CHAPTER 5

Some properties of Lorentzian para-Sasakian Manifolds
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Chapter 5
Some properties of Lorentzian para-Sasakian Manifolds

5.1 Introduction

In the previous chapter (Chapter-4), we have studied a type of semi-symmetric non-metric connection defined by S. Kumar and J. Upreti [27] in a Lorentzian para-Sasakian manifold. Some interesting properties of this connection in LP-Sasakian manifold have also been discussed. We also find the expression for curvature tensor with respect to the semi-symmetric non-metric connection and establish relations between curvature tensor with respect to the semi-symmetric non-metric connection and curvature tensor with respect to Levi-Civita connection.

In this chapter we have studied projective curvature tensor of the semi-symmetric non-metric connection in LP-Sasakian manifold. The concircular curvature tensor, conformal curvature tensor, quasi-conformal curvature tensor of a LP-Sasakian manifold with respect to the semi-symmetric non-metric connection have also been investigated. Later this connection has been applied in weakly Ricci symmetric LP-Sasakian manifolds and generalised Ricci-recurrent LP-Sasakian manifolds with cyclic Ricci tensor &
Codazzi type Ricci tensor.

5.2 Preliminaries

Consider a differentiable manifold $M_n$ of dimension $n$. It is called Lorentzian Para-Sasakian (briefly LP-Sasakian) ([30], [32]), if it admits a $(1, 1)$-tensor field $\phi$, a 1-form $\eta$, the associated vector field $\xi$ and a Lorentzian metric $g$ satisfying the following equations

$$\eta(\xi) = -1, \quad (5.2.1)$$

$$\phi^2(X) = X + \eta(X)\xi, \quad (5.2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (5.2.3)$$

$$g(X, \xi) = \eta(X), \quad (5.2.4)$$

$$\nabla_X \xi = \phi X, \quad (5.2.5)$$

$$(\nabla_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (5.2.6)$$

for arbitrary vector fields $X$ and $Y$, where $\nabla$ denotes covariant differentiation with respect to $g$.

In a LP-Sasakian manifold with structure $(\phi, \xi, \eta, g)$ following relation...
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hold

\[ (a) \phi \xi = 0, \quad (b) \eta(\phi X) = 0, \]  

\[ \text{rank}(\phi) = n - 1. \]  

If we put

\[ F'(X, Y) = g(\phi X, Y), \]  

then the tensor field \( F'(X, Y) \) is a symmetric \((0, 2)\) tensor field [30] and it can be easily seen that ([30], [33])

\[ (a) \quad F'(X, Y) = F'(Y, X) = F'(\phi X, \phi Y) \]

\[ = g(\phi X, Y) = g(\phi Y, X). \]

\[ (b) \quad F'(X, Y) = (\nabla_X \eta)(Y) = g(\phi X, Y). \]

LP-Sasakian manifold also satisfies the following properties ([31], [33])

\[ (a) \quad g(R(X, Y, Z), \xi) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \]
\[ = \eta(R(X, Y, Z)), \]

\[ (b) \quad R(\xi, X, Y) = g(X, Y)\xi - \eta(Y)X, \]

\[ (c) \quad R(X, Y, \xi) = \eta(Y)X - \eta(X)Y, \]

\[ (d) \quad R(\xi, X, \xi) = X + \eta(X)\xi = \phi^2 X, \]

\[ (e) \quad \hat{S}(X, \xi) = (n - 1)\eta(X), \]

\[ (f) \quad \hat{S}(\phi X, \phi Y) = \hat{S}(X, Y) + (n - 1)\eta(X)\eta(Y), \]
for any vector fields $X, Y, Z$. Where $R$ and $\hat{S}$ are the Riemannian curvature and the Ricci tensor of $M_n$, respectively.

5.3 A semi-symmetric non-metric connection $E$ in a LP-Sasakian manifold

Consider an $n-$dimensional LP-Sasakian manifold $M_n$ equipped with the Levi-Civita connection $\nabla$ of its Lorentzian metric $g$. Then for a semi-symmetric non-metric connection $E$ [27] in $M_n$, we have

(a) $E_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi + g(\phi X,Y)\xi,$ \hspace{1cm} (5.3.1)
(b) $E_X g = 2\eta(X)g,$
(c) $g(\phi X,Y) - g(\phi Y,X) = 0.$

The torsion tensor $S$ of $M_n$ with respect to connection $E$ is given by

$$S(X,Y) = \eta(Y)X - \eta(X)Y,$$ \hspace{1cm} (5.3.2)

For LP Saskian manifold, we have

(a) $(E_X g)(Y,Z) = -\eta(Z)g(\phi X,Y) - \eta(Y)g(\phi X,Z),$ \hspace{1cm} (5.3.3)
\hspace{5cm} = 2\eta(X)g(Y,Z).
(b) $(E_X \eta)(\phi Y) = 2g(\phi X,\phi Y) - g(\phi X,Y) = (E_Y \eta)X,$
(c) $E_X(\phi Y) = \nabla_X(\phi Y) - g(X,\phi Y)\xi + g(\phi X,\phi Y)\xi.$
5.4 Curvature tensor of $M_n$ with respect to connection $E$

According to Chapter-4, the curvature tensor $\tilde{R}(X,Y,Z)$ in a LP-Sasakian manifold admitting a semi-symmetric non-metric connection $E$, is given by

\[
\tilde{R}(X,Y,Z) = R(X,Y,Z) + 2g(\phi X,Z)Y - 2g(\phi Y,Z)X - g(Y,Z)(\phi X) + g(X,Z)(\phi Y) + g(\phi Y,Z)(\phi X) - g(\phi X,Z)(\phi Y) \]
\[
+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)X - g(X,Z)Y - \eta(X)g(\phi Y,Z)\xi + \eta(Y)g(\phi X,Z)\xi, \tag{5.4.1}
\]

Where $R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$ is curvature tensor of $M_n$ with respect to the Riemannian connection $\nabla$.

Consider $K$ and $\tilde{K}$ be the curvature tensors of type $(0,4)$ given by

\[
K(X,Y,Z,U) = g(R(X,Y,Z),U),
\]

and

\[
\tilde{K}(X,Y,Z,U) = g(R(X,Y,Z),U).
\]

Then from (5.4.1), we have
\[\tilde{K}(X, Y, Z, U) = K(X, Y, Z)U + 2g(\phi X, Z)g(Y, U)\]
\[-2g(\phi Y, Z)g(X, U) - g(Y, Z)g(\phi X, U)\]
\[+ g(X, Z)g(\phi Y, U) + g(\phi Y, Z)g(\phi X, U)\]
\[-g(\phi X, Z)g(\phi Y, U) + \eta(Y)\eta(Z)g(X, U)\]
\[-\eta(X)\eta(Z)g(Y, U) + g(Y, Z)g(X, U)\]
\[-g(X, Z)g(Y, U) - \eta(X)g(\phi Y, Z)\eta(U)\]
\[+ \eta(Y)g(\phi X, Z)\eta(U).\] (5.4.2)

In LP-Sasakian manifold with connection \(E\), we also have

(a) \[\tilde{R}(X, Y, Z) + \tilde{R}(Y, Z, X) + \tilde{R}(Z, X, Y) = 0\] (5.4.3)

(b) \[\tilde{K}(X, Y, Z, U) + \tilde{K}(Y, X, Z, U) = 0.\]

Let \(M_n\) be an \(n\)-dimensional LP-Sasakian manifold. Then the Ricci tensor \(\tilde{S}\) of the manifold \(M_n\) with respect to the new connection \(E\) is defined by [54]

\[\tilde{S}(X, Y) = \sum_{i=1}^{n} \varepsilon_i g(\tilde{R}(e_i, X, Y), e_i),\] (5.4.4)

and the scalar curvature of the manifold \(M_n\) with respect to the connection \(E\) is given by

\[\tilde{r} = \sum_{i=1}^{n} \varepsilon_i \tilde{R}(e_i, e_i),\] (5.4.5)

where \(\{e_1, e_2, \ldots, e_n\}\) is an orthonormal frame and \(\varepsilon_i = g(e_i, e_i)\).

**Theorem 5.1.** In an LP-Sasakian manifold the Ricci tensor \(\tilde{S}\) and scalar
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Curvature $\tilde{r}$ of connection $E$ are given by

$$\tilde{S}(Y, Z) = \overline{S}(Y, Z) - (2n - 4)g(\phi Y, Z) + (n - 2)g(\phi Y, \phi Z)$$  \hspace{1cm} (5.4.6)

and

$$\tilde{r} = r - (3n - n^2 - 2),$$  \hspace{1cm} (5.4.7)

where $\overline{S}$ and $r$ denote the Ricci tensor and scalar curvature of Levi-Civita connection $\nabla$, respectively. Consequently, $\tilde{S}$ is symmetric.

Proof. In view of (5.4.1) and (5.4.4), we have

$$\tilde{S}(Y, Z) = \sum_{i=1}^{n} \varepsilon_i g(R(e_i, Y, Z), e_i) - (2n - 4)g(\phi Y, Z)$$

$$+ (n - 2)g(\phi Y, \phi Z).$$  \hspace{1cm} (5.4.8)

Since the Ricci tensor of Levi-Civita connection $\nabla$ is given by

$$\overline{S}(Y, Z) = \sum_{i=1}^{n} \varepsilon_i g(R(e_i, Y, Z), e_i),$$

then (5.4.8) implies (5.4.6). (5.4.7) follows from (5.4.6). Also from (5.4.6), it is obvious that $\tilde{S}$ is symmetric. \hfill \Box

In LP-Sasakian manifold with the semi-symmetric non-metric connection $E$, it can be easily shown that

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5.5 Projective Curvature tensor

The projective curvature tensor \( \tilde{P} \) of type (1,3) of \( M_n \) with respect to the semi-symmetric non-metric connection \( E \), is defined by

\[
\tilde{P}(X,Y,Z) = \tilde{R}(X,Y,Z) + \frac{1}{n+1} \{ \tilde{S}(X,Y) - \tilde{S}(Y,X) \} Z \tag{5.5.1}
\]

\[
- \frac{n}{n^2 - 1} \{ \tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y \}
\]

\[
- \frac{1}{n^2 - 1} \{ \tilde{S}(Z,Y)X - \tilde{S}(Z,X)Y \}.
\]

Theorem 5.2. In a LP-Sasakian manifold \( M_n \) the projective curvature tensor of \( M_n \) with respect to the semi-symmetric non-metric connection \( E \) is given by

\[
\tilde{P}(X,Y,Z) = P(X,Y,Z) + \frac{2}{n-1} \{ g(\phi X,Z)Y - g(\phi Y,Z)X \}
\]

\[
+ \frac{1}{n-1} \{ g(Y,Z)X - g(X,Z)Y \} + \frac{1}{n-1} \{ \eta(Y)\eta(Z)X \}
\]
where \( P \) denotes the projective curvature tensors with respect to \( \nabla \).

Proof. Since Ricci tensor \( \tilde{S} \) of the manifold with respect to the semi-symmetric non-metric connection \( E \) is symmetric, the projective curvature tensor \( \tilde{P} \) reduces to

\[
\tilde{P}(X, Y, Z) = \tilde{R}(X, Y, Z) - \frac{1}{n-1} \{ \tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y \}. \tag{5.5.3}
\]

Using (5.4.1) and (5.4.6) in the above equation, we get the result. \( \square \)

Lemma 5.3. In an \( n \)-dimensional LP-Sasakian manifold, the projective tensor \( \tilde{P} \) of the manifold with respect to the semi-symmetric non-metric connection \( M \) satisfies the followings

\[
\begin{align*}
(a) & \quad \tilde{P}(X, Y, Z) + \tilde{P}(Y, Z, X) + \tilde{P}(Z, X, Y) = 0 \tag{5.5.4} \\
(b) & \quad \tilde{P}(X, Y, Z) + \tilde{P}(Y, X, Z) = 0
\end{align*}
\]

Proof. Since we know that the Projective curvature tensor with respect to the Riemannian connection is skew-symmetric in first two slots and satisfies Bianchi’s first identity. Therefore, the proof is obvious in view of (5.5.2). \( \square \)
5.6 Concircular Curvature Tensor

Let $M_n$ be a LP-Sasakian manifold. The concircular curvature tensor of $M_n$ with respect to the semi-symmetric non-metric connection $E$, is given by

$$
\tilde{V}(X, Y, Z) = \tilde{R}(X, Y, Z) - \frac{\tilde{r}}{n(n - 1)} \{ g(Y, Z)X - g(X, Z)Y \},
$$

(5.6.1)

where $\tilde{R}$ and $\tilde{r}$ are the curvature tensor and the scalar curvature of $M_n$ with respect to the semi-symmetric non-metric connection $E$, respectively. The concircular curvature tensor can be thought as a measure of the failure of a Riemannian manifold to be of constant curvature.

**Theorem 5.4.** The concircular curvature tensors $V$ and $\tilde{V}$ of the semi-symmetric non-metric connection $E$ and of $M_n$ are related by

$$
\tilde{V}(X, Y, Z) = V(X, Y, Z) + \frac{2}{n} \{ g(Y, Z)X - g(X, Z)Y \} + 2 \{ g(\phi X, Z)Y - g(\phi Y, Z)X \} + \eta(Y)\eta(Z)X + \eta(Y)g(\phi X, Z)\xi.
$$

(5.6.2)

**Proof.** Using (5.4.1) and (5.4.7) in (5.6.1), we get the result. \qed

**Theorem 5.5.** In $M_n$ the concircular curvature tensor $\tilde{V}$ with respect to the semi-symmetric non-metric connection $E$ satisfies the following algebraic properties

$$
\tilde{V}(X, Y, Z) + \tilde{V}(Y, X, Z) = 0,
$$

(5.6.3)
and

\[ \tilde{V}(X, Y, Z) + \tilde{V}(Y, Z, X) + \tilde{V}(Z, X, Y) = 0. \] (5.6.4)

Proof. Both the results are obvious from (5.6.2). \qed

5.7 Conformal Curvature Tensor

Let \( M_n \) be a LP-Sasakian manifold. The conformal curvature tensor of \( M_n \) with respect to semi-symmetric non-metric connection \( E \) is defined by

\[
\tilde{C}(X, Y, Z, U) = \tilde{K}(X, Y, Z, U) - \frac{1}{n-2} \{ g(Y, Z)\tilde{S}(X, U) \\
- g(X, Z)\tilde{S}(Y, U) + g(X, U)\tilde{S}(Y, Z) \\
- g(Y, U)\tilde{S}(X, Z) \} + \frac{\tilde{r}}{(n-1)(n-2)} \{ g(Y, Z)g(X, U) \\
- g(X, Z)g(Y, U) \}. \] (5.7.1)

By using (5.4.2) (5.4.6) and (5.4.7) in (5.7.1), we have

\[
\tilde{C}(X, Y, Z, U) = K(X, Y, Z, U) + g(Y, Z)g(\phi X, U) - g(X, Z)g(\phi Y, U) \\
+ g(\phi Y, Z)g(\phi X, U) - g(\phi X, Z)g(\phi Y, U) \\
+ g(\phi X, Z)\eta(U)\eta(Y) - g(\phi Y, Z)\eta(U)\eta(X) \\
+ g(X, Z)\eta(Y)\eta(U) - g(Y, Z)\eta(X)\eta(U). \] (5.7.2)

Theorem 5.6. In \( M_n \), the conformal curvature tensor \( \tilde{C} \) with respect to the semi-symmetric non-metric connection \( E \) satisfies the following algebraic properties

\[ \tilde{C}(X, Y, Z, U) + \tilde{C}(Y, X, Z, U) = 0, \] (5.7.3)
\[ \tilde{C}(X,Y,Z,U) + \tilde{C}(Y,Z,U,X) \]
\[ + \tilde{C}(Z,U,X,Y) + \tilde{C}(U,X,Y,Z) = 0. \quad (5.7.4) \]

**Proof.** Both the results are obvious from (5.7.2). \(\square\)

### 5.8 Quasi-Conformal Curvature Tensor

The notion of the quasi-conformal curvature tensor was introduced by Yano and Sawaki [80]. They define a quasi-conformal curvature tensor by

\[
W(X,Y,Z) = kR(X,Y,Z) + l\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} - r\{\frac{k}{n(n-1)} + 2l\}\{g(Y,Z)X - g(X,Z)Y\},
\]

where \(k\) and \(l\) are constants such that \(kl \neq 0\), \(R\) is the Riemannian curvature tensor, \(S\) is the Ricci tensor, \(r\) is the scalar curvature and \(Q\) is the Ricci operator defined by \(g(QX,Y) = S(X,Y)\) of the manifold.

A quasi-conformal curvature tensor \(\tilde{W}\) with respect to a semi-symmetric non-metric connection in \(M_n\) is defined by

\[
\tilde{W}(X,Y,Z) = k\tilde{K}(X,Y,Z) + l\{\tilde{S}(Y,Z)g(X,U) - \tilde{S}(X,Z)g(Y,U) + g(Y,Z)\tilde{S}(X,U) - g(X,Z)\tilde{S}(Y,U)\} - \tilde{r}\{\frac{k}{n(n-1)} + 2l\}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\},
\]

where \(\tilde{S}\) the Ricci tensor and \(\tilde{r}\) is the scalar curvature of the manifold with respect to the semi-symmetric non-metric connection and
\[ \tilde{W}(X, Y, Z, U) = g(\tilde{W}(X, Y, Z), U). \quad (5.8.2) \]

Now if we use (5.4.2) (5.4.6) and (5.4.7) in (5.8.1), we have

\[
\begin{align*}
\tilde{W}(X, Y, Z, U) &= W(X, Y, Z, U) + 2\{k + (n - 2)l\}\{g(\phi X, Z)g(Y, U) \\
&- g(\phi Y, Z)g(X, U)\} + \{k + 2(n - 2)l\}. \\
&\{g(X, Z)g(\phi Y, U) - g(Y, Z)g(\phi X, U)\} \\
&+ \{2k + (n - 2)(3n - 2)l\}\{g(Y, Z)g(X, U) \\
&- g(X, Z)g(Y, U)\} + k\{g(\phi Y, Z)g(\phi X, U) \\
&- g(\phi X, Z)g(\phi Y, U) - \eta(X)\eta(U)g(\phi Y, Z) \\
&+ \eta(Y)\eta(U)g(\phi X, Z)\} + (n - 2)l\{g(\phi X, \phi U)g(Y, Z) \\
&- g(\phi Y, \phi U)g(X, Z)\} + \{k + (n - 2)l\}. \\
&\{\eta(Y)\eta(Z)g(X, U) - \eta(X)\eta(Z)g(Y, U)\}. \quad (5.8.3)
\end{align*}
\]

If \( k + (n - 2)l = 0 \); then

\[
\begin{align*}
\tilde{W}(X, Y, Z, U) &= W(X, Y, Z, U) + k\{g(\phi Y, Z)g(\phi X, U) \\
&- g(\phi X, Z)g(\phi Y, U) - \eta(X)\eta(U)g(\phi Y, Z) \\
&+ \eta(Y)\eta(U)g(\phi X, Z) + g(\phi Y, \phi U)g(X, Z) \\
&- g(\phi X, \phi U)g(Y, Z) + g(Y, Z)g(\phi X, U) \\
&- g(X, Z)g(\phi Y, U)\} + \frac{(4 - 3n)k}{n} \\
&\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}.
\end{align*}
\]

**Theorem 5.7.** In \( M_n \), the quasi-conformal curvature tensor \( \tilde{W} \) with respect to the semi-symmetric non-metric connection \( E \) satisfies the following
algebraic properties

\[ \tilde{W}(X,Y,Z,U) + \tilde{W}(Y,X,Z,U) = 0, \quad (5.8.4) \]

and

\[ \tilde{W}(X,Y,Z,U) + \tilde{W}(Y,Z,U,X) + \tilde{W}(Z,U,X,Y) + \tilde{W}(U,X,Y,Z) = 0. \quad (5.8.5) \]

\[ \]

Proof. Both the above results are obvious from (5.8.3). \qed

5.9 Weakly Ricci symmetric LP-Sasakian manifolds admitting a semi-symmetric non-metric connection \( E \)

A non flat Riemannian manifold \( M_n \) is called weakly Ricci symmetric if there exist 1-forms \( \lambda, \mu \) and \( \nu \) and Ricci tensor \( \tilde{S} \) satisfies the condition ([65],[66])

\[
(Ex\tilde{S})(Y,Z) = \lambda(X)\tilde{S}(Y,Z) + \mu(Y)\tilde{S}(X,Z) + \nu(Z)\tilde{S}(Y,X), \quad (5.9.1)
\]

for all vector fields \( X, Y, Z \), where \( \lambda, \mu \) and \( \nu \) are not simultaneously zero. We give the following definition: A non flat Riemannian manifold \( M_n \) is called weakly Ricci symmetric with respect to a semi-symmetric non metric connection \( E \) if there exist 1-forms \( \lambda, \mu \) and \( \nu \) and Ricci tensor \( \tilde{S} \) satisfies the condition

\[
(Ex\tilde{S})(Y,Z) = \lambda(X)\tilde{S}(Y,Z) + \mu(Y)\tilde{S}(X,Z) + \nu(Z)\tilde{S}(Y,X), \quad (5.9.2)
\]
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for all vector fields $X, Y, Z$.

Let us assume that $M_n$ be a weakly Ricci symmetric LP-Sasakian manifold admitting a semi-symmetric non metric connection $E$. So the equation (5.9.2) take place. Taking $Z = \xi$ in (5.9.2) we get

$$(E_X\bar{S})(Y, \xi) = \lambda(X)\bar{S}(Y, \xi) + \mu(Y)\bar{S}(X, \xi) + \nu(\xi)\bar{S}(Y, X),$$

(5.9.3)

By the virtue of (5.9.3) the above equation gives

$$\lambda(X)\bar{S}(Y, \xi) + \mu(Y)\bar{S}(X, \xi) + \nu(\xi)\bar{S}(Y, X) = (n - 1)[(E_X\eta)Y] - \bar{S}(Y, X - \bar{X}),$$

(5.9.4)

setting $X = Y = \xi$ in above equation, we obtain

$$(n - 1)\{\lambda(\xi) + \mu(\xi) + \nu(\xi)\} = 0,$$

(5.9.5)

which implies that

$$\lambda(\xi) + \mu(\xi) + \nu(\xi) = 0.$$  

(5.9.6)

Putting $X = \xi$ in (5.9.4) we get

$$\mu(Y) = -\mu(\xi)\eta(Y).$$ 

(5.9.7)

Putting $Y = \xi$ in (5.9.4) we get

$$\lambda(X) = -\lambda(\xi)\eta(X).$$ 

(5.9.8)
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Since \((\nabla_\xi \tilde{R})(\xi, X) = 0\), then from (5.9.2), it can be shown that

\[
\nu(X) = -\nu(\xi)\eta(X). \quad (5.9.9)
\]

Replacing \(Y\) by \(X\) in (5.9.7), we get

\[
\mu(X) = -\mu(\xi)\eta(X). \quad (5.9.10)
\]

Adding (5.9.8), (5.9.9) and (5.9.10), we get

\[
\lambda(X) + \mu(X) + \nu(X) = 0. \quad (5.9.11)
\]

for any vector field \(X\) on \(M_n\).

This leads to the following:

**Theorem 5.8.** There is no weakly Ricci symmetric LP-Sasakian manifolds \(M_n\) admitting a semi-symmetric non metric connection \(E\), unless \(\lambda + \mu + \nu\) vanishes everywhere.

5.10 Generalized Ricci-recurrent LP-Sasakian manifolds admitting a semi-symmetric non-metric connection \(E\)

A non flat \(n\)-dimensional differentiable manifold \(M_n\) occupied with a connection \(\nabla\), is called generalized Ricci recurrent if its Ricci tensor \(H\) satisfies the condition

\[
(\nabla_X H)(Y, Z) = A(X)H(Y, Z) + B(X)g(Y, Z), \quad (5.10.1)
\]
where $A, B$ are two 1-forms, ($B \neq 0$) defined by

$$(a) \quad A(X) = g(X, \rho_1), \quad (b) \quad B(X) = g(X, \rho_2), \quad (5.10.2)$$

and $\rho_1, \rho_2$ are vector fields related with 1-forms $A, B$ respectively. Analogous to above definition a non flat $n$-dimensional differentiable manifold $M_n$ is called generalized Ricci recurrent with respect to a semi-symmetric non metric connection $E$ if its Ricci tensor $\tilde{S}$ satisfies the condition

$$(E_X \tilde{S})(Y, Z) = A(X)\tilde{S}(Y, Z) + B(X)g(Y, Z). \quad (5.10.3)$$

Putting $Z = \xi$ in above equation, we obtain

$$(E_X \tilde{S})(Y, \xi) = A(X)\tilde{S}(Y, \xi) + B(X)g(Y, \xi),
= (n - 1)\eta(Y)A(X) + \eta(Y)B(X). \quad (5.10.4)$$

On the other hand by virtue of (5.9.3) we have

$$(E_X \tilde{S})(Y, \xi) = (n - 1)\{ (E_Y \eta)Y \} - \tilde{S}(Y, \overline{X} - \overline{X}), \quad (5.10.5)$$

Comparing equations (5.10.4) and (5.10.5), we obtain

$$(n - 1)\{ (E_Y \eta)Y \} - \tilde{S}(Y, \overline{X} - \overline{X})
= (n - 1)\eta(Y)A(X) + \eta(Y)B(X), \quad (5.10.6)$$

Taking $Y = \xi$ in above equation, we get

$$(n - 1)A(X) + B(X) = 0, \quad (5.10.7)$$
which implies that
\[(n - 1)A(X) + B(X) = 0, \quad (5.10.8)\]
for all vector field \(X\). Thus we can state that:

**Theorem 5.9.** Let \(M_n\) be a generalized Ricci recurrent LP-Sasakian manifolds admitting a semi-symmetric non-metric connection \(E\). Then the associated vector fields of the 1-form \(A\) and \(B\) are in opposite direction.

### 5.11 Generalized Ricci-recurrent LP-Sasakian manifolds with connection \(E\) admitting cyclic Ricci tensor

In this section we suppose that a generalized Ricci-recurrent LP-Sasakian manifolds \(M_n\) with connection \(E\) admits a cyclic Ricci tensor \(\tilde{S}\), i.e.
\[(E_X \tilde{S})(Y, Z) + (E_Y \tilde{S})(Z, X) + (E_Z \tilde{S})(X, Y) = 0. \quad (5.11.1)\]

Then by virtue of (5.10.3), above equation can be written as
\[
A(X)\tilde{S}(Y, Z) + B(X)g(Y, Z) + A(Y)\tilde{S}(Z, X) + B(Y)g(Z, X)
+A(Z)\tilde{S}(X, Y) + B(Z)g(X, Y) = 0, \quad (5.11.2)
\]
putting \(Z = \xi\) in above equation, we have
\[
A(X)\tilde{S}(Y, \xi) + B(X)\eta(Y) + A(Y)\tilde{S}(\xi, X) + B(Y)\eta(X)
+A(\xi)\tilde{S}(X, Y) + B(\xi)g(X, Y) = 0, \quad (5.11.3)
\]
from (5.2.4) and (5.2.11)(e), we have
\[(n - 1)A(X)\eta(Y) + B(X)\eta(Y) + (n - 1)\eta(X)A(Y) + \eta(X)B(Y) + A(\xi)\tilde{S}(X,Y) + B(\xi)g(X,Y) = 0,\] (5.11.4)

from above quotation, we have
\[\eta(Y)\{(n - 1)A(X) + B(X)\} + \eta(X)\{(n - 1)A(Y) + B(Y)\} + A(\xi)\tilde{S}(X,Y) + B(\xi)g(X,Y) = 0.\] (5.11.5)

by the virtue of (5.10.8), we have from (5.11.5)
\[A(\xi)\tilde{S}(X,Y) = -B(\xi)g(X,Y),\] (5.11.6)

\[\Rightarrow \tilde{S}(X,Y) = -\frac{B(\xi)}{A(\xi)}g(X,Y),\] (5.11.7)

or
\[\tilde{S}(X,Y) = \nu g(X,Y),\] (5.11.8)

where
\[\nu = -\frac{B(\xi)}{A(\xi)}.\] (5.11.9)

**Theorem 5.10.** If a generalised Ricci-recurrent LP-Sasakian manifold with connection $E$ admits a cyclic Ricci tensor, then it become an Einstein manifold.
5.12 Generalized Ricci-recurrent LP-Sasakian manifolds with connection $E$ admitting Codazzi type Ricci tensor

We know that

$$ (E_X \tilde{S})(Y, Z) = E_X \tilde{S}(Y, Z) - \tilde{S}(E_X Y, Z) - \tilde{S}(Y, E_X Z), \quad (5.12.1) $$

from (5.10.3) and (5.12.1), we have

$$ A(X) \tilde{S}(Y, Z) + B(X) g(Y, Z) = E_X \tilde{S}(Y, Z) - \tilde{S}(E_X Y, Z) - \tilde{S}(Y, E_X Z), \quad (5.12.2) $$

putting $Z = \xi$ in above equation, we get

$$ (n - 1) A(X) \eta(Y) + B(X) \eta(Y) = E_X \tilde{S}(Y, \xi) - \tilde{S}(E_X Y, \xi) - \tilde{S}(Y, E_X \xi). \quad (5.12.3) $$

We know that for LP-Sasakian manifold, we have

$$ E_X \xi = \overline{X} - \overline{X}. \quad (5.12.4) $$

Using above expression in (5.12.3), we have

$$ (n - 1) A(X) \eta(Y) + B(X) \eta(Y) = (n - 1)[(E_X \eta) Y] $$

$$ - \tilde{S}(Y, \overline{X} - \overline{X}), \quad (5.12.5) $$
putting $Y = \xi$ in above equation, we get

$$(n - 1)A(X) + B(X) = 0, \quad (5.12.6)$$

which implies that

$$(n - 1)A(X) + B(X) = 0. \quad (5.12.7)$$

Here we assumed that the generalised Ricci-recurrent manifold $M_n$ with connection $E$ admits Codazzi type Ricci tensor, then we have

$$(E_X\tilde{S})(Y,Z) = (E_Y\tilde{S})(X,Z). \quad (5.12.8)$$

Using (5.10.3) in above equation, we get

$$A(X)\tilde{S}(Y,Z) + B(X)g(Y,Z) = A(Y)\tilde{S}(X,Z) + B(Y)g(X,Z). \quad (5.12.9)$$

Putting $X = \xi$ in above equation we have

$$A(\xi)\tilde{S}(Y,Z) + B(\xi)g(Y,Z) = \{(n - 1)A(Y) + B(Y)\}\eta(Z), \quad (5.12.10)$$

using (5.12.7), in above equation we get

$$A(\xi)\tilde{S}(Y,Z) + B(\xi)g(Y,Z) = 0, \quad (5.12.11)$$
⇒
\[ \tilde{S}(Y, Z) = -\frac{B(\xi)}{A(\xi)} g(Y, Z), \quad (5.12.12) \]
or
\[ \tilde{S}(X, Y) = \mu g(X, Y). \quad (5.12.13) \]

Where
\[ \mu = -\frac{B(\xi)}{A(\xi)}, \quad (5.12.14) \]
i.e. \( M_n \) is an Einstein manifold.
Hence we can state that

**Theorem 5.11.** If a generalised Ricci-reccurent LP-Sasakian manifold with connection \( E \) admits a Codazzi type Ricci tensor then it be an Einstein manifold with constant \( \mu = -\frac{B(\xi)}{A(\xi)} \).