CHAPTER 4

Coding theory on the \( m \)-extension of the Fibonacci \( p \)-numbers\(^1\)

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4.1 Introduction

We introduce a new Fibonacci $G_{p,m}$ matrix for the $m$-extension of the Fibonacci $p$-numbers where $p (> 0)$ is integer and $m (> 0)$. Thereby, we discuss various properties of $G_{p,m}$ matrix and the coding theory followed from the $G_{p,m}$ matrix. In this chapter, we establish the relations among the code elements for $p (> 0)$ is integer and $m (> 0)$. We also show that the relations, among the code matrix elements for $p (> 0)$ is integer and $m = 1$, coincide with the relations among the code matrix elements [6]. We show that the correct ability of this method is independent of $m$. Also in general, when $p$ increases, the correct ability of this method increases.

4.2 Some properties of the $m$-extension of the Fibonacci $p$-numbers for a given initial terms

In this section, we calculate $m$-extension of the Fibonacci $p$-numbers [1.3.4] for a specified initial terms. For calculations of $m$-extension Fibonacci $p$-numbers for all values of $n$, we consider the recurrence relation

$$F_{p,m}(n) = mF_{p,m}(n - 1) + F_{p,m}(n - p - 1) \quad (4.1)$$

with the given initial terms

$$F_{p,m}(n) = m^{n-1}, \quad n = 1, 2, 3, 4, \cdots, p + 1. \quad (4.2)$$

where $p = 0, 1, 2, 3, \cdots, m (> 0)$ and $n = 0, \pm 1, \pm 2, \pm 3, \cdots$

When, $n = p + 1$, (4.1) gives

$$F_{p,m}(p + 1) = mF_{p,m}(p) + F_{p,m}(0) \quad (4.3)$$

Since $F_{p,m}(p + 1) = m^p$ and $F_{p,m}(p) = m^{p-1}$, so $F_{p,m}(0) = 0$

Continuing this process by substituting $n = p, p - 1, \cdots, 2$ in (4.1) we get
4.3. **GOLDEN \((P, M)\)-PROPORTION(MEAN), \(\mu_{P, M}\)**

\(F_{p,m}(0) = F_{p,m}(-1) = F_{p,m}(-2) = \cdots = F_{p,m}(-p+1) = 0.\)

When \(n = 1\), (4.1) gives

\[
F_{p,m}(1) = mF_{p,m}(0) + F_{p,m}(-p) \tag{4.4}
\]

Since \(F_{p,m}(1) = m^{1-1} = 1, \quad F_{p,m}(0) = 0, \quad \text{so} \quad F_{p,m}(-p) = 1\)

Representing \(m\)-extension of Fibonacci \(p\)-numbers

\(F_{p,m}(0), \quad F_{p,m}(-1), \quad \cdots, \quad F_{p,m}(-p+2)\) in form of (4.1) we get

\(F_{p,m}(-p-1) = F_{p,m}(-p-2) = \cdots = F_{p,m}(-2p+1) = 0\)

Also, representing \(m\)-extension of Fibonacci \(p\)-numbers \(F_{p,m}(-p+1), \quad F_{p,m}(-p), \) and

\(F_{p,m}(-p-1)\) in form of (4.1) we get

\(F_{p,m}(-2p) = -m, \quad F_{p,m}(-2p-1) = 1\) and \(F_{p,m}(-2p-2) = 0\)

We summarized above in the following Table 4.1:

<table>
<thead>
<tr>
<th>(n)</th>
<th>(-p+2)</th>
<th>(-p+1)</th>
<th>(-p)</th>
<th>(-p-1)</th>
<th>(-2p+1)</th>
<th>(-2p)</th>
<th>(-2p-1)</th>
<th>(-2p-2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F_{p,m}(n))</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(-m)</td>
<td>(1)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

Thus, we get \(m\)-extension of Fibonacci \(p\) numbers,

\[
F_{p,m}(n) = mF_{p,m}(n-1) + F_{p,m}(n-p-1) \tag{4.5}
\]

for \(p = 0, 1, 2, 3, \ldots\) and \(n = 0, \pm 1, \pm 2, \pm 3, \ldots\)

where \(F_{p,m}(n) = m^{n-1}, \quad n = 1, 2, 3, 4, \ldots, p+1\)

### 4.3 **Golden \((p, m)\)-proportion(mean), \(\mu_{p,m}\)**

The characteristic equation of the \(m\)-extension of the Fibonacci \(p\)-numbers is

\[
x^{p+1} - mx^p - 1 = 0 \tag{4.6}
\]
where \( x = \lim_{n \to \infty} \frac{F_{p, m}(n)}{F_{p, m}(n-1)} \).

The only one positive root, \( \mu_{p, m} \) of the equation (4.6) is called golden \((p, m)\)-proportion(mean) [Table 4.2].

**Table 4.2 Golden \((p, m)\)-proportion, \(\mu_{p, m} \)**

<table>
<thead>
<tr>
<th>(p = 1)</th>
<th>(m = 1)</th>
<th>(\mu_{1,1} = 1.6180)</th>
<th>(m = 1)</th>
<th>(\mu_{2,1} = 1.4656)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m = 2)</td>
<td>(\mu_{1,2} = 2.4142)</td>
<td>(m = 2)</td>
<td>(\mu_{2,2} = 2.2056)</td>
<td></td>
</tr>
<tr>
<td>(m = 3)</td>
<td>(\mu_{1,3} = 3.3028)</td>
<td>(m = 3)</td>
<td>(\mu_{2,3} = 3.1038)</td>
<td></td>
</tr>
<tr>
<td>(m = 4)</td>
<td>(\mu_{1,4} = 4.2361)</td>
<td>(m = 4)</td>
<td>(\mu_{2,4} = 4.0607)</td>
<td></td>
</tr>
<tr>
<td>(m = 5)</td>
<td>(\mu_{1,5} = 5.1926)</td>
<td>(m = 5)</td>
<td>(\mu_{2,5} = 5.0394)</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>(p = 3)</td>
<td>(m = 1)</td>
<td>(\mu_{3,1} = 1.3803)</td>
<td>(m = 1)</td>
<td>(\mu_{4,1} = 1.3247)</td>
</tr>
<tr>
<td>(m = 2)</td>
<td>(\mu_{3,2} = 2.1069)</td>
<td>(m = 2)</td>
<td>(\mu_{4,2} = 2.0560)</td>
<td></td>
</tr>
<tr>
<td>(m = 3)</td>
<td>(\mu_{3,3} = 3.0357)</td>
<td>(m = 3)</td>
<td>(\mu_{4,3} = 3.0122)</td>
<td></td>
</tr>
<tr>
<td>(m = 4)</td>
<td>(\mu_{3,4} = 4.0155)</td>
<td>(m = 4)</td>
<td>(\mu_{4,4} = 4.0039)</td>
<td></td>
</tr>
<tr>
<td>(m = 5)</td>
<td>(\mu_{3,5} = 5.0080)</td>
<td>(m = 5)</td>
<td>(\mu_{4,5} = 5.0016)</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>(p = 5)</td>
<td>(m = 1)</td>
<td>(\mu_{5,1} = 1.2852)</td>
<td>(m = 1)</td>
<td>(\mu_{6,1} = 1.2554)</td>
</tr>
<tr>
<td>(m = 2)</td>
<td>(\mu_{5,2} = 2.0291)</td>
<td>(m = 2)</td>
<td>(\mu_{6,2} = 2.0150)</td>
<td></td>
</tr>
<tr>
<td>(m = 3)</td>
<td>(\mu_{5,3} = 3.0041)</td>
<td>(m = 3)</td>
<td>(\mu_{6,3} = 3.0014)</td>
<td></td>
</tr>
<tr>
<td>(m = 4)</td>
<td>(\mu_{5,4} = 4.0010)</td>
<td>(m = 4)</td>
<td>(\mu_{6,4} = 4.0002)</td>
<td></td>
</tr>
<tr>
<td>(m = 5)</td>
<td>(\mu_{5,5} = 5.0003)</td>
<td>(m = 5)</td>
<td>(\mu_{6,5} = 5.0001)</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
4.4 Relation among $\mu_{p,m}$, $\mu_{p,1}$ and $\mu_{1,m}$

The characteristic equation of the $m$-extension of the Fibonacci $p$-numbers is

$$x^{p+1} - mx^p - 1 = 0 \quad (4.7)$$

whereas, the characteristic equation of the Fibonacci $p$-numbers is

$$x^{p+1} - x^p - 1 = 0 \quad (4.8)$$

Both the equations (4.7) and (4.8) have $(p + 1)$ roots. The only one positive root $x_3 = \mu_{p,m}$ of the equation (4.7) is called golden $(p, m)$-proportion. $\mu_{p,m}$, extends infinitely a number of new mathematical constants [Table 4.2] or in other words we can say that $\mu_{p,m}$ is a wide generalization of the classical golden mean. Also the only one positive root $x_2 = \mu_p$, golden $p$-proportion, of the equation (4.8) coincides with $\mu_{p,1}$, golden $(p, 1)$-proportion. $\mu_{1,m}$, golden $(1, m)$-proportion is the positive root $x_4$ of the characteristic equation $x^2 - mx - 1 = 0$.

It is obvious that $x_2$, $x_3$, $x_4$ satisfy the equation

$$\frac{\log x_3}{\log x_2} = \frac{\log(1 + x_4(x_3 - x_4)) - \log x_4}{\log(x_2 - 1)} \quad (4.9)$$

4.5 Fibonacci $G_{p,m}$ matrix

We define a new matrix called Fibonacci $G_{p,m}$ matrix (4.10) of order $(p + 1)$ on the $m$-extension of the Fibonacci $p$-numbers where $p (> 0)$ is integer and $m > 0$. 
Using (4.5) we can write

\[
G_{p,m} = \begin{pmatrix}
F_{p,m}(2) & F_{p,m}(1) & \ldots & F_{p,m}(3-p) & F_{p,m}(2-p) \\
F_{p,m}(2-p) & F_{p,m}(1-p) & \ldots & F_{p,m}(3-2p) & F_{p,m}(2-2p) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
F_{p,m}(0) & F_{p,m}(-1) & \ldots & F_{p,m}(1-p) & F_{p,m}(-p) \\
F_{p,m}(1) & F_{p,m}(0) & \ldots & F_{p,m}(2-p) & F_{p,m}(1-p)
\end{pmatrix}
\]

So that,

\[
G_{1,m} = \begin{pmatrix}
m & 1 \\
1 & 0
\end{pmatrix} = \begin{pmatrix} F_{1,m}(2) & F_{1,m}(1) \\
F_{1,m}(1) & F_{1,m}(0) \end{pmatrix}
\]

\[
\text{Det } G_{1,m} = -1 = (-1)^1
\]

\[
G_{2,m} = \begin{pmatrix}
m & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix} = \begin{pmatrix} F_{2,m}(2) & F_{2,m}(1) & F_{2,m}(0) \\
F_{2,m}(0) & F_{2,m}(-1) & F_{2,m}(-2) \\
F_{2,m}(1) & F_{2,m}(0) & F_{2,m}(-1) \end{pmatrix}
\]
4.6. PROPERTIES OF $G_{p,m}$

$$\text{Det } G_{2,m} = 1 = (-1)^2$$

$$G_{3,m} = \begin{pmatrix}
  m & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
  1 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
  F_{3,m}(2) & F_{3,m}(1) & F_{3,m}(0) & F_{3,m}(-1) \\
  F_{3,m}(-1) & F_{3,m}(-2) & F_{3,m}(-3) & F_{3,m}(-4) \\
  F_{3,m}(0) & F_{3,m}(-1) & F_{3,m}(-2) & F_{3,m}(-3) \\
  F_{3,m}(1) & F_{3,m}(0) & F_{3,m}(-1) & F_{3,m}(-2)
\end{pmatrix}$$

$$\text{Det } G_{3,m} = -1 = (-1)^3$$

$$G_{4,m} = \begin{pmatrix}
  m & 1 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1 \\
  1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$= \begin{pmatrix}
  F_{4,m}(2) & F_{4,m}(1) & F_{4,m}(0) & F_{4,m}(-1) & F_{4,m}(-2) \\
  F_{4,m}(-2) & F_{4,m}(-3) & F_{4,m}(-4) & F_{4,m}(-5) & F_{4,m}(-6) \\
  F_{4,m}(-1) & F_{4,m}(-2) & F_{4,m}(-3) & F_{4,m}(-4) & F_{4,m}(-5) \\
  F_{4,m}(0) & F_{4,m}(-1) & F_{4,m}(-2) & F_{4,m}(-3) & F_{4,m}(-4) \\
  F_{4,m}(1) & F_{4,m}(0) & F_{4,m}(-1) & F_{4,m}(-2) & F_{4,m}(-3)
\end{pmatrix}$$

$$\text{Det } G_{4,m} = 1 = (-1)^4$$

and so on.

Thus, $$\text{Det } G_{p,m} = (-1)^p,$$ independent of $m$.

4.6 Properties of $G_{p,m}$

Theorem 1:

For a given integer $n$ ($n = 0, \pm 1, \pm 2, \pm 3, \ldots$), the $n$th power of the $G_{p,m}$ matrix is
given by $G_{p,m}^n =$

$$
\begin{pmatrix}
F_{p,m}(n+1) & F_{p,m}(n) & \ldots & F_{p,m}(n-p+2) & F_{p,m}(n-p+1) \\
F_{p,m}(n-p+1) & F_{p,m}(n-p) & \ldots & F_{p,m}(n-2p+2) & F_{p,m}(n-2p+1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
F_{p,m}(n-1) & F_{p,m}(n-2) & \ldots & F_{p,m}(n-p) & F_{p,m}(n-p-1) \\
F_{p,m}(n) & F_{p,m}(n-1) & \ldots & F_{p,m}(n+1) & F_{p,m}(n) \\
\end{pmatrix}
$$

(4.12)

where $F_{p,m}(n) = m^{n-1}$ is given in (4.5).

**Proof:** When $p = 1$, we have to prove

$$
G_{1,m}^n = \begin{pmatrix}
F_{1,m}(n+1) & F_{1,m}(n) \\
F_{1,m}(n) & F_{1,m}(n-1)
\end{pmatrix}
$$

(4.13)

We will prove it by mathematical induction.

For $n = 1$

$$
G_{1,m} = \begin{pmatrix}
m & 1 \\
1 & 0
\end{pmatrix}
$$

$$
= \begin{pmatrix}
F_{1,m}(2) & F_{1,m}(1) \\
F_{1,m}(1) & F_{1,m}(0)
\end{pmatrix}
$$

by (4.5)

which is true for $n = 1$

For $n = 2$

$$
G_{1,m}^2 = \begin{pmatrix}
m^2 + 1 & m \\
m & 1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
F_{1,m}(3) & F_{1,m}(2) \\
F_{1,m}(2) & F_{1,m}(1)
\end{pmatrix}
$$

by (4.5)
which is true for $n = 2$

Suppose (4.13) is true for integer $n = k$, then

$$G_{1,m}^k = \begin{pmatrix}
F_{1,m}(k+1) & F_{1,m}(k) \\
F_{1,m}(k) & F_{1,m}(k-1)
\end{pmatrix}$$

Now, we can write

$$G_{1,m}^{k+1} = G_{1,m}^k G_{1,m} = \begin{pmatrix}
F_{1,m}(k+1) & F_{1,m}(k) \\
F_{1,m}(k) & F_{1,m}(k-1)
\end{pmatrix} \begin{pmatrix}
m & 1 \\
1 & 0
\end{pmatrix}$$

$$= \begin{pmatrix}
F_{1,m}(k+2) & F_{1,m}(k+1) \\
F_{1,m}(k+1) & F_{1,m}(k)
\end{pmatrix} \text{ by (4.5)}$$

Hence by induction, we can write

$$G_{1,m}^n = \begin{pmatrix}
F_{1,m}(n+1) & F_{1,m}(n) \\
F_{1,m}(n) & F_{1,m}(n-1)
\end{pmatrix}$$

When $p = 2$, we have to prove

$$G_{2,m}^n = \begin{pmatrix}
F_{2,m}(n+1) & F_{2,m}(n) & F_{2,m}(n-1) \\
F_{2,m}(n-1) & F_{2,m}(n-2) & F_{2,m}(n-3) \\
F_{2,m}(n) & F_{2,m}(n-1) & F_{2,m}(n-2)
\end{pmatrix} \quad (4.14)$$

We will prove it by mathematical induction.

For $n = 1$

$$G_{2,m} = \begin{pmatrix}
m & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}$$

$$= \begin{pmatrix}
F_{2,m}(2) & F_{2,m}(1) & F_{2,m}(0) \\
F_{2,m}(0) & F_{2,m}(-1) & F_{2,m}(-2) \\
F_{2,m}(1) & F_{2,m}(0) & F_{2,m}(-1)
\end{pmatrix} \text{ by (4.5)}$$
which is true for \( n = 1 \)

For \( n = 2 \)

\[
G_{2,m}^2 = \begin{pmatrix}
m^2 & m & 1 \\
1 & 0 & 0 \\
m & 1 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
F_{2,m}(3) & F_{2,m}(2) & F_{2,m}(1) \\
F_{2,m}(1) & F_{2,m}(0) & F_{2,m}(-1) \\
F_{2,m}(2) & F_{2,m}(1) & F_{2,m}(0)
\end{pmatrix}
\]

by (4.5)

which is true for \( n = 2 \)

Suppose (4.14) is true for integer \( n = k \), then

\[
G_{2,m}^k = \begin{pmatrix}
F_{2,m}(k+1) & F_{2,m}(k) & F_{2,m}(k-1) \\
F_{2,m}(k-1) & F_{2,m}(k-2) & F_{2,m}(k-3) \\
F_{2,m}(k) & F_{2,m}(k-1) & F_{2,m}(k-2)
\end{pmatrix}
\]

Now, we can write

\[
G_{2,m}^{k+1} = G_{2,m}^k G_{2,m} = \begin{pmatrix}
F_{2,m}(k+1) & F_{2,m}(k) & F_{2,m}(k-1) \\
F_{2,m}(k-1) & F_{2,m}(k-2) & F_{2,m}(k-3) \\
F_{2,m}(k) & F_{2,m}(k-1) & F_{2,m}(k-2)
\end{pmatrix} \begin{pmatrix}
m & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]

by (4.5)

Hence by induction, we can write

\[
G_{2,m}^n = \begin{pmatrix}
F_{2,m}(n+1) & F_{2,m}(n) & F_{2,m}(n-1) \\
F_{2,m}(n-1) & F_{2,m}(n-2) & F_{2,m}(n-3) \\
F_{2,m}(n) & F_{2,m}(n-1) & F_{2,m}(n-2)
\end{pmatrix}
\]
Therefore, by induction, it can be proved for all values of \( p \).

Hence the theorem.

**Theorem 2:**

\[ G_{p,m}^n = mG_{p,m}^{n-1} + G_{p,m}^{n-(p+1)} \]

**Proof:** By theorem 1

\[
G_{p,m}^n = \begin{pmatrix}
F_{p,m}(n+1) & F_{p,m}(n) & \ldots & F_{p,m}(n-p+2) & F_{p,m}(n-p+1) \\
F_{p,m}(n-p+1) & F_{p,m}(n-p) & \ldots & F_{p,m}(n-2p+2) & F_{p,m}(n-2p+1) \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
F_{p,m}(n-1) & F_{p,m}(n-2) & \ldots & F_{p,m}(n-p) & F_{p,m}(n-p-1) \\
F_{p,m}(n) & F_{p,m}(n-1) & \ldots & F_{p,m}(n-p+1) & F_{p,m}(n-p)
\end{pmatrix}
\]

When \( p = 1 \)

\[
G_{1,m}^n = \begin{pmatrix}
F_{1,m}(n+1) & F_{1,m}(n) \\
F_{1,m}(n) & F_{1,m}(n-1)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
mF_{1,m}(n) + F_{1,m}(n-1) & mF_{1,m}(n-1) + F_{1,m}(n-2) \\
mF_{1,m}(n-1) + F_{1,m}(n-2) & mF_{1,m}(n-2) + F_{1,m}(n-3)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
mF_{1,m}(n) & mF_{1,m}(n-1) \\
mF_{1,m}(n-1) & mF_{1,m}(n-2)
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
F_{1,m}(n-1) & F_{1,m}(n-2) \\
F_{1,m}(n-2) & F_{1,m}(n-3)
\end{pmatrix}
\]

\[= mG_{1,m}^{n-1} + G_{1,m}^{n-2} \]
When \( p = 2 \)

\[
G_{2,m}^n = \begin{pmatrix}
F_{2,m}(n + 1) & F_{2,m}(n) & F_{2,m}(n - 1) \\
F_{2,m}(n - 1) & F_{2,m}(n - 2) & F_{2,m}(n - 3) \\
F_{2,m}(n) & F_{2,m}(n - 1) & F_{2,m}(n - 2)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
mF_{2,m}(n) + F_{2,m}(n - 2) & mF_{2,m}(n - 1) + F_{2,m}(n - 3) & mF_{2,m}(n - 2) + F_{2,m}(n - 4) \\
mF_{2,m}(n - 2) + F_{2,m}(n - 4) & mF_{2,m}(n - 3) + F_{2,m}(n - 5) & mF_{2,m}(n - 4) + F_{2,m}(n - 6) \\
mF_{2,m}(n - 1) + F_{2,m}(n - 3) & mF_{2,m}(n - 2) + F_{2,m}(n - 4) & mF_{2,m}(n - 3) + F_{2,m}(n - 5)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
mF_{2,m}(n) & mF_{2,m}(n - 1) & mF_{2,m}(n - 2) \\
mF_{2,m}(n - 2) & mF_{2,m}(n - 3) & mF_{2,m}(n - 4) \\
mF_{2,m}(n - 1) & mF_{2,m}(n - 2) & mF_{2,m}(n - 3)
\end{pmatrix}
+ \begin{pmatrix}
F_{2,m}(n - 2) & F_{2,m}(n - 3) & F_{2,m}(n - 4) \\
F_{2,m}(n - 4) & F_{2,m}(n - 5) & F_{2,m}(n - 6) \\
F_{2,m}(n - 3) & F_{2,m}(n - 4) & F_{2,m}(n - 5)
\end{pmatrix}
\]

\[= mG_{2,m}^{n-1} + G_{2,m}^{n-3}\]

Similarly, we can show that,

\[G_{p,m}^n = mG_{p,m}^{n-1} + G_{p,m}^{n-(p+1)}\]

Hence the theorem.

Therefore, the properties of \( G_{p,m}^n \) can be summarized as:

(i) \( \text{Det} (G_{p,m}^n) = [\text{Det} (G_{p,m})]^n = (-1)^{pn}, \) independent of \( m \)

(ii) \( G_{p,m}^n = mG_{p,m}^{n-1} + G_{p,m}^{n-(p+1)} \)

(iii) \( G_{p,m}^{n_1}G_{p,m}^{n_2} = G_{p,m}^{n_1+n_2} \)

(iv) By inverse matrix method, we also get \( G_{p,m}^{-n} \) from \( G_{p,m}^n \)

For a particular case \( p = 1 \), \( G_{1,m}^{-n} \) is obtained as

\[
G_{1,m}^{-n} = \begin{pmatrix}
F_{2,m}(2k - 1) & -F_{2,m}(2k) \\
-F_{2,m}(2k) & F_{2,m}(2k + 1)
\end{pmatrix}, \text{ where } n = 2k, \text{ even.}
4.7. FIBONACCI $G_{p,m}$ CODING METHOD

The Fibonacci $G_{p,m}$ matrix allows developing the following applications to the coding theory. Let us represent the initial message in the form of the square matrix $M$ of order $(p + 1)$ where $p = 1, 2, 3, \ldots$. We take $G_{p,m}^n$ matrix of order $(p + 1)$ as a coding matrix. We name a transformation $M \times G_{p,m}^n = E$ as coding and $E$ is known as code matrix.

For example, consider the case $p = 1$,

then we represent the initial message in form of the $2 \times 2$ matrix

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \quad (4.15)$$

where all elements of the matrix are positive integers. i.e. $m_1, m_2, m_3, m_4 > 0$. Let us select for any value of $n$, the $G_{1,m}^n$ matrix as the coding matrix.

Hence, for $p = 1$ and $n = 3$ (say), we have

$$G_{1,m}^3 = \begin{pmatrix} m^3 + 2m & m^2 + 1 \\ m^2 + 1 & m \end{pmatrix} \quad (4.16)$$

Then the $G_{1,m}^3$ coding of the message (4.15) consists of the multiplication by the initial matrix (4.16) that is

$$M \times G_{1,m}^3 = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} m^3 + 2m & m^2 + 1 \\ m^2 + 1 & m \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} = E \quad (4.17)$$
where
\[ e_1 = m_1m^3 + 2m_1m + m_2m^2 + m_2, \quad e_2 = m_1m^2 + m_1 + m_2m \]
\[ e_3 = m_3m^3 + 2m_3m + m_4m^2 + m_4, \quad e_4 = m_3m^2 + m_3 + m_4m. \]

### 4.8 Fibonacci \( G_{p,m} \) decoding method

Now, we describe the decoding method of Fibonacci \( G_{p,m} \) coding method. The initial message, \( M \) is a square matrix of order \((p + 1)\) where \( p = 1, 2, 3, \ldots \). We take the inverse of coding matrix \( G_{p,m}^n \) as decoding matrix \( G_{p,m}^{-n} \) so that the transformation \( E \times G_{p,m}^{-n} \) is called decoding and \( E \times G_{p,m}^{-n} = M \) where \( E \) is the code matrix.

For example,

The inverse matrix of (4.16) is given by
\[
G_{1,m}^{-3} = \begin{pmatrix}
-m & m^2 + 1 \\
\frac{m^2 + 1}{m^2 + 1} & \frac{m^3 + 2m}{m^2 + 1}
\end{pmatrix}
\tag{4.18}
\]

The decoding of the code message \( E \) (4.17) is
\[
E \times G_{1,m}^{-3} = \begin{pmatrix}
e_1 & e_2 \\
e_3 & e_4
\end{pmatrix} \begin{pmatrix}
-m & m^2 + 1 \\
m^2 + 1 & m^3 + 2m
\end{pmatrix} = \begin{pmatrix}
m_1 & m_2 \\
m_3 & m_4
\end{pmatrix} = M
\]
4.9 Determinant of the code matrix \( E \)

The code matrix \( E \) is defined by the following formula: \( E = M \times G_{p,m}^n \). According to the matrix theory [28], we have

\[
\text{Det} \ E = \text{Det} (M \times G_{p,m}^n) = \text{Det} M \times \text{Det} G_{p,m}^n = \text{Det} M \times (-1)^m = (-1)^m \times \text{Det} M
\]

(4.19)

4.10 Relations among the code matrix elements for \( m > 0 \)

Case 1: \( p = 1 \),

Similar to (3.36), we obtain

\[
\frac{e_1}{e_2} \approx \mu_{1,m}, \quad \frac{e_3}{e_4} \approx \mu_{1,m}
\]

(4.20)

where \( \mu_{1,m} = \frac{m+\sqrt{m^2+4}}{2} \), \( e_1, e_2, e_3, e_4 \) are given in (4.17)

Case 2: \( p = 2 \),

In this case, let the message

\[
M = \begin{pmatrix}
  m_1 & m_2 & m_3 \\
  m_4 & m_5 & m_6 \\
  m_7 & m_8 & m_9
\end{pmatrix}
\]

then, the \( G_{2,m}^n \) coding of the message \( M \) is

\[
M \times G_{2,m}^n = \begin{pmatrix}
  e_1 & e_2 & e_3 \\
  e_4 & e_5 & e_6 \\
  e_7 & e_8 & e_9
\end{pmatrix} = E
\]

Similar to (3.36), we obtain

\[
\frac{e_1}{e_2} \approx \mu_{2,m}, \quad \frac{e_3}{e_4} \approx \mu_{2,m}, \quad \frac{e_5}{e_6} \approx \mu_{2,m}
\]

(4.21)
where \( \mu_{2,m} = \frac{s^2 + 2sm + 4m^2}{6s} \), \( s = \sqrt{108 + 8m^3 + 12\sqrt{81 + 12m^3}} \).

and so on.

In general, like (3.36), when \( p = t \) and \( n > p + 1 = t + 1 \), the generalized relations among the code matrix elements are

\[
\begin{align*}
\frac{e_1}{e_2} &\approx \mu_{t,m}; \quad \frac{e_2}{e_3} \approx \mu_{t,m}; \quad \cdots; \quad \frac{e_t}{e_{t+1}} \approx \mu_{t,m} \\
\frac{e_1}{e_3} &\approx \mu_{t,m}^2; \quad \frac{e_2}{e_4} \approx \mu_{t,m}^2; \quad \cdots; \quad \frac{e_{t-1}}{e_{t+1}} \approx \mu_{t,m}^2 \\
\cdots &
\end{align*}
\]

\[
\frac{e_1}{e_{t+1}} \approx \mu_{t,m}^t
\]  

(4.22)

where \( e_1, e_2, e_3, \cdots, e_t, e_{t+1} \) are the first row elements of the code matrix \( E \). We also obtain similar type of relations among the elements of the second row, third row, \( \cdots, (t+1) \)th row of the code matrix \( E \) where \( \mu_{t,m} \) is golden \((t,m)\)-proportion.

4.11 Error detection and correction

For the simplest case \( p = 1 \) the correct ability of the method is 93.33\% [50] which exceeds the essentially all well known correcting codes. The correct ability of the method for \( p = 2 \) is 99.80\% [2,6]. In general, for \( p = t \) and \( n > p + 1 = t + 1 \) the correct ability of the method is \( \frac{2(t+1)^2 - 2}{2(t+1)^2 - 1} \) which depends on \( p \) but not on \( m \).

Hence, for large value of \( p \) the correct ability of the method is \( \frac{2(p+1)^2 - 2}{2(p+1)^2 - 1} \approx 1 = 100\% \).