Chapter 4

Sawtooth sequences and

a subsequence extrapolation scheme
4.1 Some divergent sawtooth sequences

In lieu of introduction, we begin this chapter by considering some examples. Let us consider the sequence

\[ 1, -1, 2, -2, \ldots . \]  \hspace{1cm} (4.1)

To evaluate its limit point \( S_0 \), we may also express (4.1) in the form of a series :

\[ S_0 \equiv 1 - 2 + 3 - 4 + \cdots . \]  \hspace{1cm} (4.2)

We now note the following :

(i) The EM provides an answer by means of the association

\[ S_0(x) = 1 - 2x + 3x^2 - 4x^3 + \cdots = \frac{1}{(1 + x)^2} , \]  \hspace{1cm} (4.3)

leading to \( S_0(1) = \frac{1}{4} \).

(ii) A straightforward application of DPA on (4.1) also gives \( S_0 = \frac{1}{4} \).

(iii) The CM gives first an oscillatory \( \{1/2, 0\} \) sequence; a subsequent averaging again leads to the answer \( S_0 = \frac{1}{4} \) (see Chapter 5 for discussion).

As a second example, we consider the sequence

\[ 1, -\frac{1}{2}, 2, -\frac{3}{2}, \ldots . \]  \hspace{1cm} (4.4)

Here also, one may write \( S_0 \) as a series

\[ S_0 = 1 - \frac{3}{2} + \frac{5}{2} - \frac{7}{2} + \cdots . \]  \hspace{1cm} (4.5)

One may now observe the various ways of obtaining an answer to \( S_0 \) as follows :

(i) Writing

\[ S_0(x) = 1 - \frac{3x}{2} + \frac{5x^2}{2} - \frac{7x^3}{2} + \cdots = \frac{2 + x + x^2}{2(1 + x)^2} , \]  \hspace{1cm} (4.6)

the EM yields \( S_0(1) = \frac{1}{2} \).

(ii) The series (4.5) is also DPA-summable to the value \( S_0 = \frac{1}{2} \).
(iii) By applying the CM on (4.4), one first obtains an oscillatory \{1/4, 3/4\} sequence which should be averaged finally. This gives \(S_0 = 1/2\) again (see again Chapter 5 for an exposition on this point).

The above two sequences are both divergent and sawtooth in character. We have found that a few standard methods offer a value for \(S_0\) in each case. But there is an important problem. Except for algebraic manipulations, it is difficult to relate the answer \(S_0\) to the PS \(\{S_j\}\). In other words, that \(1/4\) would be the answer for the sequence \((1, -1, 2, -2, \ldots)\) is, in no way, apparent from a physical viewpoint. Here, our purpose would be to provide a meaning to the answer\(^3\) for a divergent sawtooth sequence. Thus, the problem of interpretation of \(S_0\) in relation to the PS would precisely concern us.

4.2 A simple subsequence extrapolation

A direct way of finding \(S_0\) of a convergent sequence is to search for a \(S_j\)-vs.-\(j\) curve that fits the sequence properly. This is commonly used in case of monotonic sequences. One also obtains rough answers to \(S_0\) by simply extrapolating the curve to \(j \to \infty\). But, this does not seem to have been extended to cases of sawtooth sequences. We propose here an extrapolation scheme\(^4\) (ES) to deal with such sequences.

The basis of the ES lies in the observation that alternate members of a sawtooth sequence follow a monotonic trend. Thus, the two subsequences \((S_1, S_3, S_5, \ldots)\) and \((S_2, S_4, S_6, \ldots)\) may be separately extrapolated in a convergent case. Ideally, the two curves (viz. \(S_{2j}\)-vs.-\(j\) and \(S_{2j+1}\)-vs.-\(j\)) meet at \(S_0\) as \(j \to \infty\).

A divergent sawtooth sequence may also be handled in a similar way. In the ES, the sawtooth PS is first split into two subsequences. The two subsequences are then fitted by two straight lines in the simplest situation. Surprisingly, these two straight lines, when extrapolated, meet at a point \(j_0 \leq 1\) which leads to the limit point \(S_0\) of the PS. In ideal situations, when the subsequences fall exactly on the two straight lines, it gives the exact value of \(S_0\). The only difference with a convergent case is that, while the separately extrapolated lines meet at \(j = \infty\) in case of a convergent PS, here the two lines meet at some finite \(j\)-value, which we call \(j_0\).
As simple illustrations of the idea, we consider again the sequences (4.1) and (4.4). In case of (4.1), we have the following pair of straight lines:

\[
S_j = \frac{j}{2} + \frac{1}{2}, \quad j = \text{odd},
\]
\[
= -\frac{j}{2}, \quad j = \text{even}.
\]

Solving, we find

\[
S_0 = \frac{1}{4}.
\]

The result agrees with the answer obtained earlier by adopting entirely different schemes. For (4.4), one finds the pair

\[
S_j = \frac{j}{2} + \frac{1}{2}, \quad j = \text{odd},
\]
\[
= -\frac{j}{2} + \frac{1}{2}, \quad j = \text{even}.
\]

Here, solution for the intersection becomes

\[
S_0 = \frac{1}{2}.
\]

This again agrees with our previous estimate of \(S_0\). But, what is new in the present approach is a transparent physical (or, geometrical) meaning of \(S_0\) in relation to the members of the PS.

The present scheme has involved so far two straight lines, i.e. two one degree equations. So, we designate it by [1/1]ES. In more complicated situations, one may have to employ higher degree of equations for the fitting purpose. Thus, generally, we may define \([p/q]\)ES when the odd members of PS are given by a \(p\)th degree equation and the even ones by a \(q\)th degree.

Although it is apparent that the above scheme applies to a restricted class of sequences for which the two subsequences admit exact fitting by polynomials of finite degrees, importance of the ES lies chiefly in providing a feeling for \(S_0\) in case of divergence. Particularly, it is now transparent that while the limit point in a convergent case refers to the point where members gradually approach, in a divergent situation it implies the point wherefrom the PS actually emerges.
4.3 The notion of summability

The ultimate goal of any STT is to obtain the limit point of a sequence or, equivalently, the sum of the infinite series that corresponds to the sequence. Obviously, the reliability of any STT exclusively depends on whether it can exactly define the sum of a given series, at least in ideal cases. The failure of a scheme in dealing successfully with a given case indicates only its inefficiency in defining the ‘sum’ suitably. Thus, summability is a crucial test. Hence, we are now interested to know whether our scheme of obtaining $S_0$ of a divergent sawtooth sequence by employing ES is correct, so far as summability is concerned.

For brevity, we consider only [1/1]ES, i.e. linear extrapolations (LE). Figure 4.1 shows how our scheme works in providing $S_0$ for sequences (4.1) and (4.4). We may now assert that, divergent sawtooth sequences for which alternate members lie exactly on straight lines should thus be summable. In order to verify this, we compare the results of LE scheme for some arbitrary sequence with the values obtained by some other standard means. In case, agreement is not found or these standard methods fail in certain situations, we try to provide proper interpretation of the LE sum.

An example would now be appropriate. Consider the sequence

\[1, \frac{1}{4}, 2, \frac{1}{4}, \ldots\] \hspace{0.5cm} (4.11)

This is equivalent to the series

\[S_0 = 1 - \frac{3}{4} + \frac{7}{4} - \frac{7}{4} + \frac{11}{4} - \frac{11}{4} + \ldots\] \hspace{0.5cm} (4.12)

Let us note that this sequence does not readily conform to any standard summability scheme that we have considered previously. The LE sum is, however, easily obtained from Figure 4.2 by a simple backward extrapolation. The result is, $S_0 = 1/4$. The example thus provides an interesting case. The standard methods fail to sum (4.12). But, the LE scheme gives a finite answer. Now, the question is, how far this result is significant. To get the answer, here we take the help of Hardy's axioms. These may be stated as follows:

(a) if $\sum a_n = S$, $\sum a_n = aS$

(b) if $\sum a_n = S$ and $\sum b_n = T$, $\sum (a_n + b_n) = S + T$

(c) if $a_1 + a_2 + a_3 + \cdots = S$, $\alpha + a_1 + a_2 + a_3 + \cdots = \alpha + S$ and vice versa.
Figure 4.1 — Testing summability by LE: sequences (4.1) [AB–CD] and (4.4) [AB–EF]

Figure 4.2 — Employment of LE method to estimate limit points of sequence (4.11)
Let us suppose,

\[ \frac{7}{4} + 0 + \frac{11}{4} + 0 + \cdots = x \]  

(4.13)

Then, with the help of axiom (c), we can write

\[ 0 + \frac{7}{4} + 0 + \frac{11}{4} + 0 + \cdots = x \]

(4.14)

Subtracting (4.14) from (4.13) by (b), we get

\[ \frac{7}{4} - \frac{7}{4} + \frac{11}{4} - \frac{11}{4} + \cdots = 0 \]

(4.15)

Now if we introduce parentheses in the PS, viz. \( 1 - \frac{x}{2} + \left( \frac{7}{4} - \frac{7}{4} + \frac{11}{4} - \frac{11}{4} + \cdots \right) \), then by employing the axiom (c) and using (4.15), we obtain the value of PS to be \( (1 - \frac{x}{2}) = \frac{1}{2} \) which is in accordance with LE sum. In this way, the LE sum is interpretable. We then see that many divergent sequences, otherwise quite simple but not amenable to the standard methods, may be profitably handled through the LE. The strength and scope of LE may now be appreciated.

Some points regarding the LE summability may now be in order. (i) No convergent sequence will be LE summable exactly (though convergence may be accelerated) because, in that case, it would lead to some finite \( j_0 > 0 \), which is impossible. Actually, one should have \( j_0 = +\infty \). (ii) An oscillatory sequence is also not LE summable, for here the two straight lines would be parallel and hence cannot meet at some finite \( j_0 \). (iii) Selective choice in the case of divergent sequences cannot be made arbitrarily. Thus, we should not read our sequence as \( S_1, S_2, S_3, \ldots \). This is because, then, no unique answer can be obtained. The classical Riemann rearrangement theorem is essentially similar.

Now, we come to the final point of practical implementation of the LE scheme. In practice, the two subsequences obtained for a sawtooth sequence may not conform to \([p/q]\)ES. But, one may find an approximate \( p \)th degree polynomial to fit the members \( (S_1, S_3, S_5, \ldots) \) in a least squares sense. Similarly, another \( q \)th degree polynomial may be approximately assigned to the subsequence \( (S_2, S_4, \ldots) \). Their intersection may give us at least a rough estimate of \( S_0 \). We shall see later how far such a scheme is tenable and efficient. In such a situation, result will depend on the total number of members involved in calculations. By a \([p/q]\)ES\( (n) \), we mean \( n \) input data are employed, where \( n \geq p + q + 2 \). Specifically, a least square method is adopted when \( n \geq p + q + 4 \).
4.4 Sequential extrapolation – a further step

The aforesaid strategy, being the most simple version, may not give results with sufficient accuracy for the sequences in real situations. Hence, to improve further the quality of results, we propose here what we call a sequential extrapolation scheme (SES). In the simplest version of this SES, instead of taking all the members of the PS, we consider only 6 consecutive members (3 odd and 3 even), viz. $S_k$ to $S_{k+6}$ at a time, to construct two straight lines, $k$ being 1, 2, 3, .... It is remarkable that the sequence that emerges from these intersecting points again shows an alternating character to bracket the limit point $S_0$, but with tighter bounds than the PS. Thus, the sequences obtained from $(1, 6), (3, 8), (5, 10), \ldots$ and $(2, 7), (4, 9), (6, 11), \ldots$, approach monotonically towards $S_0$, but from opposite sides. This sequential scheme, which we designate by $[1/1] SES(6)$, may further be strengthened by invoking a standard STT. Here, important is the choice of the third point. Though two points are sufficient to draw each straight line, the role of the third point is to properly shape the slope of the line (by following a least squares procedure) with respect to the characteristics of the PS.

A further extension of $[1/1] SES$ can be made by choosing nonlinear extrapolations. In general, one can build a $[p/q] SES$ by taking $(p + q + 4)$ members at a time where $(p + 2)$ members refer to odd $\{S_j\}$ and $(q + 2)$ members correspond to the even $\{S_j\}$:

\[
S_j = \sum_{m=0}^{p} A_m j^m \quad (j = \text{odd})
\]

and

\[
S_j = \sum_{m=0}^{q} B_m j^m \quad (j = \text{even})
\]

(4.16)

But, such a higher order scheme generally involves considerable round off errors and also gives multivalued solutions for $(j_0, S_0)$. This we have checked. Thus, such nonlinear procedures cost a basic loss of simplicity. So, we do not follow the scheme unless the situation is otherwise a very complicated one. Particularly, in the context of lattice-sum sequences considered here, it may be seen that one need not opt for higher order extrapolations.
4.5 Some results

To employ our extrapolation scheme, we choose here the two basic three-dimensional cubic lattice-sums $S(I)$ and $S(III)$, considered also in Chapter 3. These are given by

\[
S(I) = \lim_{j \to \infty} S_j(I), \quad S_j(I) = \sum_{M,N,P=-j}^{j} \frac{(-1)^M}{\sqrt{M^2 + N^2 + P^2}}
\]

\[
S(III) = \lim_{j \to \infty} S_j(III), \quad S_j(III) = \sum_{M,N,P=-j}^{j} \frac{(-1)^{M+N+P}}{\sqrt{M^2 + N^2 + P^2}}
\]

and an auxiliary one $[S(II)]$, given by

\[
S(II) = \lim_{j \to \infty} S_j(II), \quad S_j(II) = \frac{3S_j(I)}{4} + \frac{S_j(III)}{4}
\]

The reason for choosing these two basic sequences $[S(I)$ and $S(III)]$ lies in the fact that the Madelung constants of a number of ionic crystals may be obtained from these sums by linear combinations. Towards the end of Chapter 3, this may be found. One more reason is, sufficiently good-quality data for such sequences have already been obtained and the efficacy of Padé-like methods tested. Hence, here we can also assess the efficiency of ES or SES relative to a few other standard methods.

The sequences of partial sums $\{S_j\}$ for $S(I)$ and $S(II)$ are divergent in nature, while the same for $S(III)$ is slowly convergent. However, for all the three sequences, the alternate $S_j$'s are almost linear in nature. So, we first employ the [1/1]ES($n$) on all the three sets with $n = 23$. The results, shown in Table 4.1, are quite encouraging in view of the extreme simplicity of the method. Here, a solution with $j_0 > 1$ refers to a convergent case while $j_0 < 1$ signifies divergence.

Estimates obtained by following the [1/1]ES($n$) scheme usually improve with $n$ in a gradual manner if the least squared deviations are comparable. This is evident from Table 4.2 for a sample case study of the CaCl LS. This observation is otherwise important. It signifies, in spite of the fact that $S_j$ increasingly departs from $S_0$ as $j \to \infty$, one obtains gradually 'improved' estimates with increasing $j$ in the sense that information of higher $S_j$ values helps us in evaluating $S_0$ to a better degree. Thus, higher-order terms of a divergent PS are actually more significant.
Table 4.1. Location and estimates of limit point by [1/1]ES(23).

<table>
<thead>
<tr>
<th>Set</th>
<th>$j_0$</th>
<th>$S(ES)$</th>
<th>$S_E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>-0.48</td>
<td>-0.772</td>
<td>-0.774</td>
</tr>
<tr>
<td>II</td>
<td>-0.50</td>
<td>-1.023</td>
<td>-1.018</td>
</tr>
<tr>
<td>III</td>
<td>20.63</td>
<td>-1.743</td>
<td>-1.748</td>
</tr>
</tbody>
</table>

Table 4.2. Variation of [1/1]ES(r) results with input information: the CsCl lattice [S(II)] case.

<table>
<thead>
<tr>
<th>$r$</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j_0$</td>
<td>-0.5116</td>
<td>-0.5063</td>
<td>-0.5043</td>
<td>-0.5033</td>
<td>-0.5027</td>
</tr>
<tr>
<td>$S(ES)$</td>
<td>-1.0440</td>
<td>-1.0317</td>
<td>-1.0273</td>
<td>-1.0250</td>
<td>-1.0236</td>
</tr>
</tbody>
</table>
With a view to improving the above estimates, one may employ \( [1/1] \text{SES}(6) \) and obtain two subsequences, which approach \( S_0 \). These results are shown in Table 4.3. The result \( S \) corresponds to the average of the last members of the two subsequences. However, a better approach is to employ a suitable STT on the two subsequences. For the present situation, we find MPA to be quite fitting. The limiting values \( S^L \) obtained from the two subsequences are shown. They are also subsequently averaged, leading to sufficiently accurate values for \( S_0 \), denoted by \( \bar{S}^L \). The estimates so obtained are comparable in quality with those obtained by a few other standard methods and literature values.

The object behind our inclination towards the MPA-assisted SES has been primarily to assess the potency and authenticity of the present extrapolation scheme. Otherwise, ES or SES might appear too simple to be dependable. Table 4.4 provides a strong support in this regard. The table shows the \( \mu \)-values of a number of crystals, calculated from the estimates for \( S(I), S(II) \) and \( S(III) \) obtained via \( [1/1] \text{ES}(23) \) [cf. Table 4.1] and also from \( [1/1] \text{SES}(6) \) [cf. Table 4.3]. Comparing with the exact result \( S_E \) or \( \mu_E \), one finds that the approximate values are quite reasonable and even \( [1/1] \text{ES}(n) \) scheme correctly predicts the trend, though it is essentially equivalent to a manual linear graphical extrapolation.

The success of SES is most remarkable in studying \( \mu \) of a two-dimensional square lattice\(^{49} \). It may be expressed as

\[
\mu = \lim_{j \to \infty} \sum_{-j}^{j} \frac{(-1)^{M+N}}{\sqrt{M^2 + N^2}}, \quad -j \leq M, N \leq j .
\]  

In terms of standard functions, one may write \( \mu \) as

\[
\mu = -4(1 - \sqrt{2}) \left( \left( \frac{1}{2} \right)^{\beta\left( \frac{1}{2} \right)} \right) = -1.615 \, 542 \, 626 \, 7 .
\]  

An alternative way is to express \( \mu \) in the form

\[
\mu = -4 \ln 2 + 4S(IV) ,
\]

where

\[
S(IV) = \lim_{j \to \infty} S_j(IV) ; \quad S_j(IV) = \sum_{M,N=1}^{j} \frac{(-1)^{M+N}}{\sqrt{M^2 + N^2}} .
\]

Here, the sequence obtained from \( S(IV) \) is a slowly convergent sawtooth sequence which does not conform to an easy Padé summability. This is evident from Table 4.5. Even IDPA also fails to improve the result.
Table 4.3. Madelung constants of some three-dimensional lattices from \([1/1]ES(23)\) and \([1/1]SES(6)\).

<table>
<thead>
<tr>
<th>System</th>
<th>(\mu(ES))</th>
<th>(\mu(SES))</th>
<th>(\mu_E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NaCl</td>
<td>3.486</td>
<td>3.495 129 19</td>
<td>3.495 129 19</td>
</tr>
<tr>
<td>ZnS</td>
<td>3.773</td>
<td>3.782 926 30</td>
<td>3.782 926 10</td>
</tr>
<tr>
<td>CaF(_2)</td>
<td>11.60</td>
<td>11.636 576 0</td>
<td>11.636 575 2</td>
</tr>
<tr>
<td>BiF(_3)</td>
<td>22.06</td>
<td>22.121 963 6</td>
<td>22.121 962 8</td>
</tr>
<tr>
<td>NbO</td>
<td>3.000</td>
<td>3.008 540 03</td>
<td>3.008 539 96</td>
</tr>
<tr>
<td>Li(_2)O</td>
<td>10.74</td>
<td>10.773 184 7</td>
<td>10.773 184 5</td>
</tr>
<tr>
<td>CsCl</td>
<td>2.030</td>
<td>2.035 361 71</td>
<td>2.035 361 51</td>
</tr>
<tr>
<td>CsCl(II)</td>
<td>2.045</td>
<td>2.035 361 47</td>
<td>2.035 361 51</td>
</tr>
</tbody>
</table>

Table 4.4 Performance of \([1/1]SES(6)\) for some three-dimensional sequences.

<table>
<thead>
<tr>
<th>Set</th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S(SES)</td>
<td>S(SES)</td>
<td>S(SES)</td>
</tr>
<tr>
<td></td>
<td>((2j - 1, 2j + 4))</td>
<td>((2j, 2j + 5))</td>
<td>((2j, 2j + 5))</td>
</tr>
<tr>
<td>(j)</td>
<td>(S_L)</td>
<td>(S)</td>
<td>(S_L)</td>
</tr>
<tr>
<td>1</td>
<td>-0.773</td>
<td>-0.786</td>
<td>-1.036 0</td>
</tr>
<tr>
<td>2</td>
<td>-0.768</td>
<td>-0.779</td>
<td>-1.018 0</td>
</tr>
<tr>
<td>3</td>
<td>-0.770 9</td>
<td>-0.777 0</td>
<td>-1.017 74</td>
</tr>
<tr>
<td>5</td>
<td>-0.773 0</td>
<td>-0.775 6</td>
<td>-1.017 690</td>
</tr>
<tr>
<td>7</td>
<td>-0.773 6</td>
<td>-0.775 1</td>
<td>-1.017 683</td>
</tr>
<tr>
<td>9</td>
<td>-0.773 9</td>
<td>-0.774 8</td>
<td>-1.017 681 8</td>
</tr>
<tr>
<td>(S_L)</td>
<td>-0.774 386 5</td>
<td>-0.774 386 0</td>
<td>-1.017 680 66</td>
</tr>
<tr>
<td>(S)</td>
<td>-0.774 36</td>
<td>-1.017 680 85</td>
<td>-1.747 50</td>
</tr>
<tr>
<td>(S_L)</td>
<td>-0.774 386 3</td>
<td>-1.017 680 75</td>
<td>-1.747 564 94 2</td>
</tr>
<tr>
<td>(S_E)</td>
<td>-0.774 386 1</td>
<td>-1.017 680 75</td>
<td>-1.747 564 94 6</td>
</tr>
</tbody>
</table>
Table 4.5. Behaviour of the parent, DPA, IDPA and [1/1]SES(6) sequences for the two-dimensional square lattice $S(IV)$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$S_{2j+1}$</th>
<th>S(DPA)</th>
<th>S(IDPA)</th>
<th>$(2j-1, 2j+4)$</th>
<th>$(2j, 2j+5)$</th>
<th>$S$(SES)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.48</td>
<td>0.36</td>
<td></td>
<td>0.26</td>
<td>0.31</td>
<td>0.284</td>
</tr>
<tr>
<td>2</td>
<td>0.41</td>
<td>0.33</td>
<td></td>
<td>0.276</td>
<td>0.299</td>
<td>0.288</td>
</tr>
<tr>
<td>5</td>
<td>0.35</td>
<td>0.298</td>
<td>0.296</td>
<td>0.286</td>
<td>0.292</td>
<td>0.289 1</td>
</tr>
<tr>
<td>10</td>
<td>0.321</td>
<td>0.292</td>
<td>0.290 9</td>
<td>0.288 36</td>
<td>0.290 10</td>
<td>0.289 230</td>
</tr>
<tr>
<td>12</td>
<td>0.316</td>
<td>0.291 3</td>
<td>0.289 47</td>
<td>0.288 62</td>
<td>0.289 87</td>
<td>0.289 243</td>
</tr>
<tr>
<td>15</td>
<td>0.311</td>
<td>0.290 6</td>
<td>0.289 497</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$S^L$ 0.289 261 524 5 0.289 261 525 1
$S^L$ 0.289 261 524 8
$S_F$ 0.289 261 523 6
However, application of \([1/1]\)SES(6) leads, as usual, to two sequences, which smoothly proceeds from opposite sides. Averaging the members, stability up to only fourth decimal place is ensured. On the other hand, the MPA-limits \(S^L\) are found to be almost equal. Thus, the \(S^L\) value and the exact one \((S_E)\) become virtually the same which reveals the strength of the scheme once again.

4.6 Concluding remarks

The object of our present work has been to explore the efficiency of an extrapolation technique in providing limit points of sawtooth sequences by separating the subsequences. We hope to have achieve the end.

The preliminary version of the scheme, viz. \([1/1]\)ES is too simple and becomes fruitful only for those divergent sawtooth sequences for which alternate members maintain linearity. For these sequences, the method not only offers accurate estimate of their limit points, but also gives a physical picture of the position of \(S_0\) in relation to the PS, thereby assigning a meaning to the limit point of a divergent sequence. Even when standard methods fail, LE scheme works effectively. The LE sums are also interpretable in terms of Hardy's axioms. But, the scheme as such is not applicable to convergent sawtooth sequences or oscillatory sequences.

An improved version of the scheme, viz. \([1/1]\)SES has been proposed to offer better quality results for almost-linear divergent cases and also to handle the convergent sawtooth sequences. A further improvement of the results have been achieved by adopting the MPA-assisted SES.

Sequences, for which approximate linearity is not maintained by alternate members, may be tackled by adopting high-order ES or SES. The modification involved is trivial. But, we have not explicitly carried out calculations on such a strategy. This is because, sequences involved in LS (and considered here) obey a quasi-linear character.