CHAPTER 5

CODING THEORY ON THE \((m, t)\)-EXTENSION
OF THE FIBONACCI \(p\)-NUMBERS\(^1\)

\(^{1}\text{The content of this chapter is communicated to Visual Mathematics.}\)
5.1 Introduction

In this chapter, we introduce \((m, t)\)-extension of the Fibonacci \(p\)-numbers, \(F_{p,m,t}(n)\) and golden \((p, m, t)\)-proportion, \(\mu_{p,m,t}\). Also we establish a relation among golden \((p, m, t)\)-proportion, golden \((p, m)\)-proportion and golden \(p\)-proportion. Thereby, we define a new Fibonacci \(G_{p,m,t}\) matrix. Then we show that by proper selection of the initial terms for the \((m, t)\)-extension of the Fibonacci \(p\)-numbers, we can apply Fibonacci coding/decoding in \(G_{p,m,t}\) matrix. Also for \(t = 1\), the relations among the code elements for \(p > 0\) is integer and \(m > 0\) coincide with the relations among the code matrix elements for \(p > 0\) is integer and \(m > 0\) with the same initial terms [4].

5.2 \((m, t)\)-extension of Fibonacci \(p\)-numbers, \(F_{p,m,t}(n)\)

Basu M. et. al [5] introduce the \((m, t)\)-extension of Fibonacci \(p\)-numbers, \(F_{p,m,t}(n)\) which satisfy the recurrence relation

\[
F_{p,m,t}(n) = mF_{p,m,t}(n-1) + tF_{p,m,t}(n-p-1)
\]  \hspace{1cm} (5.1)

with initial terms

\[
F_{p,m,t}(1) = b_1, F_{p,m,t}(2) = b_2, F_{p,m,t}(3) = b_3, \ldots, F_{p,m,t}(p+1) = b_{p+1}
\]  \hspace{1cm} (5.2)

where \(p \geq 0\) is integer, \(m > 0\), \(t > 0\), \(n > p + 1\) and \(b_1, b_2, b_3, \ldots, b_{p+1}\) are arbitrary real or complex numbers.

Naturally, \(F_{p,m,1}(n) = F_{p,m}(n), F_{1,m,1}(n) = F_{1,m}(n)\) and \(F_{1,1,1}(n) = F_n\)

For calculation of \((m, t)\)-extension of Fibonacci \(p\)-numbers for all values of \(n\), we consider the recurrence relation

\[
F_{p,m,t}(n) = mF_{p,m,t}(n-1) + tF_{p,m,t}(n-p-1)
\]  \hspace{1cm} (5.3)

with initial terms

\[
F_{p,m,t}(1) = b_1, F_{p,m,t}(2) = b_2, F_{p,m,t}(3) = b_3, \ldots, F_{p,m,t}(p+1) = b_{p+1}
\]  \hspace{1cm} (5.4)
where \( n = 0, \pm 1, \pm 2, \pm 3, \cdots \), \( p = 0, 1, 2, 3, \cdots \), \( m > 0 \), \( t > 0 \) and \( b_1, b_2, b_3, \cdots, b_{p+1} \) are arbitrary real or complex numbers.

### 5.3 Golden \((p, m, t)\)-proportion(mean), \( \mu_{p,m,t} \)

The characteristic equation of the \((m, t)\)-extension of the Fibonacci \(p\)-numbers is

\[
x^{p+1} - mx^p - t = 0
\]

where \( x = \lim_{n \to \infty} \frac{F_{p,m,t}(n)}{F_{p,m,t}(n-1)} \).

The only one positive root, \( \mu_{p,m,t} \) of the equation (5.5) is called golden \((p, m, t)\)-proportion(mean) [Table 5.1]. From the [Table 5.1] it is easily verified that \( \mu_{p,m,1} = \mu_{p,m}, \mu_{p,1,1} = \mu_p, \mu_{1,1,1} = \mu \).

### 5.4 Relations among golden \((p, m, t)\)-proportion, golden \((p, m)\)-proportion and golden \(p\)-proportion

The characteristic equation of the \((m, t)\)-extension of the Fibonacci \(p\)-numbers is

\[
x^{p+1} - mx^p - t = 0
\]

whereas, the characteristic equation of the \(m\)-extension of the Fibonacci \(p\)-numbers is

\[
x^{p+1} - mx^p - 1 = 0
\]

Also, the characteristic equation of the Fibonacci \(p\)-numbers is

\[
x^{p+1} - x^p - 1 = 0
\]

Each of the equations (5.6), (5.7) and (5.8) has \((p + 1)\) roots. The only one positive root \( x_5 = \mu_{p,m,t} \) of the equation (5.6) is called golden \((p, m, t)\)-proportion. \( \mu_{p,m,t} \)
Table 5.1 Golden \((p, m, t)\)-proportion, \(\mu_{p,m,t}\)

<table>
<thead>
<tr>
<th>(p = 1)</th>
<th>(m = 1)</th>
<th>(m = 2)</th>
<th>(m = 3)</th>
<th>(m = 1)</th>
<th>(m = 2)</th>
<th>(m = 3)</th>
<th>(m = 1)</th>
<th>(m = 2)</th>
<th>(m = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t = 1)</td>
<td>(\mu_{1,1,1} = 1.6180)</td>
<td>(\mu_{1,2,1} = 2.4142)</td>
<td>(\mu_{1,3,1} = 3.3028)</td>
<td>(\mu_{1,2,1} = 2.0000)</td>
<td>(\mu_{1,2,2} = 2.7321)</td>
<td>(\mu_{1,3,2} = 3.5616)</td>
<td>(\mu_{1,1,2} = 2.5616)</td>
<td>(\mu_{1,2,4} = 3.2361)</td>
<td>(\mu_{1,3,4} = 4.0000)</td>
</tr>
<tr>
<td>(t = 3)</td>
<td>(\mu_{1,3,3} = 3.7913)</td>
<td>(\mu_{2,1,3} = 2.3028)</td>
<td>(\mu_{2,2,3} = 3.0000)</td>
<td>(\mu_{2,3,3} = 2.3958)</td>
<td>(\mu_{2,1,2} = 1.6956)</td>
<td>(\mu_{2,2,2} = 2.3593)</td>
<td>(\mu_{2,3,2} = 3.1958)</td>
<td>(\mu_{2,1,4} = 2.0000)</td>
<td>(\mu_{2,2,4} = 2.5943)</td>
</tr>
</tbody>
</table>

[Table 5.1] extends infinitely a number of new mathematical constants. We can say that golden \((p, m, t)\)-proportion is a wide generalization of golden \((p, m)\)-proportion [Table 4.2]. Also the only one positive root \(x_3 = \mu_{p,m}\), golden \((p, m)\)-proportion of the equation (5.7) coincides with \(\mu_{p,m,1}\), \((p, m, 1)\)-proportion. Let \(x_2 = \mu_p\), golden \(p\)-proportion, be the positive root of the characteristic equation (5.8). Then it is obvious that \(x_5, x_3, x_2\) satisfy the equation

\[
x_3 - x_5 = x_3 \frac{\log(x_2-1)}{\log x_2} - t x_5 \frac{\log(x_2-1)}{\log x_2}
\]  

(5.9)
5.5 Fibonacci $G_{p,m,t}$ matrix

In this section, we define a new matrix called Fibonacci $G_{p,m,t}$ matrix of order $(p+1)$ on the $(m,t)$-extension of the Fibonacci $p$-numbers where $p \geq 0$ is integer and $m > 0$, $t > 0$.

$$G_{p,m,t} = \begin{pmatrix}
F_{p,m,t}(2) & F_{p,m,t}(1) & \cdots & F_{p,m,t}(2-p) \\
F_{p,m,t}(2-p) & F_{p,m,t}(1-p) & \cdots & F_{p,m,t}(2-2p) \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \\
F_{p,m,t}(0) & F_{p,m,t}(-1) & \cdots & F_{p,m,t}(-p) \\
F_{p,m,t}(1) & F_{p,m,t}(0) & \cdots & F_{p,m,t}(1-p)
\end{pmatrix}$$

(5.10)

where the initial terms $b_1, b_2, b_3, \ldots, b_{p+1}$ (5.4) are such that

$$\text{Det } G_{p,m,t} = (-1)^p, \text{ independent of } m \text{ and } t.$$  

Also the $n$th power of $G_{p,m,t}$ is

$$G_{p,m,t}^n = \begin{pmatrix}
F_{p,m,t}(n+1) & F_{p,m,t}(n) & \cdots & F_{p,m,t}(n-p+1) \\
F_{p,m,t}(n-p+1) & F_{p,m,t}(n-p) & \cdots & F_{p,m,t}(n-2p+1) \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \\
F_{p,m,t}(n-1) & F_{p,m,t}(n-2) & \cdots & F_{p,m,t}(n-p-1) \\
F_{p,m,t}(n) & F_{p,m,t}(n-1) & \cdots & F_{p,m,t}(n-p)
\end{pmatrix}$$

(5.11)

so that $\text{Det } G_{p,m,t}^n = (-1)^{np}$. After proper selection of $b_1, b_2, b_3, \ldots, b_{p+1}$, when $G_{p,m,t}$ satisfies (5.10) along with (5.11) then $G_{p,m,t}$ can be applicable for Fibonacci
5.5. FIBONACCI $G_{p,m,t}$ MATRIX

coding/decoding. When $t = 1$, equations (5.10) and (5.11) automatically satisfy (4.11) and (4.12) for considering $b_i = m^{i-1}$, $i = 1, 2, 3, \ldots, p + 1$.

We know that the correct ability of this method increases as $p$ increases and it is independent of $m, t$. And for large value of $p$, it is approximately to 100%. For $t = 1$, properties of $G_{p,m,t}$, $G_{p,m,t}^n$ matrix coincide with the properties of $G_{p,m}$, $G_{p,m}^n$ matrix respectively in chapter 4. Also, for $t = 1$, the relations among the code elements for $p (> 0)$ is integer and $m (> 0)$ coincide with the relations among the code matrix elements for $p (> 0)$ is integer and $m (> 0)$ with the same initial terms [4].