CHAPTER 5

SECONDARY k-GENERALIZED INVERSES OF s-k-NORMAL MATRICES

A generalization of the inverse of some special type of non singular matrices is described. The concept of s-k-generalized inverse of a given matrix is introduced and its characterizations are obtained. The existence and uniqueness of secondary k-generalized inverse of a given matrix are derived. s-k-hermitian idempotent matrices are defined. The condition for s-k-normal matrices to be diagonal is investigated.
5.1. SECONDARY k-GENERALIZED INVERSES OF MATRICES

In this section, we describe a generalization of the inverse of some special type of non-singular square matrices as the unique solution of a certain set of equations. The concept of secondary k-generalized inverse of a given square matrix is introduced. The existence of secondary k-generalized inverse of a given matrix is also proved (cf.[42]).

Throughout this chapter, if \( A \in C_{n\times n} \), then we assume that if \( A \neq 0 \) then \( A(KVA^*VK) \neq 0 \)

i.e., \( A(KVA^*VK) = 0 \) \( \Rightarrow A = 0 \) \hspace{1cm} (5.1.1)

It is clear that the conjugate secondary k-transpose satisfies the following properties.

\[
KV(A + B)^*VK = (KVA^*VK) + (KVB^*VK)
\]

\[
KV(\lambda A)^*VK = \overline{\lambda}(KVA^*VK)
\]

\[
KV(BA)^*VK = (KVA^*VK)(KVB^*VK)
\]

Now if \( BA(KVA^*VK) = CA(KVA^*VK) \), then by (5.1.1)

\[
BA(KVA^*VK) - CA(KVA^*VK) = 0
\]

\( \Rightarrow (BA(KVA^*VK) - CA(KVA^*VK))(KV(B - C)^*VK) = 0 \)

\( \Rightarrow (BA - CA)(KV(BA - CA)^*VK) = 0 \)

\( \Rightarrow (BA - CA) = 0 \)

\( \Rightarrow BA = CA \)

Therefore \( BA(KVA^*VK) = CA(KVA^*VK) \Rightarrow BA = CA \) \hspace{1cm} (5.1.2)

Similarly, \( B(KVA^*VK)A = C(KVA^*VK)A \Rightarrow B(KVA^*VK) = C(KVA^*VK) \) \hspace{1cm} (5.1.3)
Theorem 5.1.4

For any \( A \in C_{n \times n} \), the four equations

\[
AXA = A \quad (5.1.5)
\]

\[
XAX = X \quad (5.1.6)
\]

\[
KV(AX)^{\dagger}VK = AX \quad (5.1.7)
\]

\[
KV(XA)^{\dagger}VK =XA \quad (5.1.8)
\]

have a unique solution.

Proof

First, we shall show that equations (5.1.6) and (5.1.7) are equivalent to the single equation

\[
XKV(AX)^{\dagger}VK = X \quad (5.1.9)
\]

From equations (5.1.6) and (5.1.7), (5.1.9) follows, since it is merely (5.1.7) substituted in (5.1.6)

Conversely, equation (5.1.9) implies

\[
AXKV(AX)^{\dagger}VK = AX
\]

\[
AXAX = AX
\]

Since the left hand side is s-k-hermitian, (5.1.7) follows. By substituting (5.1.7) in (5.1.9), we get \( XAX = X \) which is actually (5.1.6). Therefore (5.1.6) and (5.1.8) are equivalent to (5.1.9). Similarly, (5.1.5) and (5.1.8) are equivalent to the equation

\[
XA(KVA^{\dagger}VK) = KVA^{\dagger}VK \quad (5.1.10)
\]

Thus to find a solution for the given set of equations, it is enough to find an \( X \) satisfying (5.1.9) and (5.1.10). Now the expressions \((KVA^{\dagger}VK)A, (KVA^{\dagger}VK)A^2, (KVA^{\dagger}VK)A^3\ldots\) cannot all be linearly independent (i.e., there exists a relation
\[ \lambda_1((KVA^\dagger VK)A) + \lambda_2((KVA^\dagger VK)A)^2 + \ldots + \lambda_m((KVA^\dagger VK)A)^m = 0 \quad (5.1.11) \]

Where \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are not all zero. Let \( \lambda_r \) be the first non zero \( \lambda \).

(i.e) \( \lambda_1 = \lambda_2 = \ldots \lambda_{r-1} = 0 \).

Therefore (5.1.11) implies that

\[ \lambda_r((KVA^\dagger VK)A)^r = -\left\{ \lambda_{r-1}((KVA^\dagger VK)A)^{r-1} + \ldots + \lambda_m((KVA^\dagger VK)A)^m \right\} \]

If we take

\[ B = -\lambda_r^{-1}\left\{ \lambda_{r+1}I + \lambda_{r+2}((KVA^\dagger VK)A) + \ldots + \lambda_m((KVA^\dagger VK)A)^m \right\} \]

Then

\[ B((KVA^\dagger VK)A)^{r+1} = -\lambda_r^{-1}\left\{ \lambda_{r+1}((KVA^\dagger VK)A)^{r+1} + \ldots + \lambda_m((KVA^\dagger VK)A)^m \right\} \]

\[ B((KVA^\dagger VK)A)^{r+1} = (KVA^\dagger VK)A^r \]

By using (5.1.2) and (5.1.3) this becomes, \( B((KVA^\dagger VK)A)^r = ((KVA^\dagger VK)A)^{r-1} \).

Applying (5.1.2) and (5.1.3) repeatedly, we get

\[ B(KVA^\dagger VK)A(KVA^\dagger VK) = KVA^\dagger VK \quad (5.1.12) \]

Now if we take \( X = B(KVA^\dagger VK) \) then (5.1.12) implies that this \( X \) satisfies (5.1.10) implies (5.1.8), we have

\[ (KV(XA)^{\dagger VK})(KVA^\dagger VK) = KVA^\dagger VK \]

\[ \Rightarrow B(KV(XA)^{\dagger VK})(KVA^\dagger VK) = B(KVA^\dagger VK) \]

Therefore \( X = B(KVA^\dagger VK) \) satisfies (5.1.9).

Thus \( X = B(KVA^\dagger VK) \) is a solution for the given set of equations.
Now let us prove that this $X$ is unique. Suppose that $X$ and $Y$ satisfy \((5.1.9)\) and \((5.1.10)\). Then by substituting \((5.1.8)\) in \((5.1.6)\) and \((5.1.7)\) in \((5.1.5)\), we obtain

\[(KV(XA)^{T}VK)X = X \quad \text{and} \quad (KV(AX)^{T}VK)A = A\]

Also, $Y = (KV(YA)^{T}VK)Y$ and

\[KVA^{T}VK = (KVA^{T}VK)AY\]

Now $X = X(KVX^{T}VK)(KVA^{T}VK)$

\[= X(KVX^{T}VK)(KVA^{T}VK)AY\]

\[= X(KV(AX)^{T}VK)AY\]

\[= XAY\]

\[=XA(KV(YA)^{T}VK)Y\]

\[= XA(KVA^{T}VK)(KVV^{T}VK)Y\]

\[= (KVA^{T}VK)(KVV^{T}VK)Y\]

\[= (KV(YA)^{T}VK)Y\]

\[X = Y\]

Therefore $X$ is unique.

Hence the theorem.

**Definition 5.1.13**

Let $A \in C_{n \times n}$. The unique solution of \((5.1.5)\), \((5.1.6)\), \((5.1.7)\) and \((5.1.8)\) is called secondary k-generalized inverse (s-k-g-inverse) of $A$ and is written as $A^{+sk}$. 

95
Example 5.1.14

If \( A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \) then \( A^{\dagger k} = \begin{pmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{pmatrix} \)

Remark 5.1.15

By using (5.1.8) in (5.1.6), (5.1.7) in (5.1.5) and from (5.1.9) and (5.1.10) we obtain

\[
A^{\dagger k} (KV(A^{\dagger k})^* VK)(KVA^* VK) = A^{\dagger k} = (KVA^* VK)(KV(A^{\dagger k})^* VK)A^{\dagger k} \\
A^{\dagger k} A(KVA^* VK) = (KVA^* VK) = (KVA^* VK)AA^{\dagger k}
\]  

(5.1.16)

If \( \lambda \) is a scalar, then \( \lambda^{\dagger k} \) means \( \lambda^{-1} \) when \( \lambda \neq 0 \) and zero when \( \lambda = 0 \).
5.2. SECONDARY-k-GENERALIZED INVERSES OF s-k-NORMAL MATRICES

In this section, characterizations of s-k-generalized inverse (s-k-g-inverse) of a matrix are obtained. s-k-g-inverse of s-k-normal matrices are discussed. s-k-herimitian matrices are defined and the condition for s-k-normal matrices to be diagonalable is investigated.

Lemma 5.2.1

For \( A \in C_{\text{non}} \),

(i) \( \left( A^{\dagger_{sk}} \right)^{\dagger_{sk}} = A \).

(ii) \( K V \left( A^{\dagger_{sk}} \right)^{\dagger_{sk}} V K = \left( K V \left( A^{\dagger_{sk}} \right)^{\dagger_{sk}} V K \right) \).

(iii) If \( A \) is non singular, then \( A^{\dagger_{sk}} = A^{-1} \).

(iv) \( \left( \lambda A \right)^{\dagger_{sk}} = A^{\dagger_{sk}} A^{\dagger_{sk}} \).

(v) \( \left( \left( K V A^{\dagger_{sk}} V K \right) A^{\dagger_{sk}} = A^{\dagger_{sk}} \left( K V \left( A^{\dagger_{sk}} \right)^{\dagger_{sk}} V K \right) \right) \).

Proof

Let \( A \in C_{\text{non}} \),

(i) By the definition of s-k-g-inverse, we have

\[ A^{\dagger_{sk}} A A^{\dagger_{sk}} = A^{\dagger_{sk}} \quad \text{and} \]

\[ A^{\dagger_{sk}} \left( A^{\dagger_{sk}} \right)^{\dagger_{sk}} A^{\dagger_{sk}} = A^{\dagger_{sk}} \]

These two equations imply that

\[ \left( A^{\dagger_{sk}} \right)^{\dagger_{sk}} = A \]
(ii) From the definition of $\uparrow_{sk}$, we have

$$AA^{\uparrow_{sk}}A = A$$

$$\Rightarrow (KVA^*VK)(KV(A^{\uparrow_{sk}})^*VK)(KVA^*VK) = KVA^*VK$$

Also  $$(KVA^*VK)(KV(A^*)^{\uparrow_{sk}}VK)(KVA^*VK) = KVA^*VK$$

From these two equations, we have

$$(KV(A^{\uparrow_{sk}})^*VK) = KV(A^*)^{\uparrow_{sk}}VK$$

(iii) Since $A$ is non-singular, $A^{-1}$ exists

Now $AA^{\uparrow_{sk}}A = A$ (By definition of $\uparrow_{sk}$)

Pre multiplying and post multiplying by $A^{-1}$, we have

$$A^{\uparrow_{sk}} = A^{-1}$$

(iv) The equations,

$$AA^{\uparrow_{sk}}A = A$$

$$(\hat{\lambda}A)(\hat{\lambda}A)^{\uparrow_{sk}}(\hat{\lambda}A) = (\hat{\lambda}A)$$ imply that

$$\hat{\lambda}(\hat{\lambda}A)^{\uparrow_{sk}} = A^{\uparrow_{sk}}$$

$$\Rightarrow (\hat{\lambda}A)^{\uparrow_{sk}} = \hat{\lambda}^{\uparrow_{sk}}A^{\uparrow_{sk}}, \text{ where } \hat{\lambda}^{\uparrow_{sk}} = \hat{\lambda}^{-1}$$

(v) From (5.1.16), we have

$$A^{\uparrow_{sk}}(KV(A^{\uparrow_{sk}})^*VK)(KVA^*VK) = A^{\uparrow_{sk}}$$

Also $ AA^{\uparrow_{sk}}A = A.$
Therefore \[ AA^\dagger_A (KV(A^\dagger_A)^\dagger VK)(KVA^\dagger VK) A = A \]

Substitute this in the right hand side of the defining relation, we get

\[ ((KVA^\dagger VK) A)^\dagger_A = A^\dagger_A (KV(A^\dagger)^\dagger VK) \]

**Theorem 5.2.2**

For \( A \in C_{n \times n} \),

(i) If \( U \) and \( W \) are s-k-unitary \( (UAW)^\dagger_A = (KVW^*VK) A^\dagger_A (KUU^*VK) \)

(ii) If \( A = \sum A_i \), where \( A_i (KVA^*_VK) = 0 \) and \( (KVA^*_VK) A_j = 0 \) whenever \( i \neq j \), then \( A^\dagger_A = \sum A_i^\dagger_A \).

**Proof**

Let \( A \in C_{n \times n} \),

(i) Since \( U \) and \( W \) are s-k-unitary

\[ U (KVU^*VK) = (KVU^*VK) U = I \]
and

\[ W (KVW^*VK) = (KVW^*VK) W = I. \]

Therefore \( AA^\dagger_A A = A \) implies that

\[ AW (KVW^*VK) A^\dagger_A (KVU^*VK) U A = A \] (5.2.3)

By the definition of s-k-g-inverse

\[ (UAW)(UAW)^\dagger_A (UAW) = UAW \] (5.2.4)

From the equations (5.2.3) and (5.2.4) we get

\[ (UAW)^\dagger_A = (KVW^*VK) A^\dagger_A (KVU^*VK) \]
Let $A = \sum A_i$ such that

$$A_i \left( KVA_i^* VK \right) = 0 \text{ and } \left( KVA_i^* VK \right) A_j = 0 \text{ if } i \neq j.$$

Therefore from equations (5.1.16), we have

$$A_i^{+a} = A_i^{+a} \left( KV \left( A_i^{+a} \right)^* VK \right) \left( KVA_i^* VK \right)$$

and

$$A_j^{+a} = \left( KVA_j^* VK \right) \left( KV \left( A_j^{+a} \right)^* VK \right) A_j^{+a}.$$

Hence by our assumption,

$$A_i A_j^{+a} = 0 \text{ and } A_j^{+a} A_i = 0 \text{ if } i \neq j \quad (5.2.5)$$

Since $A_i^{+a}$ satisfies the equation $AXA = A$ and $A_j^{+a}$ satisfies the equation $A_j X_j A_j = A$.

We get

$$\left( \sum A_i \right) \left( \sum A_i \right)^{+a} \left( \sum A_i \right) = \sum A_i$$

$$= \sum A_i A_i^{+a} A_i$$

$$= \left( \sum A_i \right) \left( \sum A_i^{+a} \right) \left( \sum A_i \right) \quad \text{by (5.2.5)}$$

$$\Rightarrow \left( \sum A_i \right)^{+a} = \sum A_i^{+a}$$

(ie,)

$$A^{+a} = \sum A_i^{+a}$$

Hence the theorem.

**Theorem 5.2.6**

If $A$ is s-k-normal then $A^{+a} A = AA^{+a}$ and $\left( A^n \right)^{+a} = \left( A^{+a} \right)^n$. 

100
Proof

Since $A$ is s-k-normal, \( A \left( KVA^*VK \right) = \left( KVA^*VK \right) A \).

Now \( A^\dagger_a A = A^\dagger_a \left( KV \left( A^\dagger_a \right)^* VK \right) \left( KVA^*VK \right) A \) \hspace{1cm} \text{by (5.1.16)}

\[ A^\dagger_a A = \left( \left( KVA^*VK \right) A \right)^\dagger_a \left( KVA^*VK \right) A \] \hspace{1cm} (5.2.7)

(By (v) of lemma 5.2.1)

Also, \( \left( A \left( KVA^*VK \right) \right)^\dagger_a \left( A \left( KVA^*VK \right) \right) = \left( KVA^*VK \right)^\dagger_a \left( KVA^*VK \right) \) \hspace{1cm} \text{By (5.1.2)}

\[ = \left( KV \left( A^* \right)^\dagger_a VK \right) \left( KVA^*VK \right) \]

\[ = KV \left( AA^\dagger_a \right)^* VK \]

\[ \text{(Since } KV \left( A^\dagger_a \right)^* VK = KV \left( A^* \right)^\dagger_a VK \text{)} \]

\[ \left( A \left( KVA^*VK \right) \right)^\dagger_a \left( A \left( KVA^*VK \right) \right) = AA^\dagger_a \] \hspace{1cm} (5.2.8)

(by (5.1.7))

Since $A$ is s-k-normal, equation (5.2.7) and (5.2.8) imply that $A^\dagger_a A = AA^\dagger_a$.

By the definition of s-k-g-inverse,

\[ A^n (A^n)^\dagger_a A^n = A^n \]

\[ = \left( AA^\dagger_a A \right)^n \]

\[ = A^n \left( A^\dagger_a \right)^n A^n \text{ which gives} \]

\[ \left( A^n \right)^\dagger_a = \left( A^\dagger_a \right)^n. \]

Hence the theorem

Theorem 5.2.9

A necessary and sufficient condition for the equation $AXB = D$ to have a solution is $AA^{\dagger_{sk}} DB^{\dagger_{sk}} B = D$, in which case the general solution is $X = A^{\dagger_{sk}} DB^{\dagger_{sk}} + Y - A^{\dagger_{sk}} AYBB^{\dagger_{sk}}$, where $Y$ is arbitrary.
Proof

Let us assume that $X$ satisfies the equation $AXB = D$. Then

$$D = AXB$$

$$= AA^\dagger_{sk} AXBB^\dagger_{sk} B$$

$$D = AA^\dagger_{sk} DB^\dagger_{sk} B$$  (By the definition of $\dagger_{sk}$)

Conversely, assume that $D = AA^\dagger_{sk} DB^\dagger_{sk} B$

If we take $X = A^\dagger_{sk} DB^\dagger_{sk}$, then it is a particular solution of $AXB = D$.

For $AXB = AA^\dagger_{sk} DB^\dagger_{sk} B = D$

Thus, $AXB = D$.

Now let us find the general solution

If $Y \in C_{non}$, then any expression of the form

$$X = A^\dagger_{sk} DB^\dagger_{sk} + Y - A^\dagger_{sk} AYBB^\dagger_{sk}$$

is a solution of $AXB = D$.

For $AXB = AA^\dagger_{sk} DB^\dagger_{sk} B + AYB - AA^\dagger_{sk} AYBB^\dagger_{sk} B$

$$= D + AYB - AYB$$  (By definition of $\dagger_{sk}$)

$$AXB = D$$

Conversely, if $X$ is a solution of $AXB = D$, then

$$X = A^\dagger_{sk} DB^\dagger_{sk} + X - A^\dagger_{sk} AXBB^\dagger_{sk}$$

satisfies $AXB = D$.

Hence the theorem

Corollary 5.2.10

The matrix equations $AX = B$ and $XD = E$ have a common solution if and only if each equation has a solution and $AE = BD$. 

102
Proof

If the equations \( AX = B \) and \( XD = E \) have a common solution, then each will individually have a solution.

Since \( AX = B \) and \( XD = E \), we have \( AXD = BD \) and \( AXD = AE \).

Therefore \( BD = AE \).

Conversely, assume that the equations \( AX = B \) and \( XD = E \) have a solution individually and that \( AE = BD \).

Since \( AX = B \) has a solution, by theorem 5.2.9,

\[
B = AA^\dagger_a B I^\dagger_a
\]

Therefore \( A^\dagger_a B I^\dagger_a \) is a solution of \( AXI = B \).

Since \( I^\dagger_a = I \), \( A^\dagger_a B \) is a solution of \( AX = B \).

Similarly \( ED^\dagger_a \) is a solution of \( XD = E \).

Therefore \( AA^\dagger_a B = B \) and

\[
ED^\dagger_a D = E .
\]

Also \( AE = BD \).

Now if we take \( X = A^\dagger_a B + ED^\dagger_a D - A^\dagger_a AED^\dagger_a \), then

\[
XD = A^\dagger_a BD + ED^\dagger_a D - A^\dagger_a AED^\dagger_a D
\]

\[
= A^\dagger_a BD + ED^\dagger_a D - A^\dagger_a BD
\]

(By our assumption \( AE = BD \) and definition of \( \dagger_{sk} \))

Therefore \( XD = E \), since \( ED^\dagger_a D = E \)

Similarly, it can be prove that \( AX = B \)

Thus \( AX = B \) and \( XD = E \) have \( X = A^\dagger_a B + ED^\dagger_a D - A^\dagger_a AED^\dagger_a \) as a common solution.

Hence the theorem.
Definition 5.2.11

A matrix $E \in C_{n \times n}$ is said to be s-k-hermitian idempotent matrix (s-k-h.i) if $E(KVE^*VK) = E$ (i.e) $E = KVE^*VK$ and $E^2 = E$. (s-k-hermitian idempotent)

Theorem 5.2.12

(i) $A^\dagger_{sk} A$, $AA^\dagger_{sk}$, $I - A^\dagger_{sk} A$, $I - AA^\dagger_{sk}$ are all the s-k-hermitian idempotent.

(ii) $J$ is idempotent $\iff$ there exist s-k-hermitian idempotent’s $E$ and $F$ such that $J = (FE)^{\dagger_{sk}}$ in which case $J = EJF$.

Proof

First we shall prove that $A^\dagger_{sk} A$ is s-k-hermitian idempotent.

$$
\left( A^\dagger_{sk} A \right) \left( K V \left( A^\dagger_{sk} A \right)^* V K \right) = A^\dagger_{sk} A \left( K V A^* \left( A^\dagger_{sk} \right)^* V K \right)
$$

$$
= A^\dagger_{sk} A \left( K V A^* V K \right) \left( K V \left( A^\dagger_{sk} \right)^* V K \right) \quad \text{(by definition of } \dagger_{sk} \text{)}
$$

$$
= A^\dagger_{sk} A
$$

Therefore $A^\dagger_{sk} A$ is s-k-hermitian idempotent.

Similarly $A^\dagger_{sk} A$ is s-k-hermitian idempotent.

Let us prove that $I - A^\dagger_{sk} A$ is s-k-hermitian idempotent.

$$
\left( I - A^\dagger_{sk} A \right) \left( K V \left( I - A^\dagger_{sk} A \right)^* V K \right) = \left( I - A^\dagger_{sk} A \right) \left( I - A^\dagger_{sk} A \right) \quad \text{(By definition of } \dagger_{sk} \text{)}
$$

$$
= I - 2 A^\dagger_{sk} A + A^\dagger_{sk} A \quad \text{(By definition of } \dagger_{sk} \text{)}
$$

$$
= I - A^\dagger_{sk} A .
$$

Therefore $I - A^\dagger_{sk} A$ is s-k-hermitian idempotent.

Similarly we can prove that $A^\dagger_{sk} A - I$ is s-k-hermitian idempotent.
(i) Since $J$ is idempotent, $J^2 = J$.

Now\[\left\{(J^\dagger_J)(JJ^\dagger_a)\right\}^\dagger_a = \left\{J^\dagger_a JJJ^\dagger_a\right\}^\dagger_a\quad \{\therefore J^2 = J\}\]

\[= \left\{J^\dagger_a JJJ^\dagger_a\right\}^\dagger_a\]

\[= \left\{J^\dagger_a\right\}^\dagger_a\quad \text{(By definition of $\dagger_{sk}$)}\]

\[\left\{(J^\dagger_a J)(JJ^\dagger_a)\right\}^\dagger_a = J\quad (5.2.13)\]

If we take $E = JJ^\dagger_a$ and $F = J^\dagger_a J$ then

\[KVE^*VK = KV\left(JJ^\dagger_a\right)^* VK\]

\[= JJ^\dagger_a\quad \text{(By definition of $\dagger_{sk}$)}\]

\[KVE^*VK = E\]

Also\[E^2 = EE\]

\[= JJ^\dagger_a JJ^\dagger_a\]

\[= JJ^\dagger_a\quad \text{(By definition of $\dagger_{sk}$)}\]

\[E^2 = E\]

Similarly, it can be seen that $KVF^*VK = F$ and $F^2 = F$.

Therefore $E$ and $F$ are s-k-hermitian idempotent.

Moreover from equation (5.2.13) we have $J = (FE)^\dagger_a$.

Conversely, suppose that there exist s-k-hermitian idempotent’s $E$ and $F$ such that $J = (FE)^\dagger_a$.

By equation (5.1.16) we have

\[\left(KVQ^*VK\right)\left(KV\left(Q^\dagger_a\right)^* VK\right)Q^\dagger_a\left(KV\left(Q^\dagger_a\right)^* VK\right)KVO^*VK = Q^\dagger_a\]

105
By using this fact, $J = (FE)^{\dagger}$ implies that $J = EFPEF$.

Where

$$P = KV \left( (FE)^{\dagger} \right)^* VK \left( FE \right)^{\dagger} KV \left( (FE)^{\dagger} \right)^* VK$$

Also

$$EJF = E^2FPEF^2$$

$$= EFPEF$$

$$EJF = J$$

Now let us prove that $J$ is idempotent

$$J^2 = JJ$$

$$J^2 = E (FE)^{\dagger} FE (FE)^{\dagger} F$$

(Since $K = (FE)^{\dagger}$ and $EJF = J$)

$$= E (FE)^{\dagger} F$$

(by definition of $\dagger_{sk}$)

$$J^2 = J$$

Therefore $J$ is idempotent.

Hence the theorem.

**Theorem 5.2.14**

(i) s-k-hermitian idempotent matrices are s-k-normal matrices.

(ii) The s-k-g inverse of an s-k-hermitian idempotent matrix is also s-k-hermitian idempotent matrix.

**Proof**

(i) s-k-hermitian idempotent matrices are s-k-hermitian matrices and hence s-k-normal matrices.

(ii) Let $E$ be an s-k-hermitian idempotent matrix.

$$KVE^*VK = E \text{ and } E^2 = E$$

Also by (i), $E$ is s-k-normal.
Now \( KV(E^t)^*VK = KV(E)^t VK \) \( \) (By (ii) of Lemma 5.2.1)

\[
KV(E^t)^*VK = E^t
\]

Moreover \( (E^t)^2 = (E^2)^t \) \( \) (By Theorem 5.2.6)

\[
(E^t)^2 = E^t
\]

Thus \( E^t \) is also an s-k-hermitian idempotent matrix.

**Definition 5.2.15**

For any square matrix \( A \) there exists a unique set of matrices \( J_\lambda \) defined for each complex number \( \lambda \) such that

\[
J_\lambda J_\mu = \delta_{\mu \lambda} J_\lambda \quad \text{(5.2.16)}
\]

\[
\sum J_\lambda = 1 \quad \text{(5.2.17)}
\]

\[
AJ_\lambda = J_\lambda A \quad \text{(5.2.18)}
\]

\[
(A - \lambda I) J_\lambda \quad \text{is nilpotent.} \quad \text{(5.2.19)}
\]

Then the non zero \( J_\lambda \)'s are called the principal idempotent elements of \( A \).

**Theorem 5.2.20**

If \( E_\lambda = 1 - \{(A - \lambda I)^n\}^{t_{1k}} (A - \lambda I)^n \) \( \) and

\[
F_\lambda = 1 - (A - \lambda I)^n \{(A - \lambda I)^n\}^{t_{1k}},
\]

where \( n \) is sufficiently large, then the principal idempotent elements of \( A \) are given by \( J_\lambda = \{F_\lambda E_\lambda\}^{t_{1k}} \) and \( n \) can be taken as unity if and only if \( A \) is diagnable.

107
Proof

Assume that $A$ is diagonalable.

Let

$$E_\lambda = 1 - \{(A - \lambda I)^{\dagger k} (A - \lambda I)\}, \text{ and}$$

$$F_\lambda = 1 - (A - \lambda I)\{(A - \lambda I)^{\dagger k}\}$$

Then by (i) of theorem 5.2.12, $E_\lambda$ and $F_\lambda$ are s-k-hermitian idempotent matrices.

If $\lambda$ is not an eigen value of $A$, then $|A - \lambda I| \neq 0$ and hence $F_\lambda$ and $E_\lambda$ are zero by (iii) of lemma 5.2.1.

Now by definitions of $E_\lambda$, $F_\lambda$ and $\dagger_{sk}$, we have

$$(A - \mu I)E_\mu = (A - \mu I) - (A - \mu I)(A - \mu I)^{\dagger k} (A - \mu I)$$

$$\Rightarrow \begin{cases} (A - \mu I)E_\mu = 0 \\ F_\lambda (A - \lambda I) = 0 \end{cases} \quad (5.2.21)$$

$$\Rightarrow \mu F_\lambda E_\mu = F_\lambda \mu E_\mu$$

$$= F_\lambda AE_\mu$$

$$= \lambda F_\lambda E_\mu$$

$$\Rightarrow F_\lambda E_\mu = 0 \text{ if } \lambda \neq \mu \quad (5.2.22)$$

Now if we take $J_\lambda = \{F_\lambda E_\lambda\}^{\dagger k}$, then by (ii) of theorem 5.2.12

$$J_\lambda = E_\lambda \{F_\lambda E_\lambda\}^{\dagger k} F_\lambda \quad (5.2.23)$$

$$\Rightarrow J_\lambda J_\mu = 0 \text{ if } \lambda \neq \mu \quad \text{(By (5.2.22))}$$

Also $J_\lambda J_\mu = J_\lambda$ if $\lambda = \mu \quad \text{(By 5.2.22 and definition of } \dagger_{sk})$
Thus \( J_{\lambda} J_{\mu} = \delta_{\lambda\mu} J_{\lambda} \)

From equation (5.2.22) we have

\[
F_{\lambda} J_{\mu} E_{\gamma} = \delta_{\lambda\mu} \delta_{\mu\gamma} F_{\lambda} E_{\lambda} \tag{5.2.24}
\]

If \( Z_{\alpha} \) is an eigen vector of \( A \) corresponding to the eigen value \( \alpha \), then

\[
E_{\alpha} Z_{\alpha} = Z_{\alpha} - (A - \alpha I)^{\dagger}k (A - \alpha I)Z_{\alpha}
\]

\[
= Z_{\alpha} - (A - \alpha I)^{\dagger}k (\alpha Z_{\alpha} - \alpha Z_{\alpha}) \quad \text{(Since } AZ_{\alpha} = \alpha Z_{\alpha})
\]

\[
E_{\alpha} Z_{\alpha} = Z_{\alpha}.
\]

Since \( A \) is diagonal, any column vector \( X \) conformable with \( A \) is expressible as a sum of eigen vectors (i.e) it is expressible in the form \( X = \sum E_{\lambda} X_{\lambda} \). This is a finite sum over all complex \( \lambda \). (Because if \( \lambda \) is not an eigen value of \( A \), then \( E_{\alpha} = 0 \)).

Similarly, if \( Y^* \) is conformable with \( A \), it is expressible as

\[
Y^* = \sum Y^* F_{\lambda}
\]

Now by equations (5.2.22) and (5.2.24)

\[
Y^* (\sum J_{\mu}) X = (\sum Y^* F_{\lambda}) (\sum J_{\mu}) (\sum E_{\gamma} X_{\lambda})
\]

\[
= \sum Y^* F_{\lambda} E_{\lambda} X_{\lambda}
\]

\[
= (\sum Y^* F_{\lambda}) (\sum E_{\gamma} X_{\gamma})
\]

\[
Y^* (\sum J_{\mu}) X = Y^* X
\]

\[
\Rightarrow \quad Y^* (\sum J_{\mu} - I) X = 0
\]

\[
\Rightarrow \quad \sum J_{\mu} = I
\]

Equations (5.2.21) implies that \( AE_{\lambda} = \lambda E_{\lambda} \)
Thus
\[ AJ_\lambda = \mathcal{A} E_\lambda (F_\lambda E_\lambda)^{1:n} F_\lambda \]
\[ AJ_\lambda = \lambda E_\lambda (F_\lambda E_\lambda)^{1:n} F_\lambda \]
\[ AJ_\lambda = \lambda J_\lambda \]
Similarly
\[ J_\lambda A = \lambda J_\lambda \]
Therefore
\[ AJ_\lambda = \lambda J_\lambda = J_\lambda A \tag{5.2.25} \]
This implies \((A - \lambda I)J_\lambda\) is nilpotent.

Thus (5.2.18) and (5.2.19) are satisfied.

Moreover \[ A = \sum \lambda J_\lambda \tag{5.2.26} \]

Conversely, if \( \sum J_\lambda = I \) and A is not diagonable \((n=I)\), then \( X = \sum J_\lambda X \) gives X as a sum of eigen vectors of A, since (5.2.25) was derived without assuming the diagonability of A. If A is not diagonable. It seems more convenient simply to prove that for any set of \( J_\lambda \)’s satisfying (5.2.16), (5.2.17) (5.2.18) and (5.2.19) each \( J_\lambda = (F_\lambda E_\lambda)^{1:n} \)
where \( F_\lambda \) and \( E_\lambda \) are defined as in the theorem.

If the \( J_\lambda \)'s satisfy (5.2.16), (5.2.17), (5.2.18) and (5.2.19),
\[ \sum J_\lambda = I \]
and
\[ (A - \lambda I)^n J_\lambda = 0 = J_\lambda (A - \lambda I)^n \tag{5.2.27} \]
which comes by using the fact that \((A - \lambda I)J_\lambda\) is nilpotent, where n is sufficiently large.

From (5.2.27) and the definition of \( E_\lambda \) and \( F_\lambda \), we have
\[ E_\lambda F_\lambda = J_\lambda = J_\lambda F_\lambda \tag{5.2.28} \]
By using Euclid’s algorithm, there exist P and Q which are polynomials in A such that
\[ I = (A - \lambda I)^n P + Q(A - \mu I)^n \quad \text{if} \quad \lambda \neq \mu. \]
Now \( F_\lambda(A - \lambda I)^n = 0 \)

\[(A - \mu I)^n E_\mu = 0 \]

This implies \( F_\lambda E_\mu = 0 \) if \( \lambda \neq \mu \)

From (5.2.28),

\[ F_\lambda J_\mu = 0 = J_\lambda E_\mu \] if \( \lambda \neq \mu \)

Since \( \sum J_\mu = I \), we get

\[
\begin{align*}
F_\lambda J_\lambda &= F_\lambda \text{ and} \\
J_\lambda E_\lambda &= E_\lambda
\end{align*}
\]

(5.2.29)

By using (5.2.28) and (5.2.29), it is easy to see that

\[ \{F_\lambda E_\lambda\}^{1*} = J_\lambda. \]

**Theorem 5.2.30**

If \( A \) is s-k-normal, it is diagonal and its principal idempotent elements are s-k-hermitian.

**Proof**

Since \( A \) is s-k-normal, then \( (A - \lambda I) \) is s-k-normal. By using theorem (5.2.6) in the definition of \( E_\lambda \) and \( F_\lambda \) of theorem (5.2.20) we obtain

\[ E_\lambda = 1 - \{(A - \lambda I)^{1*} (A - \lambda I) \} \] and \[ F_\lambda = 1 - (A - \lambda I) \{(A - \lambda I)^{1*} \}. \]

Hence by theorem (5.2.20), \( A \) is diagonalable.

Now \( E_\lambda = F_\lambda \) \hspace{1cm} (By (5.2.25))

\[ \Rightarrow J_\lambda = E_\lambda \]

\[ \Rightarrow J_\lambda \text{ is s-k hermitian.} \]

Hence the theorem.
CONCLUSION

The notion of s-k-normal matrices reflects fact that it is a generalization of normal matrices. Using this concept we may apply these results to con-s-k-normal matrices, s-k-unitary matrices, con-s-k-unitary matrices, s-k-orthogonal matrices. The new generalized s-k-normal matrices concept can be applied to the above mentioned matrices.