Stochastic Analysis Of A Cold Standby System With Server Failure and Conditional Arrival Time

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Abstract
A redundant system of two-identical units has been analyzed stochastically in detail by considering server failure while performing jobs. Initially one unit is operative and other is kept as spare in cold standby. There is a single server who takes some time to arrive at the system with the condition that he has to attend the system immediately at its complete failure. The failed server undergoes for treatment and resumes jobs with full efficiency after treatment. The random variables associated with the failure rate of the unit and server, repair rate of the unit, treatment rate and arrival rate of the server are statistically independent. The distributions for failure time of the unit and the server are taken as negative exponential while that of repair rate, treatment rate and arrival time of the server are arbitrary with different probability density functions. The semi-Markov process and regenerative point technique are adopted to derive the expressions for some measures of system effectiveness in steady state. Graphs are drawn to depict the graphical behavior of some important performance measures for arbitrary values of various parameters and costs.

Keywords: Redundant System, Server Failure, Treatment, Repair, Conditional Arrival Time and Stochastic Analysis.

1. Introduction
The method of redundancy has been proved as an effective strategy for improving durability and performance of repairable systems. Therefore, stochastic modeling of such systems has been done by the researchers including Murari and Goyal (1984) and Cao and Wu (1989) under different repair polices. But, most of these redundant systems have been investigated under a common assumption that repair facility neither fails nor deteriorates during jobs. Infect, this assumption seems to unrealistic when repair facility meets with an accident due to the reasons.
like mishandling, electric shocks and carelessness on the part of the server, etc. In such a situation, treatment may be given to the server in order to resume the job. Dhankhar and Malik (2010) studied a single-unit system with the concept of server failure during repair. Later on, they (2013) carried out cost analysis of that system with replacement of the server and unit subject to inspection. Further, it is not always possible for a server to reach at the system immediately when required may because of his pre occupations. In such a situation server may take some time (called arrival time) to reach at the system with the condition that he has to attend the system immediately when system has no standby unit to work. Malik and Bhardwaj (2007) probed a 2-out-of-3 redundant system with conditional arrival time of server.

Keeping in view the practical situations, the purpose of the present study is to make stochastic analysis of a cold standby system of two identical units with server failure. There is a single server who is allowed to take some time to arrive at the system to carry out repair activities subject to the condition that he has to attend the system immediately when there is no stand by unit for working in the system. The server is subjected to failure while performing jobs and goes for treatment. And, the failed unit waits for repair till the server becomes in good condition after receiving treatment. The unit works as new after repair. The random variables associated with the failure rate, treatment and arrival rates are statistically independent. The distributions for failure time of the unit and the server are taken as negative exponential while that of repair rate, treatment rate and arrival time of the server are arbitrary with different probability density functions. The semi Markov process and regenerative point technique are adopted to derive the expressions for some measures of system effectiveness in steady state such as transition probabilities, mean sojourn times, mean time to system failure (MTSF), availability, busy period of the server, expected number of repairs and treatments and profit functions. The graphical behavior of some important reliability characteristic has been observed for arbitrary values of the various parameters and costs.

2. Notations

\( E \) : Set of regenerative states
\( \bar{E} \) : Set of non-regenerative states
\( \Lambda \) : Constant failure rate
\( \text{FUr/FWr} \) : The unit is failed and under repair/waiting for repair
\( \text{SFUt/SFUT} \) : The server is failed and under treatment/continuously under treatment from previous state
FUR/FWR: The unit is failed and under repair / waiting for repair continuously from previous state

g(t)/G(t): pdf/cdf of repair time of the unit

f(t)/F(t): pdf/cdf of preventive maintenance time of the unit

w(t)/W(t): pdf/cdf of arrival time of the server

q_{ij}(t)/Q_{ij}(t): pdf/cdf of passage time from regenerative state S_i to a regenerative state S_j or to a failed state S_j without visiting any other regenerative state in (0, t]

q_{ij,k}(t)/Q_{ij,k}(t): pdf/cdf of direct transition time from regenerative state S_i to a regenerative state S_j or to a failed state S_j visiting state S_k, S_r once in (0, t]

M_i(t): Probability that the system up initially in state S_i ∈ E is up at time t without visiting to any regenerative state

W_i(t): Probability that the server is busy in the state S_i up to time ‘t’ without making any transition to any other regenerative state or returning to the same state via one or more non-regenerative states.

\( \mu_i \): The mean sojourn time in state \( S_i \) which is given by

\[ \mu_i = E(T) = \int_0^\infty P(T > t) \, dt = \sum_j m_{ij}, \]

where \( T \) denotes the time to system failure.

\( m_{ij} \): Contribution to mean sojourn time (\( \mu_i \)) in state \( S_i \) when system transits directly to state \( S_j \) so that \( \mu_i = \sum_j m_{ij} \) and \( m_{ij} = \int_0^t dQ_{ij}(t) = -q_{ij}^*(0) \)

\&/©: Symbol for Laplace-Stieltjes convolution/Laplace convolution

*/**: Symbol for Laplace Transformation /Laplace Stieltjes Transformation

Fig.: 1State Transition Diagram
3. Transition Probabilities and Mean Sojourn Times

Simple probabilistic considerations yield the following expressions for the non-zero elements.

\[ p_{ij} = Q_{ij}(\infty) = \int_{0}^{\infty} q_{ij}(t) \, dt \]

\[ p_{01} = 1 \quad p_{12} = \frac{\mu}{\mu + \lambda} (1 - g^*(\mu + \lambda)) \quad p_{13} = 1 - w^*(\lambda) \]

\[ p_{20} = g^*(\mu + \lambda) \quad p_{24} = \frac{\mu - \lambda}{\mu + \lambda} \]

\[ p_{32} = g^*(\mu) \quad p_{37} = 1 - g^*(\mu) \quad p_{42} = f^*(\lambda) \]

\[ p_{45} = 1 - f^*(\lambda) \quad p_{58} = f^*(0) \quad p_{62} = g^*(\mu) \]

\[ p_{67} = 1 - g^*(\mu) \quad p_{78} = f^*(0) \quad p_{82} = g^*(\mu) \]

\[ p_{87} = 1 - g^*(\mu) \]

For \( w(t) = \theta e^{-\theta t} \), \( g(t) = \beta e^{-\beta t} \) and \( f(t) = \alpha e^{-\alpha t} \)

\( \mu_0 = 1 \quad \mu_1 = \frac{1}{\theta + \lambda} \quad \mu_2 = \frac{1}{\beta + \lambda + \mu} \quad \mu_3 = \frac{1}{\beta + \mu} \quad \mu_4 = \frac{1}{\alpha + \lambda} \)

\( \mu_5 = 1 \quad \mu_6 = \frac{1}{\beta + \mu} \quad \mu_7 = 1 \quad \mu_8 = \frac{1}{\beta + \mu} \)

The mean sojourn times \( (\mu_i) \) in the state \( S_i \) are given by

\[ \phi_0(t) = Q_{01}(t) \quad \phi_1(t) = Q_{12}(t) + Q_{13}(t) \]

4. Reliability and Mean Time To System Failure (MTSF)

Let \( \phi_i(t) \) be the cdf of first passage time from regenerative state \( S_i \) to a failed state. Regarding the failed state as absorbing state, we have the following recursive relations for \( \phi_i(t) \) as:

\[ \phi_0(t) = Q_{01}(t) & \phi_1(t) \]

\[ \phi_1(t) = Q_{12}(t) + Q_{13}(t) \]
\[ \phi_2(t) = Q_{20}(t) \& \phi_0(t) + Q_{24}(t) & \phi_4(t) + Q_{26}(t) \]
\[ \phi_4(t) = Q_{42}(t) \& \phi_2(t) + Q_{45}(t) \]  \hspace{1cm} (4)

Taking \ LST\ of\ equation\ (4)\ and\ solving\ for\ \phi_0^*(s)

We have

\[ R^*(s) = \frac{1-\phi_0^*(s)}{s} \]  \hspace{1cm} (5)

The reliability of the system model can be obtained by taking Laplace Inverse Transform of the equation (5). The mean time to system failure (MTSF) is given by

\[ MTSF = \lim_{s \to 0} \frac{1-\phi_0^*(s)}{s} = \frac{N_1}{D_1} \]  \hspace{1cm} (6)

Where, \[ N_1 = (1 - p_{24}p_{42})(\mu_0 + \mu_1) + p_{12}(\mu_2 + p_{24}\mu_4) \]

And \[ D_1 = 1 - p_{24}p_{42} - p_{20}p_{12}p_{10} \]  \hspace{1cm} (7)

5. Steady State Availability

Let \( A_i(t) \) be the probability that the system is in up-state at an instant ‘t’ given that the system entered regenerative state \( S_i \) at \( t = 0 \). The recursive relations for \( A_i(t) \) are given as:

\[ A_0(t) = M_0(t) + q_{01}(t)\circ A_1(t) \]
\[ A_1(t) = M_1(t) + q_{12}(t)\circ A_2(t) + q_{13}(t)\circ A_3(t) \]
\[ A_2(t) = M_2(t) + q_{20}(t)\circ A_0(t) + \left( q_{22.6}(t) + q_{22.6(7,8)^n}(t) \right) \circ A_2(t) + q_{24}(t)\circ A_4(t) \]
\[ A_3(t) = \left( q_{32}(t) + q_{32(7,8)^n}(t) \right) \circ A_2(t) \]
\[ A_4(t) = M_4(t) + \left( q_{42}(t) + q_{42.58}(t) + q_{42.58(8,7)^n}(t) \right) \circ A_2(t) \]  \hspace{1cm} (8)

Where,

\[ M_0(t) = e^{-\lambda t} \]
\[ M_1(t) = e^{-\lambda t\bar{W}(t)} \]
\[ M_2(t) = e^{-(\mu_2+\lambda)t \bar{G}(t)} \]
\[ M_4(t) = e^{-\lambda t\bar{F}(t)} \]

Taking \ LST\ of\ equation\ (8)\ and\ solving\ for\ \phi_0^*(s),\ the\ steady\ state\ availability\ is\ given\ by

\[ A_0(\infty) = \lim_{s \to 0} \angle A_0(s) = \frac{N_2}{D_2} \]  \hspace{1cm} (9)

Where, \[ N_2 = p_{20}(\mu_0 + \mu_1 + \mu_3p_{13}) + \mu_2 + \mu_4p_{24} \]

And \[ D_2 = p_{20}(\mu_0 + \mu_1 + p_{13}\mu_3) + \mu_2 + p_{24}\mu_4 \]  \hspace{1cm} (10)

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6. Busy Period of the Server Due to Repair
Let $B_i^R(t)$ be the probability that the server is busy in repair the unit at an instant ‘$t$’ given that the system entered regenerative state $S_i$ at $t = 0$. The recursive relations for $B_i^R(t)$ are as follows:

\[ B_0^R(t) = q_{01}(t) \odot B_1^R(t) \]
\[ B_1^R(t) = q_{12}(t) \odot B_2^R(t) + q_{13}(t) \odot B_3^R(t) \]
\[ B_2^R(t) = W_2^R(t) + q_{20}(t) \odot B_0^R(t) + \left( q_{22.6}(t) + q_{22.6(7,8)^n}(t) \right) \odot B_2^R(t) + q_{24}(t) \odot B_4^R(t) \]
\[ B_3^R(t) = W_3^R(t) + (q_{32}(t) + q_{32(7,8)^n}(t)) \odot B_2^R(t) \]
\[ B_4^R(t) = \left( q_{42}(t) + q_{42.58}(t) + q_{42.5(8,7)^n}(t) \right) \odot B_2^R(t) \]

Where, $W_2^R(t) = e^{-(\mu + \lambda)t} \overline{G(t)} + (\lambda e^{-(\mu + \lambda)t} \odot 1) \overline{G(t)}$

$W_3^R(t) = e^{-\mu t} \overline{G(t)} + (g(t)e^{-\mu t} \odot 1)$

Taking LT of the above relations (11) and solving for $B_0^R(s)$. The time for which server is busy due to repair is given by

\[ B_0^R(\infty) = \lim_{s \to 0} sB_0^{**}(s) = \frac{N_3}{D_2} \]  

\[ N_3 = p_{01}(W_2^*(0) + W_3^*(0)p_{13}p_{20}) \text{ and } D_2 \text{ is already specified.} \] (13)

### 7. Expected Number of Repairs of the Unit

Let $R_i(t)$ be the expected number of repairs by the server in (0, $t$] given that the system entered the regenerative state $S_i$ at $t = 0$. The recursive relations for $R_i(t)$ are given as:

\[ R_0(t) = Q_{01}(t) \oslash R_1(t) \]
\[ R_1(t) = Q_{12}(t) \oslash R_2(t) + Q_{13}(t) \oslash R_3(t) \]
\[ R_2(t) = Q_{20}(t) \oslash (1 + R_0(t)) + \left( Q_{22.6}(t) + Q_{22.6(7,8)^n}(t) \right) \oslash (1 + R_2(t)) + Q_{24}(t) \oslash R_4(t) \]
\[ R_3(t) = Q_{32}(t) + \left( Q_{32(7,8)^n}(t) \right) \oslash (1 + R_2(t)) \]
\[ R_4(t) = Q_{42}(t) \oslash R_2(t) + \left( Q_{42.58}(t) + Q_{42.5(8,7)^n}(t) \right) \oslash (1 + R_2(t)) \] (14)

Taking LST of the relation (14) and solving for $R_0^{**}(s)$, we get the expected number of repairs of the unit per unit time as

\[ R_0 = \lim_{s \to 0} sR_0^{**}(s) = \frac{N_4}{D_2} \] (15)

Where, $N_4 = 1 + p_{14}p_{20} - p_{23}p_{32}$ and $D_2$ is already specified. (16)

### 8. Expected Number of Treatments Given to the Server
Let $T_i(t)$ be the expected number of treatments given to the server in $(0, t]$ given that the system entered the regenerative state $S_i$ at $t = 0$. The recursive relations for $T_i(t)$ are given as:

\[
T_0(t) = Q_{01}(t)S_i T_1(t)
\]
\[
T_1(t) = Q_{12}(t)S_i T_2(t) + Q_{13}(t)S_i T_3(t)
\]
\[
T_2(t) = Q_{20}(t)S_i T_0(t) + Q_{22.6}(t)S_i T_2(t) + Q_{22.6(7,8)}^n(t)(1 + T_2(t)) + Q_{24}(t)S_i T_4(t)
\]
\[
T_3(t) = Q_{32}(t)S_i T_2(t) + Q_{32(7,8)}^n(t)(1 + T_2(t))
\]
\[
T_4(t) = \left( Q_{42}(t) + Q_{42.58}(t) + Q_{42.5(8,7)}^n(t) \right) S_i (1 + T_2(t))
\]  \hfill (17)

Taking LST of relations (17) and solving for $T_0^*(s)$, we get the expected number of treatments of the unit per unit time as

\[
T_0(\infty) = \lim_{s \to 0} sT_0^*(s) = \frac{N_5}{D_2}
\]  \hfill (18)

Where, $D_2$ is already mentioned.

9. Profit Analysis

The profit incurred to the system model in steady state can be obtained as:

\[
P = K_0A_0 - K_1B_0^R - K_2R_0 - K_3T_0
\]  \hfill (20)

Where

- $P$ = Profit of the system model
- $K_0$ = Revenue per unit up – time of the system
- $K_1$ = Cost per unit time for which server is busy due to repair
- $K_2$ = Cost per unit time repair of the unit
- $K_3$ = Cost per unit time treatment given to the server

10. Conclusion

For the particular case, $W(t) = \theta e^{-\theta t}$, $g(t) = \beta e^{-\beta t}$ and $f(t) = \alpha e^{-\alpha t}$ the values of some important reliability measures including mean time to system failure (MTSF), availability and profit function are obtained for arbitrary values of the parameters. The behavior of these measures with respect to failure rate is shown respectively in figures 2, 3 and 4. It observed that MTSF, availability and profit keep on decreasing with the increase of failure rates of the unit and server. However, they follow an upward trend with the increase of repair rate ($\beta$), treatment rate ($\alpha$) and arrival rate ($\theta$) of the server. Further, it can also be seen that MTSF declines rapidly with a slight positive change in failure rate. Hence, the study reveals that a cold stand by system of two identical units can be made more available and profitable to use either by calling the server immediately to rectify the faults or by increasing the repair rate of the failed unit in case server takes some time to arrive the system.

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Fig.2: MTSF Vs Failure Rate
Fig. 3: Availability Vs Failure Rate

Fig. 4: Profit Vs Failure Rate