5.1 Introduction

In the first section of this chapter we examine the characteristics of fuzzy detour boundary vertices. It is seen by an example that a fuzzy cut vertex may or may not be a fuzzy detour boundary vertex in a fuzzy graph in contrast to the concept that no cut vertex is a boundary vertex in a crisp graph. The fuzzy detour boundary vertices of G are the fuzzy detour boundary vertices of the blocks of G and conversely. Also, it is observed that every vertex \( v \) of a fuzzy graph is a fuzzy detour boundary vertex of G unless \( v \) is the unique vertex having fuzzy detour eccentricity less than the fuzzy detour diameter.
In the second section it is noted that every fuzzy detour from some vertex to its fuzzy detour boundary vertex passes through some fuzzy detour neighbour of that vertex. Also every fuzzy detour of a fuzzy graph extends to a fuzzy detour between two fuzzy detour boundary vertices. Properties relating fuzzy detour neighbours and fuzzy detour boundary vertices are also studied. Certain properties of fuzzy detour neighbours are presented.

### 5.2 Fuzzy Detour Neighbour and Fuzzy Detour Boundary Vertices

**Definition 5.2.1**

Let $G$ be a nontrivial connected fuzzy graph $G: (\sigma, \mu)$. For a vertex $v$ in $G$, define $\Delta^-(v) = \min \{\Delta(w, v) \mid w \in V(G) - \{v\}\}$.

A vertex $w(\not= v)$ is called a **Fuzzy detour neighbour** of $v$ if $\Delta(w, v) = \Delta^-(v)$.

Let $\bar{N}_\Delta(v)$ denote the set of all fuzzy detour neighbours of $v$ in $G$. If $v$ is a fuzzy detour eccentric vertex of $u$ in a connected fuzzy graph $G$, then no vertex of $G$ is farther, in the fuzzy detour sense, from $u$ than $v$ is. In particular, if $w$ is a fuzzy detour neighbor of $v$, then $\Delta(u, w) \leq \Delta(u, v)$. 
Definition 5.2.2

A vertex \( v \) in a connected fuzzy graph \( G \) is a **Fuzzy detour boundary vertex of a vertex** \( u \) if \( \Delta(u, w) \leq \Delta(u, v) \) for every fuzzy detour neighbour \( w \) of \( v \).

A vertex \( v \) is called a **fuzzy detour boundary vertex of a fuzzy graph** \( G \) if \( v \) is a fuzzy detour boundary of some vertex of \( G \).

Example 5.2.3

From figure 5.1, we determine the following:

**Fuzzy Detour Distances:**

\[
\begin{align*}
\Delta(a, b) &= 13, \quad a-u-v-b; \\
\Delta(a, u) &= 13, \quad a-v-b-u; \\
\Delta(a, v) &= 14, \quad a-u-b-v; \\
\Delta(b, u) &= 11, \quad b-v-a-u; \\
\Delta(b, v) &= 10, \quad b-u-a-v; \\
\Delta(u, v) &= 11, \quad u-b-v.
\end{align*}
\]
Fuzzy Detour Neighbours:

$\Delta^-(a) = 13 = \Delta(a, b) = \Delta(a, u)$; $u, b$ are fuzzy detour neighbours of $a$.

$\Delta^-(b) = 10 = \Delta(b, v)$; $v$ is a fuzzy detour neighbour of $b$.

$\Delta^-(u) = 11 = \Delta(u, b) = \Delta(u, v)$; $b$ and $v$ are fuzzy detour neighbours of $u$.

$\Delta^-(v) = 10 = \Delta(v, b)$; $b$ is a fuzzy detour neighbour of $v$.

Fuzzy Detour Boundary Vertices:

$\Delta(v, u) \leq \Delta(v, a)$ and $\Delta(v, b) \leq \Delta(v, a)$. Therefore $a$ is a fuzzy detour boundary vertex of $v$.

$\Delta(u, v) = \Delta(u, b)$. Therefore $b$ is a fuzzy detour boundary vertex of $u$.

$\Delta(a, b) \leq \Delta(a, v)$ and $\Delta(u, b) \leq \Delta(u, v)$. Therefore $v$ is a fuzzy detour boundary vertex of $u$ and $a$.

$\Delta(a, b) \leq \Delta(a, u)$ and $\Delta(a, v) \not\leq \Delta(a, u)$. Therefore $u$ is not a fuzzy detour boundary vertex of $a$, for its every fuzzy detour neighbour $b$ and $v$.

For the fuzzy graph $G$ of Fig. 5.1, the vertices $u$ and $b$ are fuzzy detour neighbours of $a$, $v$ is a fuzzy detour neighbour of $b$; $b$ and $v$ are fuzzy detour neighbours of $u$ and $b$ is the fuzzy detour neighbour of $v$. 
a is the fuzzy detour boundary vertex of v, b is a fuzzy detour boundary vertex of u, v is the fuzzy detour boundary vertex of a and u but u is not the fuzzy detour boundary vertex of a. Therefore a, b and v are fuzzy detour boundary vertices of the fuzzy graph G. Also a and u are cut vertices of G and they are also fuzzy detour boundary vertices of G.

In a crisp graph no cut vertex of a connected graph G is a boundary vertex of G but in fuzzy graphs it is not true with regard to fuzzy detour $\mu$-distances.

**Example 5.2.4**

![Figure 5.2](image-url)
**Fuzzy Detour Distances:**

\[
\begin{align*}
\Delta(a, b) &= 6, \quad \Delta(a, c) = 28, \quad \Delta(a, d) = 30, \quad \Delta(a, x) = 6, \quad \Delta(a, y) = 20, \\
\Delta(a, u) &= 23, \quad \Delta(a, v) = 25; \\
\Delta(b, c) &= 30, \quad \Delta(b, d) = 32, \quad \Delta(b, x) = 8, \quad \Delta(b, y) = 22, \quad \Delta(b, u) = 25, \\
\Delta(b, v) &= 27; \\
\Delta(c, d) &= 8, \quad \Delta(c, x) = 22, \quad \Delta(c, y) = 8, \quad \Delta(c, u) = 25, \quad \Delta(c, v) = 27; \\
\Delta(d, x) &= 24, \quad \Delta(d, y) = 10, \quad \Delta(d, u) = 27, \quad \Delta(d, v) = 29; \\
\Delta(x, y) &= 14, \quad \Delta(x, u) = 17, \quad \Delta(x, v) = 19; \\
\Delta(y, u) &= 17, \quad \Delta(y, v) = 19; \\
\Delta(u, v) &= 17.
\end{align*}
\]

**Fuzzy Detour Neighbours :**

\[
\begin{align*}
\Delta^{-}(a) &= 6 = \Delta(a, x) = \Delta(a, b); \quad x \text{ and } b \text{ are fuzzy detour neighbours of } a. \\
\Delta^{-}(b) &= 8 = \Delta(b, x); \quad x \text{ is the fuzzy detour neighbour of } b. \\
\Delta^{-}(c) &= 8 = \Delta(c, d) = \Delta(c, y); \quad d \text{ and } y \text{ are fuzzy detour neighbours of } c. \\
\Delta^{-}(d) &= 8 = \Delta(c, d); \quad c \text{ is the fuzzy detour neighbour of } d. \\
\Delta^{-}(x) &= 6 = \Delta(a, x); \quad a \text{ is the fuzzy detour neighbour of } x. \\
\Delta^{-}(y) &= 8 = \Delta(c, y); \quad c \text{ is the fuzzy detour neighbour of } y.
\end{align*}
\]
\[ \Delta^-(u) = 17 = \Delta(u, v) = \Delta(u, x) = \Delta(u, y); \] v, x and y are fuzzy detour neighbours of u.

\[ \Delta^-(v) = 17 = \Delta(u, v); \] u is the fuzzy detour neighbour of v.

**Fuzzy Detour Boundary Vertices:**

a is a fuzzy detour boundary vertex of c,d,y,u,v with respect to the neighbour x.

a is not a fuzzy detour boundary vertex of any vertex of G with respect to the neighbour b.

b is a fuzzy detour boundary vertex of a,c,d,y,u,v with respect to the neighbour x.

c is not a fuzzy detour boundary vertex of any vertex of G with respect to the neighbour d.

c is a fuzzy detour boundary vertex of a,b,x,u,v with respect to the neighbour y.

d is a fuzzy detour boundary vertex of a,b,x,y,u,v with respect to the neighbour c.

x is a fuzzy detour boundary vertex of b with respect to the neighbour a.

y is a fuzzy detour boundary vertex of d with respect to the neighbour c.
\(u\) is not a fuzzy detour boundary vertex of any vertex of \(G\) with respect to the neighbour \(v\).

\(u\) is a fuzzy detour boundary vertex of \(a,b,c,d,y\) with respect to the neighbour \(x\).

\(u\) is a fuzzy detour boundary vertex of \(a,b,c,d,x\) with respect to the neighbour \(y\).

\(v\) is a fuzzy detour boundary vertex of \(a,b,c,d,x,y\) with respect to the neighbour \(u\).

**Theorem 5.2.7**

Let \(G\) be a nontrivial connected fuzzy graph and let \(u\) be a vertex of \(G\). Every vertex of \(G\) distinct from \(u\) is a fuzzy detour boundary vertex of \(u\) if and only if \(e_\Delta(u) = a, 1 \leq a < \infty\).

**Proof**

Assume first that \(e_\Delta(u) = a\) and let \(v\) be a vertex of \(G\) distinct from \(u\) such that \(\Delta(u, v) = a\). That is, \(v\) is the farthest vertex from \(u\) in the fuzzy detour sense. Let \(w\) be a fuzzy detour neighbour of \(v\), Therefore, \(\Delta(u, w) \leq \Delta(u, v)\) and hence \(v\) is a fuzzy detour boundary vertex of \(u\).
Conversely, we assume, on the contrary, that every vertex of G different from u is a fuzzy detour boundary vertex of u but \( e_\Delta(u) \neq a \). Let x be a fuzzy detour boundary vertex such that \( e_\Delta(u) = \Delta(u, x) = a + k \), where k is such that \( \Delta(u, y) = k \), for some y in G.

Also we assume that u is a fuzzy detour neighbour of y in G. Therefore, \( \Delta(u, y) = k < a + k = \Delta(u, x) \). That is, \( \Delta(u, x) > \Delta(u, y) \). This shows that y is not a fuzzy detour boundary vertex of G. Thus, \( e_\Delta(u) = a \).

There are certain vertices in a connected fuzzy graph G that have a close connection with fuzzy detour boundary vertices. A vertex y distinct from x and z is said to lie between x and z if \( \Delta(x, z) = \Delta(x, y) + \Delta(y, z) \). That is, the triangle inequality becomes equality.

**Definition 5.2.8**

A vertex v is a **Fuzzy detour interior vertex** of G if for every vertex u distinct from v, there exists a vertex w such that v lies on the fuzzy detour from u to w. The fuzzy detour interior \( \text{Int}_\Delta(G) \) of G is the fuzzy subgraph of G induced by the fuzzy detour interior vertices. That is,
\[ \text{Int}_\Delta(G) = \bigcup_{u \in V(G)} \text{Int}_\Delta(u) = \{ v \in V(G) / v \text{ lies on the } u - w \text{ fuzzydetour and } v \neq w \} \]

We now see that the fuzzy detour interior vertices are precisely those vertices that are not fuzzy detour boundary vertices.

**Theorem 5.2.9**

Let G be a connected fuzzy graph. A vertex v is a fuzzy detour boundary vertex of G if and only if v is not a fuzzy detour interior vertex of G.

**Proof**

Let v be a fuzzy detour boundary vertex of G and v is a fuzzy detour boundary vertex of the vertex u. Assume, to the contrary, that v is a fuzzy detour interior vertex of G. Therefore there exists a vertex w distinct from u and v such that v lies on the fuzzy detour from u to w.

Let \( P : u = v_1, v_2, v_3, \ldots, v_i = v, v_{i+1}, \ldots, v_j = w, 1 < i < j \), be the u to w fuzzy detour. Let \( v_{i+1} \) be the fuzzy detour neighbor of v. That is,

\[ \Delta(u, v_{i+1}) \leq \Delta(u, v). \]  

Therefore,

\[ \Delta(u, v_{i+1}) = \Delta(u, v) + \Delta(v, v_{i+1}), \]

\[ = \Delta(u, v) + k. \]

That is, \( \Delta(u, v_{i+1}) > \Delta(u, v) \), a contradiction. Therefore, v is not a fuzzy detour interior vertex.
Conversely, assume that \( v \) is a vertex that is not a fuzzy detour interior vertex of \( G \). Then there exists some vertex \( u \) such that for every vertex \( w \) distinct from \( u \) and \( v \), the vertex \( v \) does not lie on the \( u-w \) fuzzy detour. Let \( x \) be a fuzzy detour neighbour of \( v \). Then, 
\[
\Delta(u, x) \leq \Delta(u, v) + \Delta(v, x) = \Delta(u, v) + k, \text{where } \Delta(v, x) = k \text{ and } 1 \leq k \leq \infty. 
\]
Since \( v \) does not lie on the \( u - x \) fuzzy detour and every fuzzy detour from \( u \) to \( w \) must pass through fuzzy detour neighbour \( x \) of \( v \), we must have \( \Delta(u, x) < \Delta(u, v) + k \). That is, \( \Delta(u, x) \leq \Delta(u, v) \). Thus, \( v \) is a fuzzy detour boundary vertex of \( u \).

The relationship between a fuzzy graph and its blocks can be established through the fuzzy detour boundary vertices. This can be observed in proving the following theorem.

**Theorem 5.2.10**

Let \( v \) be a vertex in a connected fuzzy graph \( G \) such that \( v \) belongs to a block \( B \) and \( v \) is not a fuzzy cut vertex of \( G \). Then \( v \) is a fuzzy detour boundary vertex of \( G \) if and only if \( v \) is a fuzzy detour boundary vertex of \( B \).
**Proof**

Let $v$ be a fuzzy detour boundary vertex of a block $B$. Therefore $v$ is a fuzzy detour boundary vertex of some vertex of $B$. Since $B$ is in $G$, $v$ is a fuzzy detour boundary vertex of $G$.

Conversely, let $v$ be a fuzzy detour boundary vertex of $G$. Therefore $v$ is a fuzzy detour boundary vertex of some vertex $w$ of $G$. Since $v$ is not a cut vertex it belongs to a unique block $B$ of $G$. If $w \in B$, then the proof is complete. Thus, we may assume that $w \in B' \neq B$. For each $y$ in $B$, the $w$-$y$ fuzzy detour must pass through the cut vertex $x$ common to both $B$ and $B'$ (by the theorem 3.2.1). Therefore, in particular for $y = v \in B$, we have

$$\Delta(w, v) = \Delta(w, x) + \Delta(x, v) \quad \ldots \quad (5.1)$$

Let $u$ be a fuzzy detour neighbour of $v$. Then $u \in B$ and so

$$\Delta(w, u) = \Delta(w, x) + \Delta(x, u) \quad \ldots \quad (5.2)$$

Since $v$ is a boundary vertex of $w$ in $G$, by the definition of boundary vertex, $\Delta(w, u) \leq \Delta(w, v)$. Using this relation in (5.1), we have $\Delta(w, u) \leq \Delta(w, x) + \Delta(x, v)$ and comparing this with (5.2), we arrive at the inequality $\Delta(w, x) + \Delta(x, u) \leq \Delta(w, x) + \Delta(x, v)$. 

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Hence, $\Delta(x, u) \leq \Delta(x, v)$. That is, $v$ is a boundary vertex of $x$ and hence of $B$.

\textbf{Theorem 5.2.11}

Let $G$ be a connected fuzzy graph with fuzzy detour diameter $b$. Then every vertex $v$ is a fuzzy detour boundary vertex of $G$ unless $v$ is the unique vertex of $G$ having fuzzy detour eccentricity less than $b$.

\textbf{Proof}

Let $v$ be a vertex in $G$. If $e_{\Delta}(v) \not< b$, then $e_{\Delta}(v) = b$, since $\text{diam}_{\Delta}(G) = b$. Thus there is a vertex $u$ such that $\Delta(u, v) = b$. Since $\Delta(u, w) \leq b$ for all $w \in N_{\Delta}(v)$, it follows that $v$ is a fuzzy detour boundary vertex of $u$ and so $v$ is a fuzzy detour boundary vertex of $G$.

Suppose that $v$ is the unique vertex of $G$ having fuzzy detour eccentricity less than $b$. We claim that $v$ is not the fuzzy detour boundary vertex of $G$. Assume, to the contrary, that $v$ is a fuzzy detour boundary vertex of some vertex $w$, so that $\Delta(v, w) < b$. Since, $v$ is unique $e_{\Delta}(w) \not< b$. Then there exists a vertex $u$ that is a fuzzy detour neighbour of $v$ so that $\Delta(u, w) \not< b$, otherwise, $u$ and $v$ are fuzzy detour boundary vertices, contradicting that $v$ is unique. Hence $\Delta(u, w) \geq b > \Delta(v, w)$. 

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That is, we have $\Delta(w, v) < \Delta(u, w)$, contradicting the assumption that $v$ is a fuzzy detour boundary vertex of some vertex $w$.  

**Definition 5.2.12**

The fuzzy sub graph of $G$ induced by its fuzzy detour boundary vertices is called the *Fuzzy detour boundary graph* $\overline{\Delta}(G)$ of $G$.

The fuzzy graph $H$ is called a *fuzzy detour boundary graph* if $H = \overline{\Delta}(G)$ for some connected fuzzy graph $G$. A connected fuzzy graph $G$ is a *self fuzzy detour boundary graph* if $G = \overline{\Delta}(G)$. Obviously every self fuzzy detour boundary graph is a fuzzy detour boundary graph.

**5.3 Properties on Fuzzy Detour Boundary Vertices**

By $B_\Delta(u)$ we denote the set of all fuzzy detour boundary vertices of $u$. A vertex $v$ is called a boundary vertex of $G$ if $v \in B_\Delta(u)$ for some vertex $u \in V(G)$. Since we only consider fuzzy detour boundary vertex sets and not their induced fuzzy sub graphs, we identify the fuzzy detour boundary of $G$, $B_\Delta(G)$, as the set of all its fuzzy detour boundary vertices:

$$B_\Delta(G) = \bigcup_{u \in V(G)} B_\Delta(u) = \{v \in V(G) / \exists u \in V(G) \text{ such that } \forall w \in N_\Delta(v), \Delta(u, w) \leq \Delta(u, v)\}$$

For a complete fuzzy graph, the fuzzy detour boundary is not the whole vertex set as can be seen from the example below:
Example 5.3.1

Figure 5.3 Fuzzy detour boundary vertices in a complete fuzzy graph

From the above figure we have \( \Delta(a, u) = 14, \ \Delta(a, v) = \Delta(a, b) = 13, \ \Delta(b, u) = \Delta(b, v) = 12, \ \Delta(u, v) = 10; \)

\( \Delta^-(a) = 13 = \Delta(a, b); \quad b \text{ is a fuzzy detour neighbour of } a. \)

\( \Delta^-(b) = 12 = \Delta(b, u) = \Delta(b, v); \quad u, v \text{ are fuzzy detour neighbours of } b. \)

\( \Delta^-(u) = 10 = \Delta(u, v); \quad v \text{ is the fuzzy detour neighbour of } u. \)

\( \Delta^-(v) = 10 = \Delta(u, v); \quad u \text{ is the fuzzy detour neighbour of } v. \)

\( \Delta(u, b) = 12 \leq \Delta(u, a) = 14; \quad a \text{ is the fuzzy detour boundary vertex of } u. \)

\( \Delta(v, b) = 12 \leq \Delta(v, a) = 13; \quad a \text{ is the fuzzy detour boundary vertex of } v. \)

\( \Delta(a, v) = 13 \ngeq \Delta(a, b) = 12; \quad b \text{ is not the fuzzy detour boundary vertex of } a. \)

\( \Delta(a, u) = 14 \ngeq \Delta(a, v) = 13; \quad v \text{ is not the fuzzy detour boundary vertex of } a. \text{ Therefore “}a\text{” is the fuzzy detour boundary vertex of the fuzzy } \)

graph } G.
Example 5.3.2

In figure 5.4, \( \Delta (a, b) = 8; \ \Delta (a, c) = \Delta (b, c) = 7; \)

\( \Delta^- (a) = 7 = \Delta (a, c); \) c is a fuzzy detour neighbour of a.

\( \Delta^- (b) = 7 = \Delta (b, c); \) c is a fuzzy detour neighbour of b.

\( \Delta^- (c) = 7 = \Delta (a, c) = \Delta (b, c); \) a and b are fuzzy detour neighbours of c.

\( \Delta (b, a) = 8 \not\leq \Delta (b, c) = 7; \) c is not the fuzzy detour boundary of b.

\( \Delta (a, b) = 8 \not\leq \Delta (a, c) = 7; \) c is not the fuzzy detour boundary of a.

\( \Delta (a, c) = 7 \leq \Delta (a, b) = 8; \) b is a fuzzy detour boundary of a.

\( \Delta (b, c) = 7 \leq \Delta (b, a) = 8; \) a is a fuzzy detour boundary of b.

In both the examples seen above we conclude that all the vertices in a complete fuzzy graph are not fuzzy detour boundary vertices. This is a deviation from the concept as far as the crisp graphs are concerned, where in a complete graph all vertices are boundary vertices.
Here, in the following theorem, we show that every vertex is either a fuzzy detour interior vertex or a fuzzy detour boundary vertex.

**Theorem 5.3.3**

Every vertex of a fuzzy graph $G$ is either a fuzzy detour boundary vertex of some vertex or a fuzzy detour interior vertex.

**Proof**

Let $z$ be a vertex of a fuzzy graph $G$ which is neither a fuzzy detour boundary vertex of $x \in V(G)$ nor a fuzzy detour interior vertex. Then $z$ is the farthest vertex, in the fuzzy detour sense, from $x$. Since $z$ lies on the $x$-$z$ fuzzy detour and $z$ is not a fuzzy detour boundary vertex of $x$, $\Delta(x, v) > \Delta(x, z)$ for some fuzzy detour neighbour $v$ of $z$. Therefore $\Delta(x, v) = \Delta(x, z) + k$ where $1 \leq k < \infty$ is the fuzzy detour $\mu$-distance from some fuzzy detour neighbour $v$ of $z$. That is, $\Delta(x, v) = \Delta(x, z) + \Delta(z, v)$.

Since $z$ is the farthest vertex from $x$, $v$ lies on some fuzzy detour from $x$ to any fuzzy detour boundary vertex of $x$. That is, $v$ lies on the $x$-$y_1$, fuzzy detour for some fuzzy detour boundary vertex $y_1$ of $x \in V(G)$. Therefore, we can write $\Delta(x, y_1) = \Delta(x, v) + \Delta(v, y_1)$. That is, $\Delta(x, y_1) = \Delta(x, z) + \Delta(z, v) + \Delta(v, y_1)$ implying that $z$ lies on some
fuzzy detour, contradicting the assumption. This contradiction establishes the result.

**Theorem 5.3.4**

In a connected fuzzy graph, every fuzzy detour from some vertex to its fuzzy detour boundary vertex passes through some fuzzy detour neighbour of that vertex.

**Proof**

Let x be a vertex of a fuzzy graph G and v be its fuzzy detour boundary vertex. Suppose that there exists a x-v fuzzy detour not passing through any of the fuzzy detour neighbours of v. Let y be some fuzzy detour neighbour of v. Then, $\Delta(x, y) \leq \Delta(x, v) + \Delta(v, y)$. That is, $\Delta(x, v) < \Delta(x, y)$. This shows that v is not a fuzzy detour boundary vertex of x, a contradiction. Thus, the theorem follows.

The fuzzy detours may be extended to fuzzy detours whose end vertices are fuzzy detour boundary vertices.

**Theorem 5.3.5**

Each fuzzy detour of a fuzzy graph extends to a fuzzy detour between two fuzzy detour boundary vertices.
Proof

Let u and v be two end vertices (fuzzy end vertices, [9], Remark 3.2) in a fuzzy graph G. If u and v are fuzzy detour boundary vertices of G then the proof is complete.

Let u and v be two end vertices of some u-v be a fuzzy detour in G and one of its end vertices, say v, which is not a fuzzy detour boundary vertex. Then the u-v fuzzy detour can be extended to a longer path, which is also a fuzzy detour between its end vertices, by the definition of fuzzy detour boundary vertex. Since the fuzzy graph is finite, it is possible to extend u-v fuzzy detour whose end vertices are fuzzy detour boundary vertices.

Definition 5.3.6

We define \( D_{\Lambda}(u, v) \) to be the set of all fuzzy detour boundary vertices with respect to their fuzzy detour neighbours. In other words,

\[
D_{\Lambda}(u, v) = \{b \in B_{\Lambda}(G): \Lambda(b, u) < \Lambda(b, v)\}; \text{ for all } u \in N_{\Lambda}(v).
\]

Theorem 5.3.7

Let u and u’ be two fuzzy detour neighbours of the vertex v in fuzzy graph G with \( D_{\Lambda}(u, v) \neq D_{\Lambda}(u’, v) \). Then \( D_{\Lambda}(u, v) \) is neither empty nor the whole set \( B_{\Lambda}(G) \).
Proof

Let \( u - u' \) be a fuzzy detour in \( G \). Then, by the above theorem 5.3.5, this fuzzy detour can be extended to a fuzzy detour \( P \) whose end vertices are the fuzzy detour boundary vertices \( b \) and \( b' \).

Let \( \Delta (u, b) < \Delta (u', b) \) and also \( v \in V(P) \). Then,

\[
\Delta (b, u') = \Delta (b, v) + \Delta (v, u') \ ; \text{That is, } \Delta (b, u') > \Delta (b, v), \text{ so that } v \notin D_\lambda (v, u'). \]

Also, \( \Delta (b, v) = \Delta (b, u) + \Delta (u, v) > \Delta (b, u) \). This shows that \( b \in D_\lambda (v, u) \). Hence, \( b \in D_\lambda (v, u) \setminus D_\lambda (v, u') \).

By interchanging the roles of \( u \) and \( u' \), we see that

\( b \in D_\lambda (v, u') \setminus D_\lambda (v, u) \).

Therefore, the boundary vertices lie either on \( D_\lambda (v, u) \) or \( D_\lambda (v, u') \).

Since the \( u - v \) fuzzy detour extends to a fuzzy detour between two boundary vertices \( b \) and \( b' \), we conclude that \( D_\lambda (u, v) \) is neither empty nor the whole set \( B_\lambda (G) \), which concludes the proof.

Definition 5.3.8

For any two vertices \( u, v \) of a fuzzy graph \( G \), the set of all vertices lying in the \( u-v \) fuzzy detour is defined by

\[
S_\lambda (u, v) = \{ x \in V(G): \text{there exists a } u-v \text{ fuzzy detour } P \text{ in } G \text{ with } x \in V(P) \}. 
\]
Note that \{u, v\} \subseteq S_{\Lambda}(u, v). The following results are observed from the above definition.

**Observations 5.3.9**

1. \(S_{\Lambda}(u, v) = \{x \in V(G): \Lambda(u, x) + \Lambda(x, v) = \Lambda(u, v)\}\)

2. Let \(x_0, x_1, \ldots, x_n\) be vertices in a connected fuzzy graph \(G\). If \(\sum_{i=1}^{n} \Lambda(x_{i-1}, x_i) = \Lambda(x_0, x_n)\) then for each \(k, l, m\) with \(0 \leq k \leq l \leq m \leq n\), \(x_l \in S_{\Lambda}(x_k, x_m)\).

**Definition 5.3.10**

For \((u, v) \in V(G) \times V(G)\), we define \(B_{\Lambda}(u, v)\) by *the set of all boundary vertices* of the fuzzy graph \(G\) in which \(v\) is an interior point of the \(u\)-\(v\) fuzzy detour. That is,

\[B_{\Lambda}(u, v) = \{x \in B_{\Lambda}(G): v \in S_{\Lambda}(u, x)\}.\]

Note that \(x \in B_{\Lambda}(u, v)\) if and only if \(x \in B_{\Lambda}(G)\) and

\[\Lambda(u, v) + \Lambda(v, x) = \Lambda(u, x).\]

The fuzzy detour interior vertices and the fuzzy detour neighbours have certain relationships with fuzzy detour boundary vertices which can be seen from the following properties.
Theorem 5.3.11

For each \((u, v) \in V(G) \times V(G)\), \(B_\Delta(u, v)\) is non empty.

Proof

Since \(x \in V(G), x \in \bigcup_{y \in B_\Delta(G, x)} S_\Delta(x, y)\); that is \(x \in S_\Delta(x, y)\) for some \(y \in B_\Delta(G, x)\). Hence \(B_\Delta(u, v)\) contains one element \(x\) and so is nonempty.

Theorem 5.3.12

For any two distinct vertices \(u, v\), \(B_\Delta(u, v) \cap B_\Delta(v, u) = \emptyset\).

Proof

Assume \(B_\Delta(u, v) \cap B_\Delta(v, u) \neq \emptyset\). Therefore we can have \(z \in B_\Delta(u, v) \cap B_\Delta(v, u)\). Then \(z \in B_\Delta(u, v)\) implies that \(\Delta(u, v) + \Delta(v, z) = \Delta(u, z)\) and since \(z\) is also in \(B_\Delta(v, u)\) implies that \(\Delta(v, u) + \Delta(u, z) = \Delta(v, z)\). Adding these two results we have

\[2\Delta(u, v) + \Delta(v, z) + \Delta(u, z) = \Delta(u, z) + \Delta(v, z).\]

That is, we have \(\Delta(u, v) = 0\). By the definition of metric, \(\Delta(u, v) = 0\) whenever \(u = v\), that is, \(u\) and \(v\) coincide. Since \(u\) and \(v\) are distinct, this is a contradiction. Hence the result.
Theorem 5.3.13

For any vertex \( x \in V(G) \), \( V(G) = \bigcup_{y \in B_\Delta(G,x)} S_\Delta(x, y) \).

Proof

Assume that \( V(G) \neq \bigcup_{y \in B_\Delta(G,x)} S_\Delta(x, y) \) and let \( z \in V(G) - \bigcup_{y \in B_\Delta(G,x)} S_\Delta(x, y) \). Therefore, \( z \) is the farthest vertex from \( x \) and is not a fuzzy detour boundary vertex of \( x \). Then, there exists a fuzzy detour neighbour \( v \) of \( z \) such that \( \Delta(x, v) > \Delta(x, z) \). That is,

\[
\Delta(x, v) = \Delta(x, z) + \Delta(z, v)
\]  

Since \( \Delta(x, z) \) is the longest fuzzy detour, \( v \in \bigcup_{y \in B_\Delta(G,x)} S_\Delta(x, y) \) and hence \( v \in \mathcal{S}_\Delta(x, y_1) \) for some \( y_1 \in B_\Delta(G,x) \). By the observation 5.3.9 (1) above,

\[
\Delta(x, y_1) = \Delta(x, v) + \Delta(v, y_1) = \Delta(x, z) + \Delta(z, v) + \Delta(v, y_1)
\]  

By the observation 5.3.9 (2), \( z \in \mathcal{S}_\Delta(x, y_1) \), a contradiction and thus, the theorem follows.

Theorem 5.3.14

Let \( G \) be a connected fuzzy graph and \( N_\Delta(x) \) is the fuzzy detour neighbourhood of \( x \) in \( V(G) \). Then,

\[
\left| N_\Delta(x) \cap S_\Delta(x, y) \right| = \left| N_\Delta(x) \cap S_\Delta(x, z) \right| = 1.
\]
Proof

Let $|N_\Delta(x) \cap S_\Delta(x, y)| \geq 2$. Also let $u$ and $v$ be distinct vertices in $N_\Delta(x) \cap S_\Delta(x, y)$. Since $u \in S_\Delta(x, y)$, we have

$$\Delta(x, y) = \Delta(x, u) + \Delta(u, y) \quad \text{.......... (5.4)}$$

by observation 5.3.9 (1). On similar lines, we can have for $v \in S_\Delta(x, y)$,

$$\Delta(x, y) = \Delta(x, v) + \Delta(v, y) \quad \text{.......... (5.5)}$$

These two results (5.4) and (5.5) yield

$$\Delta(x, u) + \Delta(u, y) = \Delta(x, v) + \Delta(v, y) \quad \text{.......... (5.6)}$$

But both $u$, $v \in N_\Delta(x)$; therefore, $\Delta(x, u) = \Delta(x, v)$, using in (5.6), we have,

$$\Delta(u, y) = \Delta(v, y) \quad \text{.......... (5.7)}$$

If $y \in B_\Delta(u, v)$, then $\Delta(u, v) + \Delta(v, y) = \Delta(u, y)$. Using the result (5.7) above, we get $\Delta(u, v) = 0$, a contradiction. Thus $y \not\in B_\Delta(u, v)$.

Since $B_\Delta(u, v) \neq \emptyset$, by theorem 5.3.11, we get $B_\Delta(u, v) = \{z\}$. By applying the same argument to $(v, u)$, we also have $B_\Delta(v, w) = \{z\}$. This shows that $B_\Delta(u, v) \cap B_\Delta(v, u) = \{z\}$ which contradicts theorem 5.3.12.

Therefore, $|N_\Delta(x) \cap S_\Delta(x, y)| \leq 1$.

Since, by assumption, $\deg(x) \geq 2$ and $N_\Delta(x) \subset S_\Delta(x, y) \cup S_\Delta(x, z)$, we get

$$|N_\Delta(x) \cap S_\Delta(x, y)| = 1 = |N_\Delta(x) \cap S_\Delta(x, z)|.$$