Chapter 5

P – FUZZY SUB-BIGROUPS ITS PROPERTIES 
AND ITS RESULTS

The concept of $P$–fuzzy algebra is elaborated in this chapter with conjugate fuzzy sets, congruence and pseudo fuzzy cosets. Existence theorem in algebra is introduced under $P$–fuzzy algebra and proved. Some properties of $P$–fuzzy algebra are also discussed.

5.1. PRELIMINARIES OF PSEUDO FUZZY COSET

Theorem 5.1.1

Let $\mu$ be a positive fuzzy subgroup of a group $G$ then any two pseudo fuzzy coset of $\mu$ are either identical or disjoint.

Proof

Given that $\mu$ is a positive fuzzy subgroup of a group $G$.

Consider $(a\mu)^p$ and $(b\mu)^p$ are two pseudo fuzzy cosets defined by

$((a\mu)^p)(x) = p(a)\mu(x)$ \hspace{1cm} ... (5.1)

$((b\mu)^p)(x) = p(b)\mu(x)$

For every $x \in G$ and for some $p \in P$. 
Since this pseudo fuzzy coset belongs to the positive fuzzy subgroup \( \mu \) is a positive fuzzy subset of the \( G \).

Relating equation (5.1) and (5.2)

We get,

\[
((a\mu)^p)(x) = ((b\mu)^p)(x)
\]

\[
p(a)\mu(x) = p(b)\mu(x)
\] ...

(5.3)

Case (i)

If \( \mu(x) = \mu(e) \)

Since \( \mu(e) \) is the identity element of the positive fuzzy subgroup then the equation (5.3) becomes

\[
P(a)\mu(e) = P(b)\mu(e)
\]

\[
P(a) = P(b)
\]

\[
(a\mu)^p = (b\mu)^p
\]

\((a\mu)^p\) and \((b\mu)^p\) are identical

Case (ii)

If \( \mu(x) \not\in \mu(e) \)

Then the equation (5.3) becomes \( P(a)\mu(x) \neq P(b)\mu(x) \)

\[
(a\mu)^p \neq (b\mu)^p
\]

\((a\mu)^p\) and \((b\mu)^p\) are disjoint.
Definition 5.1.2

Let \( \mu \) and \( \lambda \) be any two fuzzy subsets of a set \( X \) and \( p \in P \).

The pseudo fuzzy double coset \( (\mu \chi \lambda)^p \) is defined by

\[
(\mu \chi \lambda)^p = (\chi \mu)^p \cap (\chi \lambda)^p \quad \text{for every } x \in X.
\]

Example

Let \( X = \{1, 2, 3\} \) be a set. Take \( \lambda \) and \( \mu \) to be any two fuzzy subsets of \( X \) given by \( \lambda(1) = 0.2, \lambda(2) = 0.8, \lambda(3) = 0.4 \) and \( \mu(1) = 0.5, \mu(2) = 0.6, \mu(3) = 0.7 \).

Then for a positive fuzzy subset \( p \) such that \( p(1) = p(2) = p(3) = 0.1 \)

The pseudo fuzzy double coset \( (\mu \chi \lambda)^p \) is

\[
(\mu \chi \lambda)^p(y) = \begin{cases} 
0.02; & \text{if } y = 1 \\
0.06; & \text{if } y = 2 \\
0.04; & \text{if } y = 3 
\end{cases}
\]

Definition 5.1.3

Let \( \mu \) be a fuzzy normal subgroup of a group \( G \) and \( \mu_t \) be a t-level congruence relation of \( \mu \) on \( G \). Let \( A \) be a nonempty subset of the group \( G \). The congruence class \( \mu_t \) containing the element \( x \) of the group \( G \) is denoted by \([x]_{\mu} \).
Theorem 5.1.4

Let $\mu$ be a fuzzy subgroup of a group $G$ then the pseudo fuzzy coset $(a\mu)^p$ is a fuzzy subgroup of the group $G$ for every $a \in G$.

Proof

This is illustrated by the following example. Let $G$ be the Klein four group. Then $G = \{e, a, b, ab\}$ where $a^2 = e = b^2, ab = ba$ and $e$ the identity element of $G$. Define $\mu : G \to [0,1]$ as follows

$$\mu(x) = \begin{cases} 
1/2 &; \text{if } x = a \\
1 &; \text{if } x = e \\
1/4 &; \text{if } x = b, ab
\end{cases}$$

Take the positive fuzzy subset as follows:

$$p(x) = \begin{cases} 
1 &; \text{if } x = e \\
1/2 &; \text{if } x = a \\
1/3 &; \text{if } x = b \\
1/4 &; \text{if } x = ab
\end{cases}$$

Now we calculate the pseudo fuzzy cosets of $\mu$. For the identity element $e$ of the group $G$ we have $(e\mu)^p = \mu$.

$$(a\mu)^p(x) = \begin{cases} 
1/2 &; \text{if } x = e \\
1/4 &; \text{if } x = a \\
1/8 &; \text{if } x = b, ab
\end{cases}$$
\[
(b^\mu)^\nu(x) = \begin{cases} 
1/3 & \text{if } x = e \\
1/6 & \text{if } x = a \\
1/12 & \text{if } x = b, ab
\end{cases}
\]

and

\[
((ab^\mu)^\nu)(x) = \begin{cases} 
1/4 & \text{if } x = e \\
1/8 & \text{if } x = a \\
1/16 & \text{if } x = b, ab
\end{cases}
\]

Here all the above pseudo fuzzy cosets of \( \mu \) are fuzzy subgroups of \( G \).

**Definition 5.1.5**

Let \( \mu \) be a fuzzy subset of a set \( X \), then \( \Sigma = \{ \lambda : \lambda \text{ is a fuzzy subset of a set } X \text{ and } \lambda \subseteq \mu \} \) is said to be a fuzzy partition of \( \mu \) if

(i) \( \bigcup \lambda = \mu \) and \( \lambda \in \Sigma \)

(ii) any two members of \( \Sigma \) are either identical or disjoint.

**Example**

Let \( X = \mathbb{N} \) be the set of all natural numbers and \( \mu \) be defined by \( \mu(x) = \frac{1}{x} \) for every \( x \in X \). Now consider the collection of fuzzy subset of \( X \) which is given by \( \{ \mu_i \}_{i=1}^n \) where \( \mu_i \)'s are such that

\[
\mu_i(x) = (1 - \frac{1}{u}) \frac{1}{x} \quad \text{for every } x \in X.
\]
For $x \in X$ we have

$$\left( \bigcup_{i=1}^{\alpha} \mu_i \right)(x) = \sup \left\{ \left( 1 - \frac{1}{i} \right) \frac{1}{x} \right\}$$

$$= \frac{1}{x} \left[ \text{as } \frac{1}{i} \rightarrow 0 \text{ as } i \rightarrow \alpha \right]$$

$$= \mu(x)$$

Hence $$\left( \bigcup_{i=1}^{\alpha} \mu_i \right)(x) = \mu(x)$$

for every $x \in X$. That is

$$\left( \bigcup_{i=1}^{\alpha} \mu_i \right) = \mu$$

If $i \neq j$ then it is easy to verify that

$$\left( 1 - \frac{1}{i} \right) \frac{1}{x} \neq \left( 1 - \frac{1}{j} \right) \frac{1}{x}$$

for every $x \in X$. This proves $\mu_i(x) \neq \mu_j(x)$ for every $x \in X$.

Hence $\{\mu_i\}_{i=1}^{\alpha}$ is the fuzzy partition of $\mu$. 
5.2. PROPERTIES OF $P$–FUZZY ALGEBRA USING PSEUDO FUZZY COSET

Theorem 5.2.1

Let $\lambda$ and $\mu$ be two fuzzy subsets in the abelian group $G$ from $P$–fuzzy algebra then $\lambda$ and $\mu$ are conjugate subsets of the group $G$ if and only if $\lambda = \mu$.

Proof

Let $G$ be a fuzzy group from $P$–fuzzy algebra on the algebra $A$. Here $\lambda$ and $\mu$ be two fuzzy subsets if the abelian group $G$ from $P$–fuzzy algebra then $\lambda$ and $\mu$ are conjugate subsets if

$$\lambda(x_1) \ast \ldots \ast \lambda(x_n) = \mu(x_1) \ast \ldots \ast \mu(x_n)$$

for all $x_1, \ldots, x_n \in A$ using conjugacy.

$$\lambda(f(x_1 \ldots x_n)) = \mu(g(x_1 \ldots x_n))$$

for all $x_1, \ldots, x_n \in A$

$$\lambda(x_1) \ast \ldots \ast \lambda(x_n) = \mu(gx_1g^{-1}) \ast \ldots \ast \mu(gx_ng^{-1})$$

relating the co-efficient of $x_1$

$$\lambda(x) = \mu(gx_1g^{-1})$$

for every $x \in G$

$$= \mu(g^{-1}gx_1)$$

for every $x \in G$

$$\lambda(x_1) = \mu(x_1)$$

for every $x \in G$
proceeding in this manner, for $n$ values and we get
\[ \lambda(x_2) = \mu(x_2), \ldots, \lambda(x_n) = \mu(x_n) \]
\[ \lambda = \mu \]

Conversely,

Given that $\lambda = \mu$ then $\lambda(x) = \mu(x)$

Extending $x$ as a function of $n$ variables, we get
\[ \lambda(f(x_1 \ldots x_n)) = \mu(g(x_1 \ldots x_n)) \]
\[ \lambda(x_1) \ast \cdots \ast \lambda(x_n) = \mu(x_1) \ast \cdots \ast \mu(x_n) \] indicates the condition for conjugate sets.

**Theorem 5.2.2**

Let $\mu$ be a fuzzy subgroup of $P$–fuzzy algebra on the algebra $A$, then the pseudo fuzzy coset $(a\mu)^P$ is a fuzzy subgroup of the $P$–fuzzy algebra $A$ for every $a \in A$.

**Proof**

Let $\mu$ be a fuzzy subgroup of $A$ then using the theorem 3.4.1. Consider a $P$–fuzzy sub algebra on the algebra $A$ then there exists a $P$–fuzzy set $\mu \in P^A$ then for

(i) $n$–ary operation
\[\mu(f(x_1,\ldots,x_n)) \geq \mu(x_1) \ast \cdots \ast \mu(x_n) \text{ for all } x_1,\ldots,x_n \in A \ldots (5.4)\]

Restrict \( n \) elements into two, \( x_1 = x \) and \( x_2 = y \)

\[\mu(f(xy)) \geq \mu(x) \ast \mu(y) \ldots (5.5)\]

from equation (5.4) and (5.5)

\[\mu(x) \ast \mu(y) = \mu(x \ast y) \geq \min\{\mu(x),\mu(y)\} \ldots (5.6)\]

Let \( \mu : X \to [0,1] \)

Here \( \mu \) is a function defined that \( X \) is a set which maps it each and every element between 0 and 1. Also the set of all elements of \( x^{-1} \) maps the values between 0 and 1.

Therefore \( \mu : x^{-1} \to [0,1] \)

\[\mu(x^{-1}) = \mu(x) \ldots (5.7)\]

(ii) nullary operation (for any constant)

If there exist any constant \( \mu(c) \) then

\[\mu(c) \ast \mu(x) = \mu(c \ast x) \geq \min\{\mu(c),\mu(x)\} = \mu(x) \ldots (5.8)\]

\( \mu(c) \geq \mu(x) \) for all \( x \in A \).

since all the elements of \( \mu(x) \) lie between 0 and 1 and \( \mu(c) \) is any constant it is equal to 1.
Using the results (5.6) (5.7) and (5.8), it is clear that every 
$P$–fuzzy subalgebra is a subgroup.

Let $x_1, x_2 \in A$

$\mu(f(x_1,x_2))^p = a \mu(f(xy^{-1})^p) \text{ using pseudo fuzzy coset } (\because x_2 = y^{-1})$

$= P(a)[\mu(x)^* \mu(y^{-1})]$ 

$\geq P(a) \mu(xy^{-1})$

$\geq P(a) \min\{\mu(x), \mu(y)\}$

$= \min\{P(a)\mu(x), P(a)\mu(y)\}$

$\geq \min\{(a\mu)^p(x), (a\mu)^p(y)\}$ for every $x, y \in G$

$\Rightarrow$ Every pseudo fuzzy coset is a fuzzy subgroup of the $P$–fuzzy algebra.

Theorem 5.2.3

Let $\lambda$ and $\mu$ be any two fuzzy subsets of a set $X$ from $P$–
fuzzy algebra $A$, then for $a \in X,(a\mu)^p \subset (a\lambda)^p$ iff $\mu \subset \lambda$.

Proof

Consider $\lambda$ and $\mu$ be two fuzzy sets.

Let $(a \mu)^p \subset (a \lambda)^p$

$[a \mu(f(x_1...x_n))]^p \subseteq [a \lambda(g(x_1...x_n))]^p$
Using pseudo fuzzy coset

\[ P(a)[\mu(x_1) \ast \cdots \ast \mu(x_n)] \subseteq P(a)[\lambda(x_1) \ast \cdots \ast \lambda(x_n)] \]

Using cancellation law

\[ \mu(x_1) \ast \cdots \ast \mu(x_n) \subseteq \lambda(x_1) \ast \cdots \ast \lambda(x_n) \]

using theorem 5.2.1

\[ \mu(x) \subseteq \lambda(X) \]

Conversely,

Let \( \mu(x) \subseteq \lambda(X) \), extending \( x \) as function of \( n \) variables, we get,

\[ \mu(x_1) \ast \cdots \ast \mu(x_n) \subseteq \lambda(x_1) \ast \cdots \ast \lambda(x_n) \]

Premultiply by \( P(a) \) on both sides

\[ P(a)[\mu(x_1) \ast \cdots \ast \mu(x_n)] \subseteq P(a)[\lambda(x_1) \ast \cdots \ast \lambda(x_n)] \text{ for all } x_1 \ldots x_n \in A \]

\[ P(a)[\mu f(x_1 \ldots x_n)] \subseteq P(a)[\lambda(g(x_1 \ldots x_n))] \text{ for all } x_1 \ldots x_n \in A \]

\[ \Rightarrow (a\mu)^P \subseteq (a\lambda)^P \]

\[ \Rightarrow \text{If } \mu \subseteq \lambda \text{ then } (a\mu)^P \subseteq (a\lambda)^P \]

**Theorem 5.2.4 (Fuzzy Existence Theorem)**

Let \( \mu \) be a fuzzy subgroup of a group \( G \) from \( P \)-fuzzy algebra \( A \). The congruence class \( [x]_\mu \) of \( \mu_x \) containing the element
$X$ of the group $G$ exists only when $\mu$ is a fuzzy normal subgroup of the group $G$.

**Proof**

Let $\mu$ be a fuzzy subgroup of a group $G$ from $P$–fuzzy algebra $A$.

If $\mu$ is a fuzzy normal subgroup of a group $G$ then the t-level relation $\mu_t$ of $\mu$ is a congruence relation on the group $G$ and hence the congruence class $[x]_{\mu_t}$ of $\mu_t$ containing the element $x$ of the group $G$ exists.

It is proved that the existence of the congruence class $[x]_{\mu_t}$ one must have the fuzzy normal subgroup of group $G$.

That is, if $\mu$ is not a fuzzy normal subgroup of the group $G$, the congruence class $[x]_{\mu_t}$ of $\mu_t$ containing the element $x$ of the group $G$ does not exists.

Consider the alternating group $A_4$ choose $P_1, P_2, P_3 \in [0,1]$ such that $1 > P_1 > P_2 > P_3 \geq 0$. Define $\mu : A_4 \to [0,1]$ by
\[ \mu(x) = \begin{cases} 
1 & \text{if } x = e \\
P_1 & \text{if } x = (12)(34) \\
P_2 & \text{if } x = (14)(23),(13)(24) \\
P_3 & \text{Otherwise} 
\end{cases} \]

\{e\}, \{e,(12),(34)\}, \{e,(12)(34),(13)(24),(14)(23)\} and \(A_4\).

\[ \mu \text{ is a fuzzy subgroup of } A_4. \]

For \(x = (123)\) \(y = (143)\)

\[ \mu(xy) = \mu((123)(143)) \]

\[ = \mu((12)(34)) \]

\[ = P_1 \]

\[ \mu(yx) = \mu((143)(123)) \]

\[ = \mu((14)(23)) \]

\[ = P_2 \]

Therefore \(\mu(xy) \neq \mu(yx)\)

For \(x = (123)\) and \(y = (143)\)

\[ \mu \text{ is not a fuzzy normal subgroup of } A_4. \]

**Definition 5.2.5**

Let \(\mu\) be a fuzzy subgroup of a group \(G\) from \(P\)-fuzzy algebra \(A\) then it is said to be a \(P\)-Positive fuzzy subgroup \(G\) if \(\mu\) is a positive fuzzy subset of the group of \(G\) of \(A\).
Example

Let $X = \{1, 2, 3\}$ be a set. Take $\lambda$ and $\mu$ to be any two fuzzy subsets of $X$ given by $\lambda(1) = 0.2, \lambda(2) = 0.8, \lambda(3) = 0.4$ and $\mu(1) = 0.5, \mu(2) = 0.6, \mu(3) = 0.7$. Then for a positive fuzzy subset $p$ such that, $p(1) = p(2) = p(3) = 0.1$.

The pseudo fuzzy double coset

\[
(\mu x \lambda)^p(y) = \begin{cases} 
0.02; & \text{if } y = 1 \\
0.06; & \text{if } y = 2 \\
0.04; & \text{if } y = 3 
\end{cases}
\]

Theorem 5.2.6

Let $\mu$ be a $P-$Positive fuzzy subgroup of a group $G$ of $P-$fuzzy algebra $A$, then any two pseudo fuzzy cosets of $\mu$ are either identical or disjoint.

Proof

Given $\mu$ be a $P-$Positive fuzzy subgroup of a group $G$ of $P-$fuzzy algebra $A$.

Consider $(a\mu)^p$ and $(b\mu)^p$ are pseudo fuzzy cosets of $\mu$ defined by $(a\mu)^p = [a\mu(f(x_1...x_n))]^p$

and $(b\mu)^p = [b\mu(f(x_1...x_n))]^p$
for every \( x_1, x_2, \ldots, x_n \in A \) and for some \( p \in P \) using pseudo fuzzy coset definition, it can be written as

\[
[a\mu(f(x_1, \ldots, x_n))]^p = P(a)[\mu(x_1) \ast \cdots \ast \mu(x_n)] \quad \text{for all} \quad x_1, x_2, \ldots, x_n \in A
\]

Similarly

\[
[b\mu(f(x_1, \ldots, x_n))]^p = P(b)[\mu(x_1) \ast \cdots \ast \mu(x_n)] \quad \text{for all} \quad x_1, x_2, \ldots, x_n \in A
\]

Since these two pseudo fuzzy cosets belong to the \( P \)–Positive fuzzy subgroup \( \mu \) is a \( P \)–Positive fuzzy subset of the group \( G \) therefore relating the two pseudo fuzzy cosets, we get

\[
(a\mu)^p = (b\mu)^p
\]

\[
[a\mu(f(x_1, x_n))]^p[b\mu(f(x_1, x_n))]^p \quad \text{for all} \quad x_1, x_2, \ldots, x_n \in A \quad \text{and for some} \quad p \in P, \\
P(a)[\mu(x_1) \ast \cdots \ast \mu(x_n)] = P(b)[\mu(x_1) \ast \cdots \ast \mu(x_n)] \quad \text{for all} \quad x_1, x_2, \ldots, x_n \in A \quad \text{and for some} \quad p \in P.
\]

In general, \( P(a)[\mu(x)] = P(b)[\mu(x)] \)

In this result two case arise,

**Case (i)**

If \( \mu(x) = \mu(e) = e \) since the two pseudo fuzzy cosets are from \( P \)–Positive fuzzy subset it has an identity element \( e \).

\[
\Rightarrow P(a)\mu(e) = P(b)\mu(e)
\]
\[ \Rightarrow P(a) = P(b) \]
\[ \Rightarrow (a\mu)^P \text{ and } (b\mu)^P \text{ are identical}. \]

Case (ii)

If \( \mu(x) \neq \mu(e) \)

If the elements relating the pseudo fuzzy coset are not identical then \( P(a) \neq P(b) \). Using the two cases, it is clear that any two pseudo fuzzy cosets of \( \mu \) from a \( P \)-Positive fuzzy subgroup are either identical or disjoint.

**Theorem 5.2.7**

Let \( \mu \) be a fuzzy subgroup of a group \( G \) with \( 3 \leq 0(\text{Im}(\mu)) < \infty \) then there exists two fuzzy subgroups \( \mu_1 \) and \( \mu_2 \) of \( \mu (\mu_1 \neq \mu \text{ and } \mu_2 \neq \mu_1) \) such that \( \mu = \mu_1 \cup \mu_2 \).

**Proof**

Let \( \mu \) be a fuzzy subgroup of a group \( G \). Suppose \( \text{Im}(\mu) = \{a_1, a_2, \ldots, a_n\} \) where \( 3 \leq n < \infty \) and \( a_1 > a_2 > \ldots > a_n \). Choose \( b_1, b_2 \in [0,1] \) be such that \( a_1 > b_1 > a_2 > b_2 > a_3 > b_3 > \ldots > a_n \) and define \( \mu_1, \mu_2 : G \to [0,1] \) by
\[
\mu_1(x) = \begin{cases} 
    a_1 & \text{if } x \in \mu a_1 \\
    b_2 & \text{if } x \in \mu a_2 / \mu a_1 \\
    \mu(x) & \text{Otherwise}
\end{cases}
\]

and

\[
\mu_2(x) = \begin{cases} 
    b_1 & \text{if } x \in \mu a_2 \\
    a_2 & \text{if } x \in \mu a_2 / \mu a_1 \\
    \mu(x) & \text{Otherwise}
\end{cases}
\]

Thus it can be easily verified that both \( \mu_1 \) and \( \mu_2 \) are fuzzy subgroups of \( \mu \). Further \( \mu_1 \neq \mu, \mu_2 \neq \mu \) and \( \mu = \mu_1 \cup \mu_2 \).

5.3. \textbf{P–FUZZY SUB-BIGROUP – DEFINITION}

Definition 5.3.1

Let \( G = (G_1 \cup G_2, \cdot, \cdot) \) be a bigroup, then \( \mu : G \to [0,1] \) is said to be \( P \)–Fuzzy sub-bigroup of the bigroup \( G \) if there exist two fuzzy subset \( \mu_1 \) of \( G_1, \mu_2 \) of \( G_2 \) such that

i. \( (\mu_1, \cdot) \) is a \( P \)–fuzzy subgroup of \( (G_1, \cdot) \)

ii. \( (\mu_1, \cdot) \) is a \( P \)–fuzzy subgroup of \( (G_1, \cdot) \)

iii. \( \mu = \mu_1 \cup \mu_2 \)

Definition 5.3.2

Let \( G = (G_1 \cup G_2, \cdot, \cdot) \) be a bigroup and \( \mu = (\mu_1 \cup \mu_2) \) be a \( P \)–fuzzy sub-bigroup of the bigroup \( G \). The bilevel subset of the
A $P$-fuzzy sub-bigroup $\mu$ of the bigroup $G$ is said to be a bigroup $G'_\mu = G'_{1\mu_1} \cup G'_{2\mu_2}$ for every $t \in [0, \min\{\mu_1(e_1), \mu_2(e_2)\}]$, where $e_1$ denotes the identity element of the group $(G_1,+)$ and $e_2$ denotes the identity element of the group $(G_2,\cdot)$.

**5.4. PROPERTIES OF $P$–FUZZY SUB-BIGROUP**

**Theorem 5.4.1**

Every $t$-level subset of a $P$–fuzzy sub-bigroup $\mu$ of a bigroup $G$ need not in general be a sub-bigroup of the bigroup $G$.

**Proof**

Let $G = \{a_1, a_2, a_3\}$ be a bigroup under the binary operation $+$ and $\cdot$ where $G_1 = \{a_2\}$ and $G_2 = \{a_1, a_3\}$ are groups respectively with respect to usual addition and usual multiplication.

Define $\mu : G \to [0, 1]$

$$\mu(x) = \begin{cases} 
  b_1 & \text{if } x = a_1, a_3 \\
  b_2 & \text{if } x = a_2 
\end{cases}$$

Then clearly $(\mu, +, \cdot)$ is a $P$–fuzzy sub-bigroup of the bigroup $(G, +, \cdot)$. Now consider the level subset $G^h_\mu$ of the $P$–fuzzy sub-bigroup $\mu$. 

\[ G^b_\mu = \{ x \in G / \mu(x) \geq b \} = \{ a_1, a_3 \} \]

\{a_1, a_3\} is not a sub-bigroup of the bigroup \((G_1, +, .)\)

\[ G^t_\mu \text{ (for } t = b) \] of the \(P\)-fuzzy sub-bigroup \(\mu\) is not a sub-bigroup of the bigroup \((G_1, +, .)\).

**Example**

\(G = \{-1, 0, 1\}\) be a bigroup under the binary operation \(+\) and \(.\) where \(G_1 = \{0\}\) and \(G_2 = \{-1, 1\}\) are groups respectively with respect to usual addition and multiplication.

Define: \(\mu : G \rightarrow [0, 1]\) by

\[ \mu(x) = \begin{cases} 
1/2; & \text{if } x = -1, 1 \\
1/4; & \text{if } x = 0 
\end{cases} \]

Then clearly \((\mu, +, .)\) is a \(P\)-fuzzy sub-bigroup of the bigroup \((G, +, .)\).

Now consider the level subset

\[ G^{1/2}_\mu = \{ x \in G / \mu(x) \geq 1/2 \} = \{-1, 1\} . \]

It is easy to verify that \(-1, 1\) is not a sub-bigroup of the bigroup \((G, +, .)\). Hence the t-level subset \(G^t_\mu\) (for \(t = 1/2\)) of the
\(P\)-fuzzy sub-bigroup \(\mu\) is not a sub-bigroup of the bigroup \((G, +, \cdot)\).

**Theorem 5.4.2**

Every bilevel subset of a \(P\)-fuzzy sub-group \(\mu\) of a bigroup \(G\) is a sub-bigroup of the bigroup \(G\).

**Proof**

Let \(\mu = \mu_1 \cup \mu_2\) be the \(P\)-fuzzy subgroup of a bigroup \((G = G_1 \cup G_2, +, \cdot)\). Consider the bilevel subset \(G'_\mu\) of the \(P\)-fuzzy sub-bigroup for every \(t \in [0, \min\{(e_1) \cup (e_2)\}]\) where \(e_1\) denotes the identity element of the group \((G_1, +)\) and \(e_2\) denotes the identity element of the group \((G_2, \cdot)\), then \(G'_\mu = G'_{\mu_1} \cup G'_{\mu_2}\) where \(G'_{\mu_1}\) and \(G'_{\mu_2}\) are subgroups of \(G_1\) and \(G_2\) respectively (since \(G'_\mu\) is a \(t\)-level subset of the group \(G_1\) and \(G'_\mu\) is a \(t\)-level subset of \(G_2\)).

**Example**

\(G = \{0, \pm 1, \pm i\}\) is a bigroup with respect to addition modulo 2 and \((\cdot)\). Clearly \(G_1 = \{0, 1\}\) and \(G_2 = \{\pm 1, \pm i\}\) are group with respect to addition modulo and \((\cdot)\) respectively.
Define: \( \mu : G \rightarrow [0, 1] \) by

\[
\mu(x) = \begin{cases} 
1; & \text{if } x = 0 \\
0.5; & \text{if } x = \pm 1 \\
0.2; & \text{if } x = \pm i 
\end{cases}
\]

Since \( \mu \) is a \( P \)-fuzzy sub-bigroup of the bigroup \( G \) as there exist two fuzzy subgroups \( \mu_1 : G \rightarrow [0,1] \) and \( \mu_2 : G \rightarrow [0,1] \) such that

\[
\mu = \mu_1 \cup \mu_2 \text{ where } \\
\mu_1(x) = \begin{cases} 
1; & \text{if } x = 0 \\
0.5; & \text{if } x = 1 
\end{cases}
\]

and

\[
\mu_2(x) = \begin{cases} 
0.5; & \text{if } x = \pm 1 \\
0.2; & \text{if } x = \pm i 
\end{cases}
\]

Now we calculate the bilevel subset \( G'_\mu \) for \( t = 0.5 \)

\[
G'_\mu = G'_{t_1} \cup G'_{t_2} = \{ x \in G_1 / \mu_1(x) \geq t \} \cup \{ x \in G_2 / \mu_2(x) \geq t \} \\
= \{ 0 \} \cup \{ \pm 1 \} \\
G'_\mu = \{ 0, \pm 1 \}
\]
Theorem 5.4.3

Let $\mu = \mu_1 \cup \mu_2$ be a $P$-fuzzy sub-bigroup of a group $G$, where $\mu_1$ and $\mu_2$ are $P$-fuzzy subgroups of the group $G$, for $t \in [0, \min\{\mu_1(e), \mu_2(e)\}]$ the level subset $G^t_\mu$ of $\mu$ can be represented as the union of two subgroups of the group $G$, that is $G^t_{\mu_1} \cup G^t_{\mu_2}$.

Proof

Let $\mu = \mu_1 \cup \mu_2$ be a $P$-fuzzy sub-bigroup of a group $G$ and for $t \in [0, \min\{\mu_1(e), \mu_2(e)\}]$. This implies that there exist $P$-fuzzy subgroups $\mu_1$ and $\mu_2$ of the group $G_1$ such that $\mu = \mu_1 \cup \mu_2$.

Let $G^t_\mu$ be the level subset of $\mu$ then

we have $x \in G^t_\mu \iff \mu(x) \geq t$

$\iff \max\{\mu_1(x), \mu_2(x)\} \geq t$

$\iff \mu_1(x) \geq t$ or $\iff \mu_2(x) \geq t$

$\iff x \in G^t_{\mu_1}$ or $\iff x \in G^t_{\mu_2}$

If and only if $x \in G^t_{\mu_1} \cup G^t_{\mu_2}$

Hence $G^t_\mu = G^t_{\mu_1} \cup G^t_{\mu_2}$
Theorem 5.4.4

Every $P$–fuzzy sub-bigroup of a group $G$ is a $P$–fuzzy subgroup of the group but not conversely.

Proof

Every $P$–fuzzy sub-bigroup of a group $G$ is a $P$–fuzzy subgroup of the group $G$.

Using the definition of $P$–fuzzy sub-bigroup, $G = (G_1 \cup G_2, +, .)$ be a bigroup, then $\mu: G \to [0, 1]$ is said to be $P$–fuzzy sub-bigroup of the bigroup $G$ if there exist two fuzzy subset $\mu_1$ of $G_1, \mu_2$ of $G_2$ such that,

i. $(\mu_1, +)$ is a $P$–fuzzy subgroup of $(G_1, +)$

ii. $(\mu_2, .)$ is a $P$–fuzzy subgroup of $(G_2, .)$

iii. $\mu = \mu_1 \cup \mu_2$.

Therefore every subset $G$ is a fuzzy subgroup of the group $G$. Conversely, if $\mu$ is a $P$–fuzzy subgroup of the group $G$ and there does not exist two $P$–fuzzy subgroups $\mu_1$ and $\mu_2$ of $\mu$($\mu_1 \neq \mu$ and $\mu_2 \neq \mu$) such that $\mu = \mu_1 \cup \mu_2$.

$\mu$ is a $P$–fuzzy sub-bigroup of the group $G$. 
Necessary not sufficient condition for a fuzzy subgroup to be a $P$–fuzzy sub-bigroup of $G$.

**Theorem 5.4.5**

Let $\mu$ be a fuzzy subset of a group $G$ with $3 \leq 0 \ (\text{Im}(\mu)) < \infty$. Then $\mu$ is a fuzzy subgroup of the group $G$ if and only if $\mu$ is a $P$–fuzzy sub-bigroup of $G$.

**Proof**

Let $\mu$ be a fuzzy subgroup of the group $G$ with $3 \leq 0 \ (\text{Im}(\mu)) < \infty$ then there exist two $P$–fuzzy subgroup $\mu$, and $\mu_e$ of $\mu(\mu_1 \neq \mu$ and $\mu_e \neq \mu$) such that $\mu = \mu_1 \cup \mu_2$.

Hence $\mu$ is a $P$–fuzzy sub-bigroup of the group $G$.

Conversely,

Let $\mu$ be a $P$–fuzzy sub-bigroup of a group $G$, we know that every $P$–fuzzy sub-bigroup of a group $G$ is a fuzzy subgroup of the group $G$ we illustrate this theorem by example.
Example

Define $\mu : G \to [0,1]$ where $G = \{1, -1, i, -i\}$ by

$$
\mu(x) = \begin{cases} 
1 & \text{if } x = 1 \\
0.9 & \text{if } x = -1 \\
0.8 & \text{if } x \pm i
\end{cases}
$$

It is easy to prove that $\mu$ is a fuzzy subgroup of the group $G$ and $O(\text{Im}(\mu)) = 3$. Further it can be easily verified that there exist two $P-$ fuzzy subgroup $\mu$ and $\mu_\circ$ of $\mu(\mu_1 \neq \mu$ and $\mu_2 \neq \mu$) such that $\mu = \mu_1 \cup \mu_2$ where $\mu_1 \cup \mu_2 : G \to [0,1]$ are defined by

$$
\mu_1(x) = \begin{cases} 
0.9 & \text{if } x = 1 \\
0.8 & \text{if } x = -1, \pm i
\end{cases}
$$

and

$$
\mu_2(x) = \begin{cases} 
1 & \text{if } x = 1 \\
0.9 & \text{if } x = -1 \\
0.7 & \text{if } x = \pm i
\end{cases}
$$