CHAPTER 6
On Slightly $b^*$-Continuous Mappings

6.1 Introduction

In the last few years different types of weak and strong continuities have been introduced. Many topologists have worked in this direction. Jain (52) made his contribution by giving the notion of slightly continuous mappings. Many weak forms of slightly continuous mappings have been investigated. For instance slightly semi-continuous (62) slightly $\beta$- continuous (65), slightly $\gamma$- continuous (68), slightly $\omega$- continuous (70) and slightly $b$-continuous mappings (73) have been studied. In this chapter we investigated one weak forms of open sets namely $b^*$open sets. By means of these sets we studied some concepts, namely slightly $b^*$-continuous mappings, slightly $b^*$-closed graphs and $b^*$-compactness.

6.2 Slight $b^*$-Continuity

Definition 6.2.1: A set $B \subseteq R$ is defined as $b^*$- closed (30) if $\text{int}(\text{cl}(B)) \subseteq G$, whenever $B \subseteq G$ and $G \in BO(R)$.

Definition 6.2.2: A mapping $\psi : R \to S$ is called slightly continuous (52) if $\forall$ set $B \in CO(R), \psi^{-1}(B) \in \tau$.

Definition 6.2.3: A mapping $\psi : R \to S$ is slightly $b^*$-continuous (103) at a point $x \in R$ if $\forall \ B \in CO(S, \psi(x) )$, $\exists A \in B^*O(R,x)$ with $\psi(A) \subseteq B$.

Definition 6.2.4: A mapping $\psi : R \to S$ is defined as:

(i) $b^*$-irresolute(103) if $\psi^{-1}(B) \in B^*O(R)$ $\forall$ $B \in B^*O(S)$

(ii) $b^*$-open(103) if $\psi(A) \in B^*O(S)$ $\forall A \in B^*O(R)$.

(iii) weakly $b^*$continuous (103) if $\forall x \in R$ and $\forall$ open subset $B$ of $S$ with $\psi(\psi(x)) \in B$, $\exists A \in B^*O(R,x)$ such that $\psi(A) \subseteq \text{cl}(B)$.

(iv) Contra $b^*$-continuous (103) if $\psi^{-1}(H) \in B^*O(R)$ $\forall$ closed set $H$ of $S$. 
**Theorem 6.2.1:** The below mentioned properties hold for a mapping $\psi : R \to S$:

1. $\psi$ is slightly $b^*$-continuous;
2. $\psi^{-1}(B) \in B^*O(R), \forall B \in CO(S)$;
3. $\psi^{-1}(B) \in B^*C(R), \forall B \in CO(S)$;
4. $\psi^{-1}(B) \in B^*O(R), \forall B \in \delta^*O(S)$;
5. $\psi^{-1}(B) \in B^*C(R), \forall B \in \delta^*C(S)$.

**Proof:**

1. $\Rightarrow (2)$: Suppose $B \in CO(S)$ with $x \in \psi^{-1}(B)$. As $\psi$ is slightly $b^*$-continuous, by (1) $\exists A \in B^*O(R, x)$ with $\psi(A) \subset B$; hence $A \subset \psi^{-1}(B)$. We have $\psi^{-1}(B) \supseteq \bigcup \{A \mid x \in \psi^{-1}(B)\}$ is $b^*$-open set. Therefore $\psi^{-1}(B) \in B^*O(R)$.

2. $\Rightarrow (3)$: Let $B \in CO(S)$, so $S - B \in CO(S)$

   By (2) $\psi^{-1}(S - B) = R - \psi^{-1}(B) \in B^*O(R)$ Therefore $\psi^{-1}(B) \in B^*C(R)$.

3. $\Rightarrow (4)$: Let $B \in CO(S)$. Then $S - B$ is clopen.

   Now $\psi^{-1}(S - B) = R - \psi^{-1}(B) \in B^*C(R)$. Also $\psi^{-1}(B) \in B^*C(R)$ Thus $\psi^{-1}(B) \subset B^*CO(R)$.

4. $\Rightarrow (5)$: Let $B \in \delta^*O(S)$ with $x \in \psi^{-1}(B)$. Therefore $\psi(x) \in B$. Since $B$ is $\delta^*$-open $\exists$, a set $F \in CO(S)$ with $\psi(x) \in F \subset B$ This implies that $x \in \psi^{-1}(F) \subset \psi^{-1}(B)$. By (4), $\psi^{-1}(F)$ is $b^*$-clopen. Hence $\psi^{-1}(B)$ is $b^*$-neighbourhood of each of its points. Consequently, $\psi^{-1}(B) \in B^*O(R)$.

5. $\Rightarrow (6)$: Let $B \in \delta^*C(S)$.

   Therefore $S - B \in \delta^*O(S)$.

So $\psi^{-1}(S - B)$ is $b^*$-open, therefore $R - \psi^{-1}(B) \in B^*O(R)$, and hence $\psi^{-1}(B) \in B^*C(R)$.

5. $\Rightarrow (1)$: Let $B \in CO(S)$.

   Therefore $B \in \delta^*O(S)$. 


So \( \psi^{-1}(B) \in B^*O(R) \). Further \( \psi^{-1}(B) \subseteq B \) Take \( \psi^{-1}(B) = A \) It follows that \( \psi(A) \subseteq B \), \( A \in B^*O(R) \).

Hence \( \psi \) is slightly \( b^* \)-continuous.

**Lemma 6.2.1:** If \( B \) and \( R_0 \) be subsets of \( R \). Let \( B \in B^*O(R) \) and \( R_0 \in \alpha \ O(R) \), then \( B \cap R_0 \in B^*O(R_0) \).

**Theorem 6.2.2:** Let \( \psi : R \to S \) be slightly \( b^* \)-continuous and \( B \in \alpha O(R) \), then \( \psi \big|_B : B \to S \) is slightly \( b^* \)-continuous.

**Proof:** Let \( W \in CO(S) \). Now \( (\psi \big|_B)^{-1}(W) = \psi^{-1}(W) \cap B \). Further \( \psi^{-1}(W) \in B^*O(R) \) and \( B \in \alpha O(R) \). So, \( \psi^{-1}(W) \cap B \in B^*O(B) \). Thus \( (\psi \big|_B)^{-1}(B) \) is \( b^* \)-open in the relative topology of \( B \). Hence \( \psi \big|_B \) is slightly \( b^* \)-continuous (by Lemma 6.2.1).

**Lemma 6.2.2:** Let \( B \subset R_0 \subset R \). If \( B \in B^*O(R_0) \) and \( R_0 \in \alpha O(R) \), then \( B \in B^*O(R) \).

**Theorem 6.2.3:** If \( \psi : R \to S \) is a mapping and \( x \in R \). And the restriction of \( \psi \) to \( B \) is slightly \( b^* \)-continuous at \( x \) where, \( B \in \alpha O(R, x) \), then \( \psi \) is slightly \( b^* \)-continuous at \( x \).

**Proof:** Suppose \( F \in CO(S, \psi(x)) \). Then, \( \exists A \in B^*O(B, x) \) with \( \psi(A) \subseteq F \). As \( A \in B^*O(B) \), \( B \in \alpha O(R) \) and \( A \subset B \subset R \). So by Lemma 6.2.2, \( A \in B^*O(R) \), from which it follows that \( \psi \) is slightly \( b^* \)-continuous at \( x \).

**Theorem 6.2.4:** Let \( \psi : R \to S \) be a mapping and \( \sigma = \{ A_i : i \in I \} \) be a cover of \( R \) by \( \alpha \)-open sets. Let \( \psi \big|_{A_i} \) be slightly \( b^* \)-continuous \( \forall i \in I \). Then \( \psi \) is slightly \( b^* \)-continuous mapping.

**Proof:** Let \( V \in CO(S) \). As \( \psi \big|_{A_i} \) is slightly \( b^* \)-continuous \( \forall i \in I \), \( (\psi \big|_{A_i})^{-1}(V) \in B^*O(A_i), \forall i \in I \). Now \( \psi^{-1}(V) = \cup_{i \in I} (\psi^{-1}(V) \cap A_i) = \cup_{i \in I} (\psi \big|_{A_i})^{-1}(V) \). Now \( A_i \in \alpha O(R) \forall i \in I \). It follows that \( \psi^{-1}(V) \in B^*O(R) \).

**Theorem 6.2.5:** Let \( \psi : R \to S \) and \( \eta : S \to T \) be mappings. The Composition \( \eta \circ \psi \) is slightly \( b^* \)-continuous if:
(1) $\psi$ is $b^*$-irresolute and $\eta$ is slightly $b^*$-continuous.

(2) If $\psi$ is slightly $b^*$-continuous and $\eta$ is slightly continuous.

**Proof:** (1) $U \in CO(T)$. By the slight $b^*$-continuity of $\eta$, $\eta^{-1}(U) \in B^*O(S)$. Since $\psi$ is $b^*$-irresolute, $\psi^{-1}(\eta^{-1}(U)) = (\eta \circ \psi)^{-1}(U)$ is $b^*$-open. Therefore, $\eta \circ \psi$ is slightly $b^*$-continuous.

(2) Let $U \in CO(T)$. By the slight continuity of $\eta$, $\eta^{-1}(U) \in CO(S)$. As $\psi$ is slightly $b^*$-continuous, $\psi^{-1}(\eta^{-1}(U)) = (\eta \circ \psi)^{-1}(U)$ is $b^*$-open. Therefore $\eta \circ \psi$ is slightly $b^*$-continuous.

**Theorem 6.2.6:** If $\psi : R \rightarrow S$ is $b^*$-open, $b^*$-irresolute surjection and $\eta : S \rightarrow T$ be any mapping. The composition $\eta \circ \psi$ is slightly $b^*$-continuous iff $\eta$ is slightly $b^*$-continuous.

**Proof:** Assume that $\eta$ is slightly $b^*$-continuous. Then $\eta \circ \psi$ is slightly $b^*$-continuous. Conversely, let $\eta \circ \psi$ be slightly $b^*$-continuous and $U \in CO(T)$. Then $(\eta \circ \psi)^{-1}(U)$ is $b^*$-open. As $\psi$ is $b^*$-open surjection, so $\psi((\eta \circ \psi)^{-1}(U)) = \eta^{-1}(U)$ is $b^*$-open in $S$. Hence $\eta$ is slightly $b^*$-continuous.

The relationship between weak $b^*$-continuity, contra $b^*$-continuity, slight continuity and slight $b^*$-continuity is:

$$b^*$-continuous $\Rightarrow$ weakly $b^*$-continuous

$$\downarrow$$

Contra $b^*$-continuous $\Rightarrow$ slightly $b^*$-continuous $\iff$ slightly continuous.

These implications need not be reversible as is clear from the following examples.

**Example 6.2.1:** The identity mapping on $(\mathbb{R}, u)$ is slightly $b^*$-continuous but not contra- $b^*$-continuous since the preimage of any singleton set is not $b^*$-open.

**Example 6.2.2:** let $R = \mathbb{R}$, $\tau = u$, $\sigma = \{S, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $S = \{a, b, c\}$.

$\psi : (R, u) \rightarrow (S, \sigma)$ be a mapping defined by
\[
\psi(x) = \begin{cases} 
  a, x \in Q & \text{is not weakly } b^*-\text{continuous but slightly } b^*-\text{continuous.} \\
  b, x \notin Q & 
\end{cases}
\]

Example 6.2.3: Let \( R = S = \mathbb{R} \), \( u = \) usual topology, \( \sigma = \) discrete topology, Let \( \psi : (\mathbb{R},u) \rightarrow (S,\sigma) \) be an identity mapping. The mapping \( \psi \) is slightly \( b^* \)-continuous, but not slightly continuous.

**Theorem 6.2.7:** Let \( \psi : R \rightarrow S \) be slightly \( b^* \)-continuous and \( S \) be extremally disconnected, then \( \psi \) is weakly \( b^* \)-continuous.

**Proof:** Suppose \( x \in R \) and \( B \) is an open subset of \( S \) with \( \psi(x) \in B \). As \( S \) is extremally disconnected so \( \text{cl}(B) \subseteq \text{CO}(S) \). As \( \psi \) is slightly \( b^* \)-continuous, \( \exists \) a set \( A \in B^{*\text{O}}(R, x) \) with \( \psi(A) \subseteq \text{cl}(B) \). It follows that \( \psi \) is weakly \( b^* \)-continuous.

**Theorem 6.2.8:** If \( \psi : R \rightarrow S \) is slightly \( b^* \)-continuous and \( S \) is locally indiscrete, then \( \psi \) is contra \( b^* \)-continuous and \( b^* \)-continuous.

**Proof:** Suppose \( U \subseteq S \) be open. As \( S \) is locally indiscrete, \( U \) is clopen and hence \( \psi^{-1}(U) \in B^{*\text{CO}}(R) \). Therefore \( \psi \) is contra-\( b^* \)-continuous and \( b^* \)-continuous.

**Theorem 6.2.9:** If \( \psi : R \rightarrow S \) is slightly \( b^* \)-continuous and \( S \) is 0-dimensional, then \( \psi \) is \( b^* \)-continuous.

**Proof:** Suppose \( x \in R \) and \( U \subseteq S \) be any open set and \( \psi(x) \in U \). Since \( S \) is 0-dimensional, \( \exists \) a set \( V \subseteq \text{CO}(S, \psi(x)) \) such that \( V \subseteq U \). Further \( \psi \) is slightly \( b^* \)-continuous, so \( \exists K \in B^{*\text{O}}(R, x) \) with \( \psi(K) \subseteq V \). Thus \( \psi(x) \in \psi(K) \subseteq U \). Hence \( \psi \) is \( b^* \)-continuous.

**Definition 6.2.5:** A set \( B \subseteq R \) is defined as \( b^{**} \)-closed if \( \forall x \in R - B, \exists V \in B^{*\text{CO}}(R, x) \) with \( V \cap B = \emptyset \).

**Definition 6.2.6:** A mapping \( \psi : R \rightarrow S \) is defined as set \( b^* \)-connected if whenever \( R \) is \( b^* \)-connected between \( A \) and \( B \), then \( \psi(R) \) is connected between \( \psi(A) \) and \( \psi(B) \) with respect to the relative topology on \( \psi(R) \).
6.3 Slightly b*-Closed Graph

**Definition 6.3.1:** For a mapping \( \psi : R \to S \) the graph \( G (\psi) (x) = \{(x, \, \psi(x)) \mid x \in R \} \) is defined as slightly b*-closed graph(104) if \( \forall (x, \, y) \in (R \times S) - G (\psi) \), \( \exists \, V \in B^*O(R, \, x) \) and a clopen set \( U \) containing \( y \) with \( (V \times U) \cap G (\psi) = \phi \).

**Lemma 6.3.1:** A mapping \( \psi : R \to S \) has slightly b*-closed graph iff \( \forall \, x \in R \) and \( y \in S \) such that \( \psi(x) \neq y \), \( \exists \, V \in B^*O(R, \, x) \) and \( U \in CO(S, \, y) \) such that \( \psi \, (V) \cap U = \phi \).

**Proof:** It can be obtained immediately from Definition 6.3.1.

**Theorem 6.3.1:** If \( \psi : R \to S \) be a mapping, then \( G (\psi) \) is slightly b*-continuous iff \( \psi \) is slightly b*-continuous.

**Proof:** Let \( U \in CO(S) \), then \( R \times U \) is clopen in \( R \times S \). As \( G (\psi) \) is slightly b*-continuous, \( \psi^{-1}(U) = G (\psi) ^{-1}(R \times U) \in B^*O(R) \). Therefore, \( \psi \) is slightly b*-continuous. Conversely, let \( x \in R \) and \( H \in CO (R \times S, \, G (\psi) \, (x)) \). Then \( H \cap (\{x\} \times S) \) is clopen in \( \{x\} \times S \) containing \( G (\psi) \, (x) \). Further \( \{x\} \times S \) is homeomorphic to \( S \). Hence \( \{y \in S \mid (x, \, y) \in H \} \in CO(S) \). As \( \psi \) is slightly b*-continuous, \( \cup \{\psi^{-1}(y) \mid (x, \, y) \in H \} \in B^*O(R) \).

Moreover \( x \in \cup \{\psi^{-1}(y) \mid (x, \, y) \in H \} \subset G (\psi) ^{-1}(H) \). Therefore \( G (\psi) ^{-1}(H) \in B^*O(R) \). Thus \( G (\psi) \) is slightly b*-continuous.

**Theorem 6.3.2:** Let \( \psi : R \to S \) be slightly b*-continuous and \( S \) be Ultra Hausdorff, then \( G (\psi) \) is slightly b*-closed.

**Proof:** Assume that \( (x, \, y) \in (R \times S) - G (\psi) \). So \( y \neq \psi (x) \). As \( S \) is Ultra Hausdorff, \( \exists \) clopen sets \( K_1 \) and \( K_2 \) in \( S \), with \( y \in K_1, \, \psi (x) \in K_2 \) and \( K_1 \cap K_2 = \phi \). Further \( \psi \) is slightly b*-continuous, So \( \exists \, U \in B^*O(R) \) with \( \psi \, (U) \subset K_2 \) Therefore \( \psi \, (U) \cap K_1 = \phi \); hence \( G (\psi) \) is slightly b*-closed.

**Theorem 6.3.3:** Let \( \psi : R \to S \) be slightly b*-continuous and \( S \) be Ultra Hausdorff space. If the product of b*-open sets is b*-open, then
(1) The graph $G(\psi)$ of $\psi$ is $b^\ast$-closed in the product space $R \times S$.

(2) The set \{(x, y)|\psi(x) = \psi(y)\} is $b^{**}$-closed in the product space $R \times R$.

**Proof:** (1) Let $(x, y) \in (R \times S) - G(\psi)$. We have $\psi(x) \neq y$. As $S$ is Ultra Hausdorff, $\exists$ clopen sets $W$ and $V$ with $y \in W$ and $\psi(x) \in V$ and $W \cap V = \emptyset$. Further $\psi$ is slightly $b^\ast$-continuous. So, $\exists$ $b^\ast$-clopen set $U$ containing $x$ with $\psi(U) \subset V$. Thus $W \cap \psi(U) = \emptyset$ and hence $(U \times W) \cap G(\psi) = \emptyset$ and $U \times W$ is $b^\ast$-clopen set of $R \times S$. It follows that $G(\psi)$ is $b^\ast$-closed in the space $R \times S$.

(2) Set $B= \{(x, y)|\psi(x) = \psi(y)\}$. Let $(x, y) \notin B$ then $\psi(x) \neq \psi(y)$. As $S$ is Ultra Hausdorff, $\exists U_1 \in CO(S, \psi(x))$ and $U_2 \in CO(S, \psi(y))$, with $U_1 \cap U_2 = \emptyset$. As $\psi$ is slightly $b^\ast$-continuous, $\exists$ $b^\ast$-clopen sets $V_1, V_2$ of $R$ with $x \in V_1$, $\psi(V_1) \subset U_1$ and $y \in V_2$, $\psi(V_2) \subset U_2$; hence $\psi(V_1) \cap \psi(V_2) = \emptyset$. Therefore $(x, y) \in V_1 \times V_2$ and $(V_1 \times V_2) \cap B = \emptyset$. Moreover $V_1 \times V_2$ is $b^\ast$-clopen in $R \times R$ containing $(x, y)$. It follows that $B$ is $b^{**}$-closed in the product space $R \times R$.

### 6.4 $b^\ast$-Connectedness

**Definition 6.4.1:** A space $R$ is defined as $b^\ast$-connected (104) between subsets $A$ and $B$ provided there is no $b^\ast$-clopen set $H$ for which $A \subset H$ and $H \cap B = \emptyset$.

**Theorem 6.4.1:** If $\psi: R \rightarrow S$ is slightly $b^\ast$ onto continuous mapping, then $S$ is connected provided $R$ is $b^\ast$-connected.

**Proof:** Assume that $S$ is a disconnected space. Then $\exists$ nonempty disjoint open sets $V$ and $U$ with $S = V \cup U$. Thus, $V$ and $U$ are clopen sets in $S$. As $\psi$ is slightly $b^\ast$-continuous, $\psi^{-1}(V)$ and $\psi^{-1}(U)$ are $b^\ast$-clopen in $R$. Further, $\psi^{-1}(V) \cap \psi^{-1}(U) = \emptyset$ and $R = \psi^{-1}(V) \cup \psi^{-1}(U)$. As $\psi$ is surjective, $\psi^{-1}(V) \neq \emptyset$ and $\psi^{-1}(U) \neq \emptyset$. Thus, $R$ is not $b^\ast$-connected. Which is a contradiction, hence $S$ is connected.

**Theorem 6.4.2:** A mapping $\psi: R \rightarrow S$ is set $b^\ast$-connected iff $\psi^{-1}(H)$ is $B^\ast CO(R)$ for every clopen subset $H$ of $\psi(R)$ (with respect to the relative topology on $\psi(R)$).

**Proof:** Assume that $H \in CO(\psi(R))$ with respect to the relative topology on $\psi(R)$. Suppose that $\psi^{-1}(H) \notin B^\ast C(R, x)$. Then $\exists x \in R$ such that $\forall$ $b^\ast$-open set $V$
with \( x \in V \), we have \( V \cap \psi^{-1}(H) \neq \emptyset \). We claim that the space \( R \) is \( b^* \)-connected between \( x \) and \( \psi^{-1}(H) \). Suppose \( \exists b^* \)-clopen \( B \) such that \( \psi^{-1}(H) \subset B \) and \( x \notin B \). Then \( x \in R - B \subset R - \psi^{-1}(H) \) and clearly \( R - B \in B^*O(R, x) \), \( (R - B) \cap \psi^{-1}(H) = \emptyset \), this contradiction implies that \( R \) is \( b^* \)-connected between \( x \) and \( \psi^{-1}(H) \). Since \( \psi \) is set \( b^* \)-connected, \( \psi (R) \) is connected between \( \psi (x) \) and \( \psi (\psi^{-1}(H)) \). But \( \psi (\psi^{-1}(H)) \subset H \) which is clopen in \( \psi (R) \) and \( \psi (x) \notin H \). This is a contradiction. Hence \( \psi^{-1}(H) \) is \( b^* \)-closed in \( R \) and similarly it can be shown that \( \psi^{-1}(H) \) is \( b^* \)-open.

Next assume that \( R \) is \( b^* \)-connected between \( A \) and \( B \), but \( \psi (R) \) is not connected between \( \psi (A) \) and \( \psi (B) \) (in the relative topology on \( \psi (R) \)). So there is a set \( H \subset \psi (R) \) that is clopen in the relative topology on \( \psi (R) \) such that \( \psi (A) \subset H \) and \( H \cap \psi (B) = \emptyset \). Then \( A \subset \psi^{-1}(H), B \cap \psi^{-1}(H) = \emptyset \) and \( \psi^{-1}(H) \) is \( b^* \)-clopen, it follows that \( R \) is not \( b^* \)-connected between \( A \) and \( B \). Hence \( \psi \) is set \( b^* \)-connected.

**Corollary 6.4.1:** Every slightly \( b^* \)-continuous surjection is set \( b^* \)-connected.

**Proof:** Assume that \( H \) is a clopen subset of \( S \). Suppose that \( \psi^{-1}(H) \) is not \( b^* \)-closed in \( R \). Then \( \exists x \in R - \psi^{-1}(H) \) such that \( \forall b^* \)-open set \( V \) with \( x \in V \) we have \( V \cap \psi^{-1}(H) = \emptyset \). We claim that the space \( R \) is \( b^* \)-connected between \( x \) and \( \psi^{-1}(H) \). Suppose \( \exists b^* \)-clopen \( B \) such that \( \psi^{-1}(H) \subset B \) and \( x \notin B \). Then \( x \in R - B \subset R - \psi^{-1}(H) \) and clearly \( R - B \in B^*O(R, x), (R - B) \cap \psi^{-1}(H) = \emptyset \), this contradiction implies that \( R \) is \( b^* \)-connected between \( x \) and \( \psi^{-1}(H) \). Since \( \psi \) is set \( b^* \)-connected, \( S \) is connected between \( \psi (x) \) and \( \psi (\psi^{-1}(H)) \). But \( \psi (\psi^{-1}(H)) \subset H \) which is clopen in \( S \) and \( \psi (x) \notin H \), which is a contradiction. Therefore \( \psi^{-1}(H) \) is \( b^* \)-closed in \( R \) and similarly it can be shown that \( \psi^{-1}(H) \) is \( b^* \)-open.

Next assume that \( R \) is \( b^* \)-connected between \( A \) and \( B \) but \( S \) is not connected between \( \psi (A) \) and \( \psi (B) \). Thus there is a set \( H \subset S \) that is clopen in \( S \) with \( \psi (A) \subset H \) and \( H \cap \psi (B) = \emptyset \). Then \( A \subset \psi^{-1}(H), B \cap \psi^{-1}(H) = \emptyset \) and \( \psi^{-1}(H) \) is \( b^* \)-clopen, it follows that \( R \) is not \( b^* \)-connected between \( A \) and \( B \). Hence \( \psi \) is set \( b^* \)-connected.
Theorem 6.4.3: Every set $b^*$-connected mapping is slightly $b^*$-continuous.

Proof: Suppose that $\psi : R \to S$ is set $b^*$-connected. Let $H \in CO(S)$. Then $H \cap \psi'(R)$ is clopen in the relative topology on $\psi'(R)$. As $\psi$ is set $b^*$-connected, by Theorem 6.4.2, $\psi^{-1}(H) = \psi^{-1}(H \cap \psi'(R)) \in B^*CO(R)$. Hence $\psi$ is slightly $b^*$-continuous.

Corollary 6.4.2: A surjective mapping is slightly $b^*$-continuous iff it is set $b^*$-connected.

6.5 Compactness

Theorem 6.5.1: Let $\psi : R \to S$ be slightly $b^*$-continuous and $H$ be $b^*$-compact relative to $R$, then $\psi(H)$ is $cl$-compact relative to $S$.

Proof: Let $\{H_\beta | \beta \in \Delta\}$ be any covering of $\psi(H)$ by clopen sets of $S$. As $\psi$ is slightly $b^*$-continuous $\{\psi^{-1}H_\beta | \beta \in \Delta\}$ is a covering of $H$ by $b^*$-open set of $R$. Further $H$ is $b^*$-compact relative to $R$, So $\exists$ a finite subset $\Delta_0$ of $\Delta$ such that $H \subseteq \cup \{\psi^{-1}(H_\beta) | \beta \in \Delta_0\}$. Thus $\psi(H) \subseteq \cup \{H_\beta | \beta \in \Delta_0\}$. It follows that $\psi(H)$ is $cl$-compact relative to $S$.

Corollary 6.5.1: Image of $b^*$-compact space under slightly $b^*$-continuous surjection is $cl$-compact

Proof: Let $\{H_\alpha | \alpha \in \Delta\}$ be any covering of $S$ by clopen sets. As $\psi$ is slightly $b^*$-continuous, By Theorem 6.2.1 $\{\psi^{-1}H_\alpha | \alpha \in \Delta\}$ is a cover of $R$ by $b^*$-open sets of $R$. As $R$ is $b^*$-compact relative to $R$, $\exists$ a finite subset $\Delta_0$ of $\Delta$ such that $R \subseteq \cup \{\psi^{-1}(H\alpha) | \alpha \in \Delta_0\}$. Thus $S \subseteq \cup \{H\alpha | \alpha \in \Delta_0\}$. This shows that $S$ is $cl$-compact.

Theorem 6.5.2: Let $\psi : R \to S$ Shave a slightly $b^*$-closed graph, Then $\psi(H)$ is $\delta^*$-closed in $S, \forall b^*$-compact subset $H$ relative to $R$.

Proof: Assume that $y \notin \psi(H)$ Then $(x, y) \in ((R \times S) - G(\psi)) \forall x \in H$. As $G(\psi)$ is slightly $b^*$-closed graph, $\exists V_x \in B^*(R, x)$ and a clopen set $U_x$ of $S$ with $\psi(V_x) \cap U_x = \phi$. The family $\{V_x | x \in R\}$ is a covering of $H$ by $b^*$-open sets of $R$. As $H$ is $b^*$-compact, $\exists$ a finite subset $H_0$ of $H$ such that $H \subseteq \bigcup \{V_x | x \in H_0\}$.
The space $A \times B$ have a slightly,

Theorem 6.6.3:

Let $\psi : R \rightarrow S$ have a slightly $b^\ast$-closed graph $G (\psi )$. The space $R$ is $b^\ast$-$T_1$ provided $\psi$ is injective.

Proof: Suppose $x \neq y \in R$. So $(x, \psi (y)) \in (R \times S) - G (\psi )$. As $G (\psi )$ is slightly $b^\ast$-closed graph, $\exists V \in B^\ast O (R, x)$ and $U \in CO(S, \psi (y))$ such that $\psi (V) \cap U = \phi$. Therefore $V \cap \psi ^{-1}(U) = \phi$. It follows that $S$ is $b^\ast$-$T_1$.

Theorem 6.6.2: If $\psi : R \rightarrow S$ have a slightly $b^\ast$-closed graph $G (\psi )$. The space $S$ is $b^\ast$-$T_2$ provided $\psi$ is surjective and $b^\ast$-open.

Proof: Suppose $y_1 \neq y_2 \in S$. As $\psi$ is surjection, $\psi (x) = y_1$ for some $x \in R$ and $(x, y_2) \in ((R \times S) - G(\psi ))$. As $\psi$ has the slightly $b^\ast$-closed graph, $\exists V \in B^\ast O(R, x)$ and a clopen set $U \in CO(S, y_2)$ with $((V \times U) \cap G(\psi )) = \phi$. So $\psi (V) \cap U = \phi$. As $\psi$ is $b^\ast$-open, $\psi (V)$ is $b^\ast$-open with $\psi (x) = y_1 \in \psi (V)$. Hence $S$ is $b^\ast$-$T_2$.

Theorem 6.6.3: If $\psi : R \rightarrow S$ is slightly $b^\ast$-continuous injection and $S$ is a 0-dimensional space. Then the below mentioned properties hold:
(1) $R$ is $b^*$-T$_2$, if $S$ is $T_2$.

(2) $R$ is $b^*$-regular, if $\psi$ is either open or closed.

(3) $R$ is $b^*$-normal, if $\psi$ is closed and $S$ is normal.

**Proof:** (1). Let $S$ be $T_2$ and $x_1 \neq x_2 \in R$ then $\psi (x_1) \neq \psi (x_2)$ as $\psi$ is injective. Also $S$ is $T_2$, so $\exists$ open sets $V_1$, $V_2$ in $S$ such that $\psi (x_1) \in V_1$, $\psi (x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$. As $S$ is a 0-dimensional space, $\exists U_1$, $U_2 \in CO(S)$ with $\psi (x_1) \in U_1 \subseteq V_1$ and $\psi (x_2) \in U_2 \subseteq V_2$.

Consequently $x_1 \in \psi^{-1}(U_1) \subseteq \psi^{-1}(V_1)$, $x_2 \in \psi^{-1}(U_2) \subseteq \psi^{-1}(V_2)$, and $\psi^{-1}(U_1) \cap \psi^{-1}(U_2) = \emptyset$. As $\psi$ is slightly $b^*$-continuous, $\psi^{-1}(U_1)$ and $\psi^{-1}(U_2)$ are $b^*$-open sets. It follows that $R$ is $b^*$-$T_2$.

(2) First assume that $\psi$ is open. Let $x \in R$ and $V$ be an open set such that $x \in V$. So $\psi (x) \in \psi (V)$ which is open in $S$, as $\psi$ is open. Further, $S$ is 0-dimensional, so $\exists U \in CO(S)$ with $\psi (x) \in U \subseteq \psi (V)$, Thus $x \in \psi^{-1}(U) \subseteq V$, as $\psi$ is injective. Since $\psi$ is slightly $b^*$-continuous, $\psi^{-1}(U)$ is an $b^*$-clopen set in $R$ and thus $x \in \psi^{-1}(U) = b^*cl (\psi^{-1}(U)) \subseteq V$. Hence $R$ is $b^*$-regular.

Now suppose $\psi$ is closed. Let $x \in R$ and $H$ be closed set of $R$ with $x \notin H$. It implies that, $\psi (x) \notin \psi (H)$ thus $\psi (x) \in S - \psi (H)$ which is an open set in $S$ as $\psi$ is closed. Now $S$ is 0-dimensional so $\exists$ a clopen set $U$ in $S$ such that $\psi (x) \in U \subseteq S - \psi (H)$. As $\psi$ is slightly $b^*$-continuous, we have $x \in \psi^{-1}(U) \in B^*O(R)$ and

$H \subseteq R - \psi^{-1}(U) \in B^*CO(R)$. Hence $R$ is $b^*$-regular.

(3) Let $H_1$ and $H_2$ be any two closed sets in $R$ with $H_1 \cap H_2 = \emptyset$. As $\psi$ is injective and closed, $\psi (H_1)$ and $\psi (H_2)$ are two closed sets in $S$ with $\psi (H_1) \cap \psi (H_2) = \emptyset$. Further, $S$ is normal, so $\exists$ two open sets $U$ and $V$ in $S$ with $\psi (H_1) \subseteq U$, $\psi (H_2) \subseteq V$ and $U \cap V = \emptyset$. Let $y \in \psi (H_1)$, then $y \in U$. As $S$ is 0-dimensional and $U$ is open in $S$, $\exists$ a clopen set $U_y$ with $y \in U$ $y \in U$. Then $\psi (H_1) \subseteq \{ U_y | U_y \in CO(S), y \in \psi (H_1) \} \subseteq U$, thus $H_1 \subseteq \bigcup \{ \psi^{-1}(U_y) | U_y \in CO(S), y \in$
$\psi(H_1) \subseteq \psi^{-1}(U)$. As $\psi$ is slightly $b^*$-continuous, $\psi^{-1}(U_y)$ is $b^*$-open $\forall U_y \in CO(S)$.

Let $K = \bigcup \{ \psi^{-1}(U_y) \mid y \in H_1 \}$ then $K$ is $b^*$-open in $R$ and $H_1 \subseteq K \subseteq \psi^{-1}(U)$. Similarly, $\exists$ a $b^*$-open set $F$ in $R$ with $H_2 \subseteq F \subseteq \psi^{-1}(V)$ and $K \cap F \subseteq \psi^{-1}(U \cap V) = \emptyset$. Hence $R$ is $b^*$-normal.