim_{n \to \infty} A(\omega, \phi_n(\omega)) = \xi^*(c, \omega). \]

From the continuity of \( A(\omega) \) and \( S(\omega) \) it follows that
\[ A(\omega, \phi^*(\omega)) = S(\omega, \phi^*(\omega))(c). \]
Thus, \( \phi^*(\omega) \) is a PPF dependent random coincidence point of random operators \( A(\omega) \) and \( S(\omega) \). This completes the proof. \( \square \)
Corollary 3.5.1. Let $T : \Omega \times E_0 \to E$ and $S : \Omega \times E_0 \to E_0$ be two continuous random operators satisfying for a given $c \in I$ and for each $\omega \in \Omega$,

$$\|S(\omega, \xi) - T(\omega, \eta)\|_E \leq \alpha(\omega) \|\xi(c) - S(\omega, \xi(c))\|_E + \beta(\omega) \|\eta(c) - T(\omega, \eta(\omega))\|_E + \gamma(\omega) \|\xi(c) - \eta(c)\|_E$$

for all $\xi, \eta \in E_0$, where $\alpha, \beta, \gamma : \Omega \to \mathbb{R}_+$ are measurable functions satisfying $\alpha(\omega) + \beta(\omega) + \gamma(\omega) < 1$ for all $\omega \in \Omega$. Further suppose that

(a) $T(\Omega \times E_0) \subset S(\Omega \times E_0)(c)$, and

(b) $S(\Omega, E_0)$ is complete.

If $R_c$ is algebraically closed w.r.t the difference, then $T(\omega)$ and $S(\omega)$ have a PPF dependent random coincidence point.

3.6 Conclusion

Finally, we conclude this chapter with the remark that common random fixed point theorems with PPF dependence proved here are very fundamental in the random fixed point theory involving geometric hypothesis of distance between the images and objects in question. However, using the principle that has been formulated in Theorems 3.4.1 and 3.5.1, several other common random fixed point theorems with PPF dependence may be proved for
the random operators with different domain and range spaces. The existence results of this chapter may also be extended to three of four random operators in Banach spaces with appropriate modifications. In a forthcoming work, we plan to prove some PPF dependent approximating random fixed point theorems for a single as well as pairs of random operators in separable Banach spaces satisfying more generalized contraction conditions via constructive method.