Chapter 3

Modified Ratio Estimators

3.1 Introduction

Sometimes the additional information regarding the parameters of auxiliary variable like coefficient of correlation, coefficient of skewness etc. is also available in advance. In such cases, the main interest of an investigator is to use that information at estimation stage. The new ratio (or product) estimators which use this additional information are generally known as modified ratio (or product) type estimators. Sisodia and Dwivedi (1981) has initiated the use of additional information in ratio estimator by assuming that coefficient of variation ($C_x$) of auxiliary variable is known. After this, a number of researchers proposed different modified estimators for population mean under SRS by assuming different type of additional information. Some modified ratio estimators as follows:

Sisodia and Dwivedi (1981)

$$
\bar{y}_{sd} = \frac{\bar{y} \mu_x + C_x}{\bar{x} + C_x}.
$$
CHAPTER 3. MODIFIED RATIO ESTIMATORS

Pandey and Dubey (1988)

\[ \bar{y}_{pd} = \bar{y} \frac{\bar{x} + C_x}{\mu_x + C_x} \]

Singh and Kakran (1993)

\[ \bar{y}_{sk} = \bar{y} \frac{\mu_x + \beta_2(x)}{\bar{x} + \beta_2(x)} , \]

where \( \beta_2(x) \) is coefficient of kurtosis of auxiliary variable.

Upadhyaya and Singh (1999)

\[ \bar{y}_{us1} = \bar{y} \frac{\beta_2(x) \mu_x + C_x}{\beta_2(x) \bar{x} + C_x} \]

\[ \bar{y}_{us2} = \bar{y} \frac{C_x \mu_x + \beta_2(x)}{C_x \bar{x} + \beta_2(x)} \]

Singh and Tailor (2003)

\[ \bar{y}_{st} = \bar{y} \frac{\mu_x + \rho}{\bar{x} + \rho} \]

Singh (2003a)

\[ \bar{y}_{s1} = \bar{y} \frac{\mu_x + S_x}{\bar{x} + S_x} \]

\[ \bar{y}_{s2} = \bar{y} \frac{\beta_1 \mu_x + S_x}{\beta_1 \bar{x} + S_x} \]

\[ \bar{y}_{s3} = \bar{y} \frac{\beta_2(x) \mu_x + S_x}{\beta_2(x) \bar{x} + S_x} \]

Al-Omari et al (2009)

\[ \bar{y}_{AO1} = \bar{y} \left( \frac{\bar{X} + q_1}{\bar{x} + q_1} \right) \]

\[ \bar{y}_{AO3} = \bar{y} \left( \frac{\bar{X} + q_3}{\bar{x} + q_3} \right) \]

Yan and Tian (2010)

\[ \bar{y}_{yt1} = \bar{y} \frac{\mu_x + \beta_1}{\bar{x} + \beta_1} , \]
\[ \bar{y}_{gt2} = \bar{y} \frac{\beta_2(x) \mu_x + \beta_1}{\beta_2(x) \bar{x} + \beta_1}, \]
\[ \bar{y}_{gt3} = \bar{y} \frac{\beta_1 \mu_x + \beta_2(x)}{\beta_1 \bar{x} + \beta_2(x)}, \]
\[ \bar{y}_{gt4} = \bar{y} \frac{C_x \mu_x + \beta_1}{C_x \bar{x} + \beta_1}, \]

where \( \beta_1 \) is the coefficient of skewness of auxiliary variable.

On the same pattern, some researchers have done work on modified ratio type estimators under RSS as follows:


\[ \bar{y}_{Jrss} = \bar{y}_{rss} \left( \frac{\mu_x + \rho}{\bar{x}_{rss} + \rho} \right) \]

Al-Omari et al (2009)

\[ \bar{y}_{AO1rss} = \bar{y}_{rss} \left( \frac{\mu_x + q_1}{\bar{x}_{rss} + q_1} \right) \]
\[ \bar{y}_{AO3rss} = \bar{y}_{rss} \left( \frac{\mu_x + q_3}{\bar{x}_{rss} + q_3} \right) \]

All the researchers obtained the approximate expressions bias and MSE for their corresponding estimators. Under some conditions these modified ratio estimator are more efficient than the usual ratio estimator.

The work of present chapter is based on our published paper Brar and Malik (2014a). In Section-3.2, the expressions for bias and MSE are obtained by considering the generalized form of modified ratio estimators under SRS and further more general form of these estimators is considered by using the power transformation. Further, we propose generalized form of modified ratio estimator for population mean under RSS in Section-3.3. Comparison of these estimators with the existing estimators and simulation study have been performed in following sections.

### 3.2 Modified Ratio Estimator under SRS

General form of modified ratio estimators may defined as

\[ \bar{y}_{R\beta} = \bar{y} \left( \frac{\mu_x + \beta}{\bar{x} + \beta} \right) \]  \( (3.1) \)
where $\beta$ is the known value of a real constant or population parameter.

One can easily obtain different estimators of population mean ($\bar{Y}$) by choosing different values of $\beta$. For example

1. When $\beta = C_x$, where $C_x$ is the coefficient of variation of auxiliary variable($x$), $\bar{y}_{R\beta}$ reduces to $\bar{y}_{SD} = \bar{y} \left( \frac{u_x + C_x}{\bar{x} + C_x} \right)$ which was defined by Sisodia and Dwivedi (1981).

2. When $\beta = \rho$, where $\rho$ is the coefficient of correlation between auxiliary variable($x$) and study variable($y$), then $\bar{y}_{R\beta}$ reduces to $\bar{y}_{ST} = \bar{y} \left( \frac{u_x + \rho}{\bar{x} + \rho} \right)$ which was proposed by Singh and Tailor (2003).

3. When $\beta = \frac{\beta_1}{C_x}$, then $\bar{y}_{R\beta}$ reduces to $\bar{y}_{y1t} = \bar{y} \left( \frac{C_x u_x + \beta_1}{C_x \bar{x} + \beta_1} \right)$ which was defined by Al-Omari et al (2009).

Equation (3.1) can be expressed in term of $\epsilon$’s as

$$\bar{y}_{R\beta} = \mu_y (1 + \epsilon_1)(1 + \epsilon_2')^{-1},$$

where

$$\epsilon_2' = \frac{\bar{x} - \mu_x}{\mu_x + \beta}, \quad E(\epsilon_2') = 0,$$

$$E(\epsilon_2'^2) = \frac{V(\bar{x})}{\mu_x + \beta}, \quad E(\epsilon_1' \epsilon_2') = \frac{\text{Cov}(\bar{x}, \bar{y})}{\mu_y (\mu_x + \beta)}.$$

To find out the expressions for bias and MSE of $\bar{y}_{R\beta}$ up to first order approximation, we assume that $|\epsilon_1| < 1$ and $|\epsilon_2'| < 1$, so one can consider term in $\epsilon$’s up to second degree as

$$\bar{y}_{R\beta} = \mu_y (1 + \epsilon_1)(1 - \epsilon_2' + \epsilon_2'^2),$$

or $$\bar{y}_{R\beta} = \mu_y (1 + \epsilon_1 - \epsilon_2' + \epsilon_2'^2 - \epsilon_1' \epsilon_2'),$$

or $$\bar{y}_{R\beta} - m u_y = \mu_y (\epsilon_1 - \epsilon_2' + \epsilon_2'^2 - \epsilon_1' \epsilon_2'), \quad (3.2)$$

After taking expectation on both sides, we get

$$\text{Bias} (\bar{y}_{R\beta}) = \left( \frac{1}{n} - \frac{1}{N} \right) \frac{1}{\mu_y} \left[ R^2 \theta_{\beta \beta} S_x^2 - R \theta_{\beta \beta} S_{yx} \right]. \quad (3.3)$$
CHAPTER 3. MODIFIED RATIO ESTIMATORS

From equation (3.2), we have

\[(\bar{y}_{R\beta} - mu_y)^2 = \mu_y^2(\epsilon_1 - \epsilon'_2 + \epsilon'_2 - \epsilon_1^2)^2,\]
\[(\bar{y}_{R\beta} - mu_y)^2 = \mu_y^2(\epsilon_1^2 + \epsilon'_2 - 2\epsilon_1\epsilon'_2),\]

we can get MSE of \(\bar{y}_{R\beta}\) after taking expectation as

\[MSE(\bar{y}_{R\beta}) = \left(\frac{1}{n} - \frac{1}{N}\right)\left[S_y^2 + R^2\theta\beta^2S_x^2 - 2R\theta\beta S_{yx}\right]\]  (3.4)

where \(K\beta = \frac{\bar{y}}{X+\beta}\) and \(\theta\beta = \frac{1}{X+\beta}.\)

After ignoring the \(fpc\), we get

\[bias(\bar{y}_{R\beta}) = \frac{1}{n} \frac{1}{\mu_y} \left[R^2\theta\beta^2\sigma_x^2 - R\theta\beta\sigma_{yx}\right]\]  (3.5)

and

\[MSE(\bar{y}_{R\beta}) = \frac{1}{n} \left[\sigma_y^2 + R^2\theta\beta^2\sigma_x^2 - 2R\theta\beta\sigma_{yx}\right]\]  (3.6)

which are required expressions for bias and MSE of \(\bar{y}_{R\beta}\).

Some more modified ratio type estimators such as
Pandey and Dubey (1988)
\[\bar{y}_{pd} = \bar{y} \frac{\bar{x} + C_x}{\mu_x + C_x}\]

Singh et al (2008)
\[\bar{y}_{s(\alpha)} = \bar{y} \left(\frac{\beta_2(x)\bar{X} + C_x}{\beta_2(x)\bar{x} + C_x}\right)^\alpha\]

where \(\alpha\) is some real constant.

These type of modified ratio (or product) estimators not included in (3.1). So, there is need to define more general form of modified ratio estimators which includes also the modified product estimators and the estimators like defined by Singh et al (2008).

A more general form of modified ratio estimators can be defined by using the power transformation as:

\[\bar{y}_{R\beta(\alpha)} = \bar{y} \left(\frac{\bar{x} + \beta}{\mu_x + \beta}\right)^\alpha\]    (3.7)

where \(\alpha\) and \(\beta\) are known constants and the optimum value of \(\alpha\) can also be
CHAPTER 3. MODIFIED RATIO ESTIMATORS

obtained by minimizing the MSE of $\bar{y}_{R\beta}(\alpha)$.

One can obtain the different estimators by choosing different values of $\alpha$ and $\beta$. For example,

1. When $\alpha = 0$; $\bar{y}_{R\beta}(\alpha)$ becomes mean per unit estimator.

2. When $\alpha = 1, \beta = 0$; $\bar{y}_{R\beta}(\alpha)$ becomes classical product estimator.

3. When $\alpha = -1$; $\bar{y}_{R\beta}(\alpha)$ reduces to $\bar{y}_{R\beta}$.

4. When $\beta = 0$; $\bar{y}_{R\beta}(\alpha)$ becomes power transformation estimator [see Srivastava (1967)].

5. When $\beta = \frac{c_x}{\beta^2(x)}$; $\bar{y}_{R\beta}(\alpha)$ reduces to $\bar{y}_{s(\alpha)}$.

To obtain the approximate expression for bias of $\bar{y}_{R\beta}(\alpha)$, first write $\bar{y}_{R\beta}(\alpha)$ in terms of $\epsilon$’s as

$$\bar{y}_{R\beta}(\alpha) = \mu_y (1 + \epsilon_1)(1 + \epsilon'_2)^\alpha,$$

$$\bar{y}_{R\beta}(\alpha) - \mu_y = \mu_y (\epsilon_1 + \alpha \epsilon'_2 + \frac{\alpha(\alpha - 1)}{2} \epsilon'_2 + \alpha \epsilon_1 \epsilon'_2),$$

(3.8)

taking expectation both sides, we get

$$Bias \left( \bar{y}_{R\beta}(\alpha) \right) = \left( \frac{1}{n} - \frac{1}{N} \right) \frac{1}{\mu_y} \frac{\alpha(\alpha - 1)}{2} \left[ R^2 \theta_{\beta}\sigma_x^2 + \alpha R \theta_{\beta} \sigma_{yx} \right]$$

(3.9)

Further, from (3.8), we have

$$(\bar{y}_{R\beta}(\alpha) - \mu_y)^2 = \mu_y^2(\epsilon_1 + \alpha \epsilon'_2 + \frac{\alpha(\alpha - 1)}{2} \epsilon'_2 + \alpha \epsilon_1 \epsilon'_2)^2$$

(3.10)

$$MSE \left( \bar{y}_{R\beta}(\alpha) \right) = \left( \frac{1}{n} - \frac{1}{N} \right) \left[ S_y^2 + \alpha^2 R^2 \theta_{\beta}^2 S_x^2 + 2\alpha R \theta_{\beta} \sigma_{yx} \right].$$

(3.11)

After ignoring fpc, we have

$$bias \left( \bar{y}_{R\beta}(\alpha) \right) = \frac{1}{n \mu_y} \left[ \frac{\alpha(\alpha - 1)}{2} R^2 \theta_{\beta}^2 \sigma_x^2 + \alpha R \theta_{\beta} \sigma_{yx} \right]$$

(3.12)
CHAPTER 3. MODIFIED RATIO ESTIMATORS

and

\[ MSE (\bar{y}_{R\beta(\alpha)}) = \frac{1}{n} \left[ \sigma_y^2 + \alpha^2 R^2 \theta^2 \sigma_x^2 + 2\alpha R\theta \sigma_{yx} \right]. \tag{3.12} \]

The value of \( \alpha \) which minimizes \( MSE (\bar{y}_{R\beta(\alpha)}) \) can be obtained after solving the following equation:

\[ \frac{\partial MSE (\bar{y}_{R\beta(\alpha)})}{\partial \alpha} = 0 \]

The optimum value of \( \alpha \) is

\[ \alpha_{opt} = -\rho \frac{S_y}{S_x} \frac{\bar{X} + \beta \bar{Y}}{\bar{Y}} = -\frac{S_y}{S_x} \frac{\rho}{K_\beta} = \alpha' \text{ (say)} \tag{3.13} \]

Substituting the value of \( \alpha = \alpha' \) in (3.10), we obtain minimum \( MSE (\bar{y}_{R\beta(\alpha)}) \) as

\[ MSE_{min} (\bar{y}_{R\beta(\alpha)}) = \left( \frac{1}{n} - \frac{1}{N} \right) (1 - \rho^2) S_y^2. \tag{3.14} \]

From equations (3.4) and (3.14), we get

\[ MSE (\bar{y}_{R\beta}) - MSE_{min} (\bar{y}_{R\beta(\alpha)}) = \left( \frac{1}{n} - \frac{1}{N} \right) [\rho S_y - R\theta \beta S_x]^2 \geq 0 \tag{3.15} \]

Above equation shows that the proposed estimator is more efficient than the estimators defined by Sisodia and Dwivedi (1981), Singh and Tailor (2003) and Al-Omari et al (2009).

The optimum value of \( \alpha \) i.e. \( \alpha' \) depends upon the values of unknown parameters. Therefore, the estimator cannot work for practical problems. Following corollary addresses this problem.

**COROLLARY 1.** The modified estimator of \( \bar{y}_{R\beta(\alpha)} \) may be used practically which can be obtained by replacing \( \alpha \) by a consistent estimator of \( \alpha' \) in (3.7) as given below

\[ \bar{y}^P_{R\beta(\alpha)} = \bar{y} \left( \frac{\bar{x} + \beta}{\mu_x + \beta} \right)^{\hat{\alpha}'} \]

where \( \hat{\alpha}' \) is a consistent estimator of \( \alpha' \). For example, \( \hat{\alpha}' = -\rho \frac{s_y}{s_x} \frac{\mu_x + \beta}{\bar{y}} \), where \( s_y \) and \( s_x \) are sample standard deviations of study and auxiliary variables respectively.

### 3.3 Modified Ratio Estimators under RSS

estimators under RSS for population mean ($\mu_y$). Generalized form of these estimators is defined as

$$\bar{y}_{R\beta rss} = \bar{y}_{rss} \left( \frac{\mu_x + \beta}{\bar{x}_{rss} + \beta} \right)$$  \hspace{1cm} (3.16)$$

where $\beta$ is a real constant or a known population parameter.

By choosing different values of $\beta$, one can obtain different estimators of population mean as follow

1. When $\beta = C_x$, $\bar{y}_{R\beta rss}$ reduces to $\bar{y}_{SDrss} = \bar{y}_{rss} \left( \frac{\mu_x + C_x}{\bar{x}_{rss} + C_x} \right)$.

2. When $\beta = \rho$, $\bar{y}_{R\beta rss}$ reduces to $\bar{y}_{Jrss} = \bar{y}_{rss} \left( \frac{\mu_x + \rho}{\bar{x}_{rss} + \rho} \right)$ which was proposed by Jemain et al (2008).

3. When $\beta = q_1$ or $q_3$, $\bar{y}_{R\beta rss}$ reduces to $\bar{y}_{AO1rss} = \bar{y}_{rss} \left( \frac{\mu_x + q_1}{\bar{x}_{rss} + q_1} \right)$ or $\bar{y}_{AO3rss} = \bar{y}_{rss} \left( \frac{\mu_x + q_3}{\bar{x}_{rss} + q_3} \right)$ respectively which were defined by Al-Omari et al (2009).

Equation (3.16) can be expressed in term of $\delta$’s as

$$\bar{y}_{R\beta rss} = \mu_y (1 + \delta_1)(1 + \delta_2')^{-1},$$

where

$$\delta'_2 = \frac{\bar{x}_{rss} - \mu_x}{\mu_x + \beta}, \hspace{1cm} E(\delta'_2) = 0,$$

$$E(\delta'_2^2) = \frac{V(\bar{x}_{rss})}{\mu_x + \beta}, \hspace{1cm} E(\delta_1 \delta'_2) = \frac{Cov(\bar{x}_{rss}, \bar{y}_{rss})}{\mu_y(\mu_x + \beta)}.$$

After considering the term of $\delta$’s upto second degree, we get

$$\bar{y}_{R\beta rss} = \mu_y (1 + \delta_1)(1 - \delta'_2 + \delta'_2^2), \hspace{1cm} \text{or} \hspace{1cm} \bar{y}_{R\beta rss} = \mu_y (1 + \delta_1 - \delta'_2 + \delta'_2^2 - \delta_1 \delta'_2),$$

$$\text{or} \hspace{1cm} \bar{y}_{R\beta rss} - \mu u_y = \mu_y (\delta_1 - \delta'_2 + \delta'_2^2 - \delta_1 \delta'_2),$$  \hspace{1cm} (3.17)$$

taking expectation on both sides, we have

$$\text{Bias} (\bar{y}_{R\beta rss}) = \frac{1}{\mu_y} \left[ R^2 \theta'_{\beta} V(\bar{x}_{rss}) - R \theta_{\beta} Cov(\bar{x}_{rss}, \bar{y}_{rss}) \right]$$  \hspace{1cm} (3.18)$$
CHAPTER 3. MODIFIED RATIO ESTIMATORS

From (3.17), we have

\[(\bar{y}_{R\beta rss} - m u_y)^2 = \mu^2_y (\delta_1 - \delta'_2 + \delta'_2 - \delta_1)^2,\]
\[(\bar{y}_{R\beta rss} - m u_y)^2 = \mu^2_y (\delta_1^2 + \delta'_2^2 - 2\delta_1\delta'_2),\]

we can get MSE of \(\bar{y}_{R\beta}\) after taking expectation as

\[MSE (\bar{y}_{R\beta rss}) = V (\bar{y}_{rss}) + R^2 \theta^2 \beta V (\bar{x}_{rss}) - 2 R \theta \beta Cov (\bar{x}_{rss}, \bar{y}_{rss}) \] (3.19)

After utilizing the expression of \(V (\bar{y}_{rss}), V (\bar{x}_{rss})\) and \(Cov (\bar{y}_{rss}, \bar{x}_{rss})\) in (3.19), we get

\[MSE (\bar{y}_{R\beta rss}) = \frac{1}{r m} \left[ \sigma^2_y + R^2 \theta^2 \beta \sigma^2_x - 2 R \theta \beta \sigma_{yx} \right] \]
\[- \frac{1}{r m^2} \left[ \sum_{i=1}^{m} \tau^2_{yi[i]} + R^2 \theta^2 \beta \sum_{i=1}^{m} \tau^2_{xi(i)} - 2 R \theta \beta \sum_{i=1}^{m} \tau_{xy[i]} \right] \]

or

\[MSE (\bar{y}_{R\beta rss}) = \frac{1}{r m} \left[ \sigma^2_y + R^2 \theta^2 \beta \sigma^2_x - 2 R \theta \beta \sigma_{yx} \right] - \frac{1}{r m^2} \sum_{i=1}^{m} \left[ \tau_{yi[i]} - R \theta \beta \tau_{xi(i)} \right]^2. \] (3.20)

From (3.6) and (3.20), we have

\[MSE (\bar{y}_{R\beta}) - MSE (\bar{y}_{R\beta rss}) = \frac{1}{r m^2} \sum_{i=1}^{m} \left[ \tau_{yi[i]} - R \theta \beta \tau_{xi(i)} \right]^2 \geq 0, \] (3.21)

which shows that \(\bar{y}_{R\beta rss}\) is always more efficient than \(\bar{y}_{R\beta}\).

3.4 Proposed Estimator

A generalized class of modified ratio type estimators for population mean under RSS is proposed as

\[\bar{y}_{prop} = \bar{y}_{rss} \left( \frac{\bar{x}_{rss} + \beta}{\mu_x + \beta} \right)^{\alpha} \] (3.22)

where \(\alpha\) and \(\beta\) are constants and optimum value of \(\alpha\) can be obtained by minimizing MSE of \(\bar{y}_{prop}\).
First write $\bar{y}_{\text{prop}}$ in terms of $\delta$’s as

\[
\bar{y}_{\text{prop}} = \mu_y (1 + \delta_1)(1 + \delta'_2)^\alpha, \\
\bar{y}_{\text{prop}} - \mu_y = \mu_y (\delta_1 + \alpha \delta'_2 + \frac{\alpha (\alpha - 1)}{2} \delta_2^2 + \alpha \delta_1 \delta'_2),
\]

(3.23)

Bias of $\bar{y}_{\text{prop}}$ can easily be obtained by taking expectation as

\[
\text{Bias}(\bar{y}_{\text{prop}}) = \frac{1}{\mu_y} \left[ \frac{\alpha (\alpha - 1)}{2} R^2 \theta_\beta^2 V(\bar{\bar{x}}_{\text{rss}}) + \alpha R \theta_\beta Cov(\bar{\bar{x}}_{\text{rss}}, \bar{y}_{\text{rss}}) \right]
\]

or

\[
\text{Bias}(\bar{y}_{\text{prop}}) = \frac{1}{r m \mu_y} \left[ \frac{\alpha (\alpha - 1)}{2} R^2 \theta_\beta^2 \sigma_x^2 + \alpha R \theta_\beta \sigma_{yx} \right] - \frac{1}{r m^2 \mu_y} \left[ \frac{\alpha (\alpha - 1)}{2} R^2 \theta_\beta^2 \sum_{i=1}^{m} \tau_{x(i)}^2 + \alpha R \theta_\beta \sum_{i=1}^{m} \tau_{xy[i]} \right] \quad (3.24)
\]

Further, from (3.23), we have

\[
(\bar{y}_{\text{prop}} - \mu_y)^2 = \mu_y^2 (\delta_1 + \alpha \delta'_2 + \frac{\alpha (\alpha - 1)}{2} \delta_2^2 + \alpha \delta_1 \delta'_2)^2 \\
(\bar{y}_{\text{prop}} - \mu_y)^2 = \mu_y^2 (\delta_1^2 + \alpha \delta'_2^2 + 2 \alpha \delta_1 \delta'_2),
\]

taking expectation, we get

\[
MSE(\bar{y}_{\text{prop}}) = V(\bar{y}_{\text{rss}}) + \alpha^2 R^2 \theta_\beta^2 V(\bar{\bar{x}}_{\text{rss}}) + 2 \alpha R \theta_\beta Cov(\bar{\bar{x}}_{\text{rss}}, \bar{y}_{\text{rss}}) \quad (3.25)
\]

After using the expressions $V(\bar{y}_{\text{rss}})$, $V(\bar{\bar{x}}_{\text{rss}})$ and $Cov(\bar{y}_{\text{rss}}, \bar{\bar{x}}_{\text{rss}})$ in (3.25), we get

\[
MSE(\bar{y}_{\text{prop}}) = \frac{1}{r m \mu_y} \left[ \sigma_y^2 + \alpha^2 R^2 \theta_\beta^2 \sigma_x^2 + 2 \alpha R \theta_\beta \sigma_{yx} \right] - \frac{1}{r m^2} \left[ \sum_{i=1}^{m} \tau_{y[i]} + \alpha^2 R^2 \theta_\beta^2 \sum_{i=1}^{m} \tau_{x(i)}^2 + 2 \alpha R \theta_\beta \sum_{i=1}^{m} \tau_{xy[i]} \right]
\]

or

\[
MSE(\bar{y}_{\text{prop}}) = \frac{1}{r m \mu_y} \left[ \sigma_y^2 + \alpha^2 R^2 \theta_\beta^2 \sigma_x^2 + 2 \alpha R \theta_\beta \sigma_{yx} \right] - \frac{1}{r m^2} \left[ \tau_{y[i]} + \alpha R \theta_\beta \tau_{x(i)} \right]^2. \quad (3.26)
\]
From (3.12) and (3.26), we get

$$MSE\left( \bar{y}_{R\beta(\alpha)} \right) - MSE\left( \bar{y}_{prop} \right) = \frac{1}{m\theta^2} \sum_{i=1}^{m} \left[ \tau_{yi[i]} + \alpha R\theta \tau_{x(i)} \right]^2 \geq 0 \quad (3.27)$$

which shows that $\bar{y}_{prop}$ is always more efficient than $\bar{y}_{R\beta(\alpha)}$ corresponding to each value of $\alpha$. The optimum value of $\alpha$ which minimizes MSE of $\bar{y}_{prop}$, can easily be obtained after solving the following equation:

$$\frac{\partial MSE\left( \bar{y}_{prop} \right)}{\partial \alpha} = 0$$

and that value of $\alpha$ is

$$\alpha_{opt} = -\frac{Cov\left( \bar{x}_{rss}, \bar{y}_{rss} \right) \mu_x + \beta}{V\left( \bar{x}_{rss} \right) \mu_y} = \frac{\left( \sigma_{yx} - \frac{1}{m} \sum_{i=1}^{m} \tau_{yx[i]} \right) \mu_x + \beta}{\left( \sigma_x^2 - \frac{1}{m} \sum_{i=1}^{m} \tau_{x(i)}^2 \right) \mu_y} = \alpha^* \quad (say)$$

After substituting $\alpha = \alpha^*$ in (3.25) or (3.26), one can obtain the expressions for minimum MSE of $\bar{y}_{prop}$ as follows:

$$MSE_{\min}\left( \bar{y}_{prop} \right) = \left( 1 - \rho_{\bar{x}_{rss}, \bar{y}_{rss}}^2 \right) V\left( \bar{y}_{rss} \right) \quad (3.29)$$

or

$$MSE\left( \bar{y}_{prop} \right) = \frac{1}{rm^2} \left[ \sigma_y^2 + \alpha^* R^2 \theta^2 \sigma_x^2 + 2\alpha^* R\theta \sigma_{yx} \right]$$

$$- \frac{1}{rm^2} \sum_{i=1}^{m} \left[ \tau_{yi[i]} + \alpha^* R\theta \tau_{x(i)} \right]^2.$$

where $\rho_{\bar{x}_{rss}, \bar{y}_{rss}}$ is the correlation coefficient between $\bar{x}_{rss}$ and $\bar{y}_{rss}$.

The optimum value of $\alpha$ (i.e. $\alpha^*$) depends upon the values of unknown parameters. So it cannot be used practically. Thus we have following corollary.

**COROLLARY 2.** The modified estimator of $\bar{y}_{prop}$ may be used practically which can be obtained by replacing $\alpha$ by a consistent estimator of $\alpha^*$ in (3.22) as given below

$$\bar{y}_{prop}^2 = \bar{y}_{rss} \left( \frac{\bar{x}_{rss} + \beta}{\mu_x + \beta} \right)^{\hat{\alpha}^*}.$$
where \( \hat{\alpha}^* \) is a consistent estimator of \( \alpha^* \). For example
\[
\hat{\alpha}^* = \frac{\sum_{i=1}^{m} s_{yx(i)} \mu_{x(i)} + \beta}{\sum_{i=1}^{m} s_{x(i)}^2 \mu_y},
\]
where \( s_{yx(i)} = \frac{1}{r-1} \sum_{j=1}^{r} \left( x_j(i) - \bar{x}(i) \right) \left( y_j[i] - \bar{y}[i] \right) \) and \( s_{x(i)}^2 = \frac{1}{r-1} \sum_{j=1}^{r} \left( x_j(i) - \bar{x}(i) \right)^2 \).

3.5 Comparison

**Theorem 3.1** \( \bar{y}_{\text{prop}} \) is always more efficient than \( \bar{y}_{Jrss} \).

**Proof:** From equations (3.19) and (3.29), we get
\[
MSE(\bar{y}_{R\beta rss}) - MSE_{\min}(\bar{y}_{\text{prop}}) = \left[ \rho \bar{x}_{rss}, \bar{y}_{rss} \sqrt{V(\bar{y}_{rss})} - R\theta \beta \sqrt{V(\bar{x}_{rss})} \right]^2 \geq 0
\]
which shows that \( \bar{y}_{\text{prop}} \) is more efficient than \( \bar{y}_{Jrss} \) at optimum value of \( \alpha \).

**Theorem 3.2** \( \bar{y}_{\text{prop}} \) is always more efficient than \( \bar{y}_{R\beta(\alpha)} \).

**Proof:** From equation (3.27), at point \( \alpha' \), we get
\[
MSE_{\min}(\bar{y}_{R\beta(\alpha)}) \geq MSE_{\alpha'}(\bar{y}_{\text{prop}}).
\]

Since \( \alpha^* \) is an optimum point for \( MSE(\bar{y}_{\text{prop}}) \), this implies
\[
MSE_{\alpha'}(\bar{y}_{\text{prop}}) \geq MSE_{\min}(\bar{y}_{\text{prop}}),
\]
From above two inequalities, we get
\[
MSE_{\min}(\bar{y}_{R\beta(\alpha)}) \geq MSE_{\min}(\bar{y}_{\text{prop}})
\]
which shows that \( \bar{y}_{\text{prop}} \) is more efficient than \( \bar{y}_{R\beta(\alpha)} \) corresponding to their optimum values of \( \alpha \).

3.6 Simulation Study

A simulation study is conducted by using software \( R \) to study the properties of the estimators. For this 1, 00,000 samples are generated from bivariate normal distribution \( BVN(100,200,4,4,\rho) \) where \( \rho = 0.3, 0.5, 0.7, 0.9 \) under
Table 3.1: Biases of the estimators for $n = 9, m = 3$ and $r = 3$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>bias$(\bar{y})$</td>
<td>0.00080</td>
<td>0.00080</td>
<td>0.00080</td>
<td>0.00080</td>
</tr>
<tr>
<td>bias$(\bar{y}_{sd})$</td>
<td>-0.00015</td>
<td>-0.00056</td>
<td>-0.00088</td>
<td>-0.00103</td>
</tr>
<tr>
<td>bias$(\bar{y}^a_{R\beta(\alpha)})$</td>
<td>0.00009</td>
<td>-0.00056</td>
<td>-0.00124</td>
<td>-0.00170</td>
</tr>
<tr>
<td>bias$(\bar{y}_{st})$</td>
<td>-0.00015</td>
<td>-0.00056</td>
<td>-0.00088</td>
<td>-0.00103</td>
</tr>
<tr>
<td>bias$(\bar{y}^a_{R\beta(\alpha)})$</td>
<td>0.00009</td>
<td>-0.00056</td>
<td>-0.00124</td>
<td>-0.00170</td>
</tr>
<tr>
<td>bias$(\bar{y}_{AO1})$</td>
<td>0.00004</td>
<td>-0.00016</td>
<td>-0.00032</td>
<td>-0.00040</td>
</tr>
<tr>
<td>bias$(\bar{y}^{a1}_{R\beta(\alpha)})$</td>
<td>-0.00007</td>
<td>-0.00084</td>
<td>-0.00163</td>
<td>-0.00220</td>
</tr>
<tr>
<td>bias$(\bar{y}_{AO3})$</td>
<td>0.00005</td>
<td>-0.00016</td>
<td>-0.00032</td>
<td>-0.00039</td>
</tr>
<tr>
<td>bias$(\bar{y}^{a3}_{R\beta(\alpha)})$</td>
<td>-0.00007</td>
<td>-0.00084</td>
<td>-0.00163</td>
<td>-0.00220</td>
</tr>
<tr>
<td>bias$(\bar{y}_{SDRss})$</td>
<td>-0.00032</td>
<td>-0.00054</td>
<td>-0.00073</td>
<td>-0.00080</td>
</tr>
<tr>
<td>bias$(\bar{y}_{Cprop})$</td>
<td>-0.00009</td>
<td>-0.00036</td>
<td>-0.00070</td>
<td>-0.00105</td>
</tr>
<tr>
<td>bias$(\bar{y}_{Jrss})$</td>
<td>-0.00032</td>
<td>-0.00054</td>
<td>-0.00073</td>
<td>-0.00080</td>
</tr>
<tr>
<td>bias$(\bar{y}_{prop})$</td>
<td>-0.00009</td>
<td>-0.00036</td>
<td>-0.00071</td>
<td>-0.00105</td>
</tr>
<tr>
<td>bias$(\bar{y}_{AO1rss})$</td>
<td>-0.00033</td>
<td>-0.00042</td>
<td>-0.00048</td>
<td>-0.00048</td>
</tr>
<tr>
<td>bias$(\bar{y}_{AO3rss})$</td>
<td>-0.00018</td>
<td>-0.00051</td>
<td>-0.00090</td>
<td>-0.00131</td>
</tr>
<tr>
<td>bias$(\bar{y}_{prop})$</td>
<td>-0.00033</td>
<td>-0.00042</td>
<td>-0.00048</td>
<td>-0.00047</td>
</tr>
</tbody>
</table>

SRS design and RSS design.

The efficiency of an estimator $\hat{\theta}_1$ with respect to $\bar{y}$ to estimate population mean $\bar{Y}$ is defined as:

$$eff \left( \hat{\theta}_1 \right) = \frac{V(\bar{y})}{MSE(\hat{\theta}_1)}.$$

The values of biases of the estimators for sample sizes 9, 12 and 15 are obtained in Table-3.1, Table-3.2 and Table-3.3 respectively and efficiencies of the estimators for sample sizes 9, 12 and 15 are obtained in Table-3.4, Table-3.5 and Table-3.6 respectively.

Some results based on Table-3.1 to Table-3.6 are as follows:
### Table 3.2: Biases of the estimators for $n = 12$, $m = 4$ and $r = 3$.  

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>bias($\bar{y}$)</td>
<td>0.00126</td>
<td>0.00126</td>
<td>0.00126</td>
<td>0.00126</td>
</tr>
<tr>
<td>bias($\bar{y}_{sd}$)</td>
<td>0.00107</td>
<td>0.00064</td>
<td>0.00024</td>
<td>-0.00013</td>
</tr>
<tr>
<td>bias($\bar{y}<em>{\rho\beta}(C)</em>{R\beta}(\alpha)$)</td>
<td>0.00105</td>
<td>0.00064</td>
<td>0.00006</td>
<td>-0.00064</td>
</tr>
<tr>
<td>bias($\bar{y}<em>{\rho\beta}(C)</em>{R\beta}(\alpha)$)</td>
<td>0.00105</td>
<td>0.00064</td>
<td>0.00006</td>
<td>-0.00064</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO1}$)</td>
<td>0.00096</td>
<td>0.00074</td>
<td>0.00054</td>
<td>0.000036</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO1}$)</td>
<td>0.00096</td>
<td>0.00074</td>
<td>0.00054</td>
<td>0.000036</td>
</tr>
<tr>
<td>bias($\bar{y}_{SDrss}$)</td>
<td>0.00056</td>
<td>0.00031</td>
<td>0.00011</td>
<td>0.00004</td>
</tr>
<tr>
<td>bias($\bar{y}_{C\times prop}$)</td>
<td>0.00116</td>
<td>0.00082</td>
<td>0.00030</td>
<td>-0.00045</td>
</tr>
<tr>
<td>bias($\bar{y}_{Jrss}$)</td>
<td>0.00056</td>
<td>0.00031</td>
<td>0.00011</td>
<td>0.00005</td>
</tr>
<tr>
<td>bias($\bar{y}_{\rho prop}$)</td>
<td>0.00116</td>
<td>0.00082</td>
<td>0.00030</td>
<td>-0.00045</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO1}$)</td>
<td>0.00083</td>
<td>0.00072</td>
<td>0.00065</td>
<td>0.000065</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO1}$)</td>
<td>0.00083</td>
<td>0.00072</td>
<td>0.00065</td>
<td>0.000066</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO3}$)</td>
<td>0.00111</td>
<td>0.00073</td>
<td>0.00018</td>
<td>-0.00061</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO3}$)</td>
<td>0.00111</td>
<td>0.00073</td>
<td>0.00018</td>
<td>-0.00061</td>
</tr>
</tbody>
</table>

### Table 3.3: Biases of the estimators for $n = 15$, $m = 5$ and $r = 3$.  

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>bias($\bar{y}$)</td>
<td>0.00138</td>
<td>0.00138</td>
<td>0.00138</td>
<td>0.00138</td>
</tr>
<tr>
<td>bias($\bar{y}_{sd}$)</td>
<td>0.00183</td>
<td>0.00139</td>
<td>0.00092</td>
<td>0.00040</td>
</tr>
<tr>
<td>bias($\bar{y}<em>{\rho\beta}(C)</em>{R\beta}(\alpha)$)</td>
<td>0.00157</td>
<td>0.00139</td>
<td>0.00092</td>
<td>0.00010</td>
</tr>
<tr>
<td>bias($\bar{y}<em>{\rho\beta}(C)</em>{R\beta}(\alpha)$)</td>
<td>0.00157</td>
<td>0.00139</td>
<td>0.00092</td>
<td>0.00010</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO1}$)</td>
<td>0.00144</td>
<td>0.00121</td>
<td>0.00098</td>
<td>0.00072</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO1}$)</td>
<td>0.00144</td>
<td>0.00121</td>
<td>0.00098</td>
<td>0.00072</td>
</tr>
<tr>
<td>bias($\bar{y}_{SDrss}$)</td>
<td>0.00136</td>
<td>0.00111</td>
<td>0.00079</td>
<td>0.00039</td>
</tr>
<tr>
<td>bias($\bar{y}_{C\times prop}$)</td>
<td>0.00071</td>
<td>0.00074</td>
<td>0.00069</td>
<td>0.00039</td>
</tr>
<tr>
<td>bias($\bar{y}_{Jrss}$)</td>
<td>0.00136</td>
<td>0.00110</td>
<td>0.00079</td>
<td>0.00039</td>
</tr>
<tr>
<td>bias($\bar{y}_{\rho prop}$)</td>
<td>0.00071</td>
<td>0.00074</td>
<td>0.00069</td>
<td>0.00039</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO1}$)</td>
<td>0.00082</td>
<td>0.00071</td>
<td>0.00057</td>
<td>0.00041</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO1}$)</td>
<td>0.00082</td>
<td>0.00071</td>
<td>0.00057</td>
<td>0.00041</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO3}$)</td>
<td>0.00067</td>
<td>0.00068</td>
<td>0.00061</td>
<td>0.00029</td>
</tr>
<tr>
<td>bias($\bar{y}_{AO3}$)</td>
<td>0.00067</td>
<td>0.00068</td>
<td>0.00061</td>
<td>0.00029</td>
</tr>
</tbody>
</table>
### Table 3.4: Efficiencies of the estimators for $n = 9$, $m = 3$ and $r = 3$.  

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{eff}(\bar{y})$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{sd})$</td>
<td>1.053248</td>
<td>1.334118</td>
<td>1.819217</td>
<td>2.858433</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{st})$</td>
<td>1.053569</td>
<td>1.334115</td>
<td>1.819217</td>
<td>2.84399</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO1})$</td>
<td>1.09644</td>
<td>1.231903</td>
<td>1.40552</td>
<td>1.636005</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO3})$</td>
<td>1.096235</td>
<td>1.230622</td>
<td>1.402527</td>
<td>1.630158</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{R\beta(\alpha)})$</td>
<td>1.099414</td>
<td>1.334132</td>
<td>1.961767</td>
<td>5.262914</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{SDrss})$</td>
<td>1.671444</td>
<td>2.100576</td>
<td>2.903828</td>
<td>4.920783</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{Jrss})$</td>
<td>1.672733</td>
<td>2.102486</td>
<td>2.904315</td>
<td>4.903447</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO1rss})$</td>
<td>2.007205</td>
<td>2.264369</td>
<td>2.613332</td>
<td>3.112515</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO3rss})$</td>
<td>2.008323</td>
<td>2.263678</td>
<td>2.609383</td>
<td>3.102326</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{prop})$</td>
<td>2.036949</td>
<td>2.273514</td>
<td>2.907244</td>
<td>6.244577</td>
</tr>
</tbody>
</table>

### Table 3.5: Efficiencies of the estimators for $n = 12$, $m = 4$ and $r = 3$.  

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{eff}(\bar{y})$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{sd})$</td>
<td>1.05258</td>
<td>1.333274</td>
<td>1.817967</td>
<td>2.856132</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{st})$</td>
<td>1.052898</td>
<td>1.333268</td>
<td>1.81568</td>
<td>2.841701</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO1})$</td>
<td>1.095674</td>
<td>1.231053</td>
<td>1.40461</td>
<td>1.635205</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO3})$</td>
<td>1.095472</td>
<td>1.229776</td>
<td>1.401623</td>
<td>1.629365</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{R\beta(\alpha)})$</td>
<td>1.098588</td>
<td>1.333279</td>
<td>1.962002</td>
<td>5.272846</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{SDrss})$</td>
<td>1.888871</td>
<td>2.366586</td>
<td>3.288925</td>
<td>5.750991</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{Jrss})$</td>
<td>1.890671</td>
<td>2.369569</td>
<td>3.291312</td>
<td>5.733676</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO1rss})$</td>
<td>2.412198</td>
<td>2.726525</td>
<td>3.164204</td>
<td>3.81702</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO3rss})$</td>
<td>2.414396</td>
<td>2.72665</td>
<td>3.160377</td>
<td>3.799801</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{prop})$</td>
<td>2.492892</td>
<td>2.72673</td>
<td>3.35366</td>
<td>6.657089</td>
</tr>
</tbody>
</table>

### Table 3.6: Efficiencies of the estimators for $n = 15$, $m = 5$ and $r = 3$.  

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{eff}(\bar{y})$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{sd})$</td>
<td>1.050763</td>
<td>1.330968</td>
<td>1.814932</td>
<td>2.852279</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{st})$</td>
<td>1.051083</td>
<td>1.330965</td>
<td>1.812657</td>
<td>2.837888</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO1})$</td>
<td>1.09457</td>
<td>1.229862</td>
<td>1.403409</td>
<td>1.634277</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO3})$</td>
<td>1.094375</td>
<td>1.229776</td>
<td>1.401623</td>
<td>1.629365</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{R\beta(\alpha)})$</td>
<td>1.097289</td>
<td>1.330973</td>
<td>1.958309</td>
<td>5.266478</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{SDrss})$</td>
<td>2.087935</td>
<td>2.610542</td>
<td>3.641135</td>
<td>6.506117</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{Jrss})$</td>
<td>2.090183</td>
<td>2.614458</td>
<td>3.645175</td>
<td>6.490633</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO1rss})$</td>
<td>2.784831</td>
<td>3.152106</td>
<td>3.672732</td>
<td>4.460187</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{AO3rss})$</td>
<td>2.788081</td>
<td>3.153052</td>
<td>3.669087</td>
<td>4.446733</td>
</tr>
<tr>
<td>$\text{eff}(\bar{y}_{prop})$</td>
<td>2.921451</td>
<td>3.162086</td>
<td>3.803403</td>
<td>7.169375</td>
</tr>
</tbody>
</table>
Figure 3.1: Comparison of different estimators for $n = 9$
Figure 3.2: Comparison of different estimators for $n = 12$
Figure 3.3: Comparison of different estimators for $n = 15$
CHAPTER 3. MODIFIED RATIO ESTIMATORS

1. \( \bar{y}_{prop} \) is always more efficient than the other estimators.

2. Efficiencies of all ratio-type estimators increases as the value of \( \rho \) increases.

3. For this particular population, all the estimators are approximately unbiased estimators of population mean.

3.7 Conclusion

The proposed ratio type estimator for population mean under SRS is always more efficient than the estimators defined by Sisodia and Dwivedi (1981), Singh and Tailor (2003) and Al-Omari et al (2009). The proposed estimator for population mean is more efficient under RSS design than under SRS design, for any value of \( \alpha \). Also, the proposed estimator is more efficient than the estimator defined by Jemain et al (2008) and Al-Omari et al (2009). Also, at the optimum value of \( \alpha \), \( \bar{y}_{prop} \) is more efficient than \( \bar{y}_{R\beta(\alpha)} \). Hence, the use of proposed estimator of population mean will always be beneficial than the existing ones.