CHAPTER 2

ON FIBONACCI CODING THEORY

\[\text{1} \text{The content of this chapter is based on "Fibonacci matrices, a generalization of the Cassini formula and a new coding theory" Chaos, Solitons and Fractals 30 (2006)56-66.}\]
2.1 Introduction

Stakhov A.P. [48] considers a new class of Fibonacci matrices $Q_p$ [1.3.8] of order $(p + 1)$. The unique property $\text{Det } Q_p^n = (-1)^{np}$ leads to a generalization of the Cassini formula (1.5) that follows from the theory of the $Q_p$ matrices. Fibonacci coding/decoding method follows from the Fibonacci matrices. For the simplest case $p = 1$, the correct ability of the method exceeds essentially all well-known correcting codes.

2.2 Fibonacci coding/decoding method

Fibonacci $Q_p$ [1.3.8] which is the generalization of $Q$ matrix [1.3.7] allows developing the following applications to the coding theory. Let us represent the initial message in the form of the square matrix $M$ of order $(p + 1)$ where $p = 0, 1, 2, 3, \cdots$. We take the Fibonacci matrix $Q_p^n$ of order $(p + 1)$ as a coding matrix and its inverse matrix $Q_p^{-n}$ as a decoding matrix. We name a transformation $M \times Q_p^n = E$ as Fibonacci coding and a transformation $E \times Q_p^{-n} = M$ as Fibonacci decoding. We define $E$ as code matrix or in other words let the message

$$M = \begin{pmatrix}
m_{11} & m_{12} & \cdots & \cdots & m_{1p+1} \\
m_{21} & m_{22} & \cdots & \cdots & m_{2p+1} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
m_{p1} & m_{p2} & \cdots & \cdots & m_{pp+1} \\
m_{p+11} & m_{p+12} & \cdots & \cdots & m_{p+1p+1}
\end{pmatrix}$$
where elements of $M$ are positive integers. Then the code matrix

$$E = M \times Q_p^n = \begin{pmatrix}
  e_{11} & e_{12} & \cdots & e_{1p+1} \\
  e_{21} & e_{22} & \cdots & e_{2p+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  e_{p1} & e_{p2} & \cdots & e_{pp+1} \\
  e_{p+11} & e_{p+12} & \cdots & e_{p+1p+1}
\end{pmatrix}$$

and for decoding

$$M = E \times Q_p^{-n}$$

### 2.2.1 Examples of the Fibonacci coding/decoding method

**Example 1**

Let the message is the sequence of the decimal numerals. We can represent the message 1235 in the matrix form $M = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$. Also, we write Fibonacci $Q = Q_1$ matrix of the second power as coding matrix

$$Q_1^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Then the inverse of $Q_1^2$ is given by

$$Q_1^{-2} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Then the code message, $E$

$$E = M \times Q_1^2 = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 11 & 8 \end{pmatrix}$$
and after decoding, the message, $M$

$$M = E \times Q_1^{-2} = \begin{pmatrix} 4 & 3 \\ 11 & 8 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

**Example 2**

If we take message 12456212 in the sequence of the decimal numerals, we can represent this message in any form of square matrix so that each elements are positive.

Let us take

$$M = \begin{pmatrix} 124 & 56 \\ 21 & 2 \end{pmatrix}$$

For $n = 3$, Fibonacci coding matrix $Q_1^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

Then the inverse of $Q_1^3$ is given by $Q_1^{-3} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$

The code matrix

$$E = M \times Q_1^3 = \begin{pmatrix} 124 & 56 \\ 21 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 484 & 304 \\ 67 & 44 \end{pmatrix}$$

and the message

$$M = E \times Q_1^{-3} = \begin{pmatrix} 484 & 304 \\ 67 & 44 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 124 & 56 \\ 21 & 2 \end{pmatrix}$$

### 2.3 Determinant of the code matrix $E$

The code matrix $E$ is defined by the following formula

$$E = M \times Q_p^n$$  \hspace{1cm} (2.1)

According to the matrix theory [28], we have

$$Det E = Det (M \times Q_p^n) = (-1)^m \times Det M$$  \hspace{1cm} (2.2)
2.4 Relations between the code matrix elements

Stakhov A.P. [50] establishes a relation between the code elements for \( p = 1 \) as follows:

\[
e_1 \approx \mu e_2, \quad e_3 \approx \mu e_4
\]  

(2.3)

where \( \mu \) is golden mean and \( e_1, e_2, e_3, e_4 \) are elements of the code matrix \( E \).

2.5 Error detection and correction

2.5.1 Error detection:

The main aims of the coding theory are the detection and correction of errors arising in the code message \( E \) under influence of noise in the communication channel. Stakhov A.P. [50] shows how to detect errors in the code message by using the Fibonacci coding. The most important idea is using the property of the matrix determinant as the check criterion of the transmitted message \( E \). We illustrate this for \( p = 1 \). Let the initial message

\[
M = \begin{pmatrix}
m_1 & m_2 \\
m_3 & m_4
\end{pmatrix}
\]  

(2.4)

where all elements \( m_1, m_2, m_3, m_4 \) of the matrix \( M \) are positive integers.

Now determinant of the message,

\[
Det M = m_1m_4 - m_2m_3
\]  

(2.5)

And the code message,

\[
E = (M \times Q_1^n)
\]  

(2.6)

So,

\[
Det E = Det (M \times Q_1^n) = (-1)^n \times Det M
\]  

(2.7)
The formula (2.7) means that the determinant of the transmitted message $E$ is equal to the determinant of the initial message $M$ by the absolute value. The sign of the determinant of the message $E$ depends on the number $n$. If the number $n$ is even then we have

$$Det E = Det M$$

(2.8)

If the number $n$ is odd, we have

$$Det E = -Det M$$

(2.9)

The identities (2.8), (2.9) underlay the basis of the new method of the error detection based on the application of the Fibonacci $Q$ matrix. The essence of the method consists of the following:

The sender calculates the determinant of the initial message $M$ represented in the matrix form (2.4) and sends it to the channel after the code message $E$ (2.6). The receiver calculates the determinant of the code message $E$ (2.6) and compares the latter with the determinant of the initial message of $M$ (2.4) received from the channel. If this comparison corresponds to (2.8) or (2.9) it means that the code message $E$ (2.6) is correct and the receiver can decode the code message $E$ (2.6) otherwise the code message $E$ (2.6) is not correct. Error detection is the first step in communication of messages.

Example:

Let us suppose that for the message 358091466725, the initial matrix has the following form:

$$M = \begin{pmatrix} 358 & 091 \\ 466 & 725 \end{pmatrix}$$

(2.10)

The determinant of $M$ is

$$Det M = (358)(725) - (466)(091) = 217144$$

(2.11)
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Then by the Fibonacci coding, the code matrix

\[ E = M \times Q_1^5 = \begin{pmatrix} 358 & 091 \\ 466 & 725 \end{pmatrix} \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 3319 & 2063 \\ 7353 & 4505 \end{pmatrix} \]  

(2.12)

and

\[ \text{Det } E = (3319)(4505) - (7353)(2063) = -217144 \]  

(2.13)

Comparing (2.11) and (2.13) we have \( \text{Det } M = -\text{Det } E = (-1)^5 \text{Det } E \). This means that the code matrix \( E \) is correct and we can decode by \( E \times Q_1^{-5} \).

2.5.2 Error correction:

Consider now a possibility of the code message \( E \) restoration by using the property of the Fibonacci \( Q_p^n \) matrix determinant.

Let us now select for any value of \( n \) Fibonacci matrix for \( p = 1 \) as the coding matrix.

Thus for \( n = 4 \)

\[ Q_1^4 = \begin{pmatrix} F_1(5) & F_1(4) \\ F_1(4) & F_1(3) \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \]  

(2.14)

Then the Fibonacci coding of the message (2.4) consists of the multiplication of the initial matrix (2.14) that is

\[ M \times Q_1^4 = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 5m_1 + 3m_2 & 3m_1 + 2m_2 \\ 5m_3 + 3m_4 & 3m_3 + 2m_4 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} = E \]  

(2.15)

where \( e_1 = 5m_1 + 3m_2, e_2 = 3m_1 + 2m_2, e_3 = 5m_3 + 3m_4, e_4 = 3m_3 + 2m_4. \)
The essence of the code matrix restoration consists of the following:

After constructing the code matrix $E$, we calculate the determinant of the initial matrix $M$ (2.4). The determinant is sent to the communication channel after the code message $E$.

Let us assume that the communication channel has the special means for the error detection in each of elements $e_1, e_2, e_3, e_4$ of the code message $E$. Let us assume that the first element $e_1$ of $E$ is received with the error. Then, we can represent the code message in the matrix form

$$E' = \begin{pmatrix} x & e_2 \\ e_3 & e_4 \end{pmatrix}$$  \hspace{1cm} (2.16)

where $x$ is the destroyed element of the code message $E$, but the rest matrix entries must be correct and equal to the following:

$$e_2 = 3m_1 + 2m_2; e_3 = 5m_3 + 3m_4; e_4 = 3m_3 + 2m_4$$  \hspace{1cm} (2.17)

Then, according to the properties of the Fibonacci coding method, we can write the following equation for calculation of $x$

$$xe_4 - e_2e_3 = x(3m_3 + 2m_4) - (3m_1 + 2m_2)(5m_3 + 3m_4) = m_1m_4 - m_2m_3$$  \hspace{1cm} (2.18)

After execution of the elementary transformations in the equation (2.18), we get the solution of the equation in the form

$$x = 5m_1 + 3m_2$$  \hspace{1cm} (2.19)

Comparing the calculated value (2.19) with the entry $e_1$ of the code matrix $E$ given with (2.15) we conclude that $x = e_1$. Thus, we have restored the code message $E$ using the property of the determinant of Fibonacci $Q^n_p$ matrix. But in the real situation usually we do not know what element of the code message is destroyed. In this case, we suppose different hypotheses about the possible destroyed elements and then we test these hypotheses. However, we have one more condition for the
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elements of the code matrix $E$ that is all its elements are integers.

Our first hypothesis is that we have the case of single error in the code matrix $E$
received from the communication channel. Clearly, there are four variants of the
single errors in the code matrix $E$:

\[
a) \begin{pmatrix} x & e_2 \\ e_3 & e_4 \end{pmatrix}, \quad b) \begin{pmatrix} e_1 & y \\ e_3 & e_4 \end{pmatrix}, \quad c) \begin{pmatrix} e_1 & e_2 \\ z & e_4 \end{pmatrix} \quad \text{and} \quad d) \begin{pmatrix} e_1 & e_2 \\ e_3 & t \end{pmatrix}
\]

where $x, y, z, t$ are destroyed elements.

In this case, we check different hypotheses (2.20). For checking the hypothesis (a),
(b), (c) and (d) we can write the following algebraic equations based on the checking
relation (2.4):

\[
x e_4 - e_2 e_3 = (-1)^n Det M(\text{a possible single error is in the element } e_1), \quad (2.21)
\]

\[
e_1 e_4 - y e_3 = (-1)^n Det M(\text{a possible single error is in the element } e_2), \quad (2.22)
\]

\[
e_1 e_4 - e_2 z = (-1)^n Det M(\text{a possible single error is in the element } e_3), \quad (2.23)
\]

\[
e_1 t - e_2 e_3 = (-1)^n Det M(\text{a possible single error is in the element } e_4), \quad (2.24)
\]

It follows from (2.21)-(2.24) four variants for calculation of the possible single errors.

\[
x = \frac{(-1)^n Det M + e_2 e_3}{e_4}, \quad (2.25)
\]

\[
y = \frac{-(1)^n Det M + e_1 e_4}{e_3}, \quad (2.26)
\]
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\[ z = \frac{(-1)^n \text{Det} M + e_1e_4}{e_2}, \quad (2.27) \]

\[ t = \frac{(-1)^n \text{Det} M + e_2e_3}{e_1}, \quad (2.28) \]

The formulae (2.25)-(2.28) give four possible variants of single error but we have to choice the correct variant only among the cases of the integer solutions \( x, y, z, t \); besides, we have to choice such solution, which satisfies the additional checking relations (2.3). If calculations by formulae (2.25)-(2.28) do not give an integer result we conclude that our hypothesis about single error is incorrect or we have error in the checking element \( \text{Det} M \). For the latter case we can use the approximate equalities (2.3) for checking a correctness of the code matrix \( E \).

By analogy, we can check all hypotheses of double error in the code matrix. As example, let us consider the following case of double errors in the code matrix \( E \)

\[
\begin{pmatrix}
x & y \\
e_3 & e_4
\end{pmatrix}
\]

(2.29)

Using the first checking relation (2.2) we can write the following algebraic equation for the matrix (2.29):

\[ xe_4 - ye_3 = (-1)^n \text{Det} M \quad (2.30) \]

However, according to the second checking relation (2.3),

\[ x \approx \mu y \quad (2.31) \]

It is important to emphasize that (2.30) is Diophantine one. As the Diophantine equation (2.30) has many solutions we have to choice such solutions \( x, y \) which satisfy the checking relation (2.31).

By analogy, one may prove that using checking relations (2.2), (2.3) by means of
solution of the Diophantine equation similar to (2.30) we can correct all possible double errors in the code matrix.

Also, we can show by using such approach there is a possibility to correct all possible triple errors in the code matrix \( E \),

For example \[
\begin{pmatrix}
x & y \\
z & e_4
\end{pmatrix}
\] etc. where \( x, y, z \) are destroyed elements.

Thus, our method of error correction is based on the verification of different hypotheses about errors in the code matrix by using the checking relations (2.2), (2.3) and by using the fact that the elements of the code matrix are integers. If all our solutions do not bring to integer solutions it means that the checking element \( \text{Det} \ M \) is erroneous or we have the case of fourfold error in the code matrix \( E \). Then we have to reject the code matrix \( E \) as defective and not correctable. Our method allows correcting 14 cases among \( (4C_1 + 4C_1 + 4C_2 + 4C_3 + 4C_4) = 2^4 - 1 = 15 \) cases.

It means that correct ability of the method is equal to \( \frac{14}{15} = 0.9333 = 99.33\% \).