7.1 Introduction

The physical significance of Transverse Momentum Dependent Parton Distribution Functions (TMD) is outlined in chapter 1 of this thesis. In this chapter, we discuss how this aspect of the proton structure function can be introduced in the self-similarity based models of proton structure functions discussed in chapter 4.
7.2 Formalism

7.2.1 Ansatz of TMD in the self-similarity based models and theoretical limitations

The simplest way to introduce TMD in the self-similarity based model is suggested in Refs. [80, 82] by redefining the magnification factor \( \left( 1 + \frac{Q^2}{Q_0^2} \right) \) by \( \left( 1 + \frac{k_t^2}{k_0^2} \right) \) and is given as

\[
\log f_i(x, k_t^2) = D_1 \log \frac{1}{x} \log \left( 1 + \frac{k_t^2}{k_0^2} \right) + D_2 \log \frac{1}{x} + D_3 \log \left( 1 + \frac{k_t^2}{k_0^2} \right) + D_0 - \log M^2 \tag{7.1}
\]

instead of Eq. 2.1 of chapter 2. Here, \( k_t^2 \) is the square of the intrinsic transverse momentum of the parton which has corresponding \( x \) as the longitudinal fraction. The parameters \( D_1, D_2, D_3 \) are determined from Deep Inelastic HERA data as earlier. Redefining the PDF of Eq. 2.2 of chapter 2 to be

\[
q_i(x, Q^2) = \int_{0}^{|k_t|^2 < Q^2} dk_t^2 f_i(x, k_t^2) \tag{7.2}
\]

with the cut off \( |k_t|^2 < Q^2 \), one can obtain the identical expression for integrated PDF and structure function (Eqs. 2.3-2.5). The unintegrated Parton Distribution Function (uPDF) \( f_i(x, Q^2) \) is now redefined as TMD: \( f_i(x, k_t^2) \). Thus this minimal extension of the approach to transverse structure of Proton keeps the results of the previous form of parton distribution and structure function unchanged.

Clearly, this can be done only in a model frame as in Refs. [120–123]. But it could be of interest to explore this approach to study \( k_t \) dependence TMD \( f_i(x, k_t^2) \) only in the specific \( x \) region where the approach works and where the parameters have been fitted. However, Eq. 7.1 has deep theoretical limitation at the level of quantum field theory as noted by Collins [12].
Further, it has been found in recent years that the DIS experiment is not sufficient to obtain full transverse structure of the nucleon. Additional information is obtained from Semi Inclusive DIS (SIDIS) [123] where one observes a hadron in the final stage. In this case, the hadron, which results from the fragmentation of a scattered quark, remembers the original motion of the quark, including its transverse motion and offers such new information through parton fragmentation process. Such process is described by a fragmentation function 

$$D_i(z_h, P_{ht}; Q^2)$$

which is analogous to the uPDF \(f_i(x, k_t; Q^2)\) discussed earlier. Here, \(z_h\) and \(P_{ht}\) are the longitudinal momentum fraction and transverse momentum of the final hadron \(h\) with respect to the fragmenting parton. The present model, however, has not accommodated the fragmentation function.

With this theoretical limitation, let us now discuss the graphical representation of TMDs in the model. First, we take the \(k_t\)-dependent version of the Lastovicka model using the previous and new data and compare their relative pattern with \(x\) and \(k^2_t\).

Using Eq. 2.1, the TMD of the two versions of Lastovicka Model are:

**Model 1:**

$$f_i(x, k^2_t) = \frac{e^{i0}}{M^2} \left( \frac{1}{x} \right)^{D_2 + D_3 \log \left( 1 + \frac{k^2}{k'^2} \right)} \left( 1 + \frac{k^2}{k'^2} \right)^{D_3}$$ (7.3)

**Model 2:**

$$f''_i(x, k^2_t) = \frac{e^{i0'}}{M^2} \left( \frac{1}{x} \right)^{D_2' + D_3' \log \left( 1 + \frac{k'^2}{k''^2} \right)} \left( 1 + \frac{k'^2}{k''^2} \right)^{D_3'}$$ (7.4)

We take the mean value of the parameters \(D_1, D_2, D_3\) from Eq. 2.12 and that of \(D_1', D_2', D_3'\) from Table 2.1 respectively. Here, for simplicity, \(Q^2\) values represent \(k^2_t\)s-values. Tables 7.1 and 7.2 give the mean values of the parameters for the models 1 and 2 respectively.

<table>
<thead>
<tr>
<th>(D_0)</th>
<th>(D_1)</th>
<th>(D_2)</th>
<th>(D_3)</th>
<th>(k^2_0)(GeV²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.339</td>
<td>0.073</td>
<td>1.013</td>
<td>-1.287</td>
<td>0.062</td>
</tr>
</tbody>
</table>
Table 7.2 Mean values taken from Table 2.1 for the parameters of model 2

<table>
<thead>
<tr>
<th>$D'_0$</th>
<th>$D'_1$</th>
<th>$D'_2$</th>
<th>$D'_3$</th>
<th>$k^2_{0}(\text{GeV}^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.354</td>
<td>0.071</td>
<td>1.032</td>
<td>-1.314</td>
<td>0.064</td>
</tr>
</tbody>
</table>

As an illustration, we use Eq. 2.8 with $N_f = 4$ and assume $u$, $d$, $s$ and $c$ are in the ratio

$$e^{D'_0} : e^{D'_d} : e^{D'_s} : e^{D'_c} = 4 : 4 : 1 : 1 \quad (7.5)$$

for definiteness. This gives $e^{D'_u} = 1.008 = e^{D'_d}$ and $e^{D'_s} = 0.252 = e^{D'_c}$. Similarly $e^{D'_{us}} = 1.024 = e^{D'_{ud}}$ and $e^{D'_{cs}} = 0.256 = e^{D'_{cd}}$.

In Fig. 7.1, TMD vs $k^2_t$ is shown using Eqs.7.3 and 7.4 for representative values of

(i) $x = 10^{-4}$ and (ii) $x = 0.01$ setting $M^2 = 1 \text{ GeV}^2$ and

(i) $k^2_t = 0.01 \text{ GeV}^2$ and (ii) $k^2_t = 0.25 \text{ GeV}^2$

considering the $x$-range: $10^{-4} \leq x \leq 0.01$ and $k^2_t$-range: $0.01 \leq k^2_t \leq 0.25 \text{ GeV}^2$ for convenient.

The present graphical analysis of TMDs (Figs. 7.1-7.2) is a comparison of both the versions of self-similar models of proton structure function. The steep rise of TMD at small $x$ is due to their growth as power law in $\left(\frac{1}{x}\right)$ as evidence from Eqs. 7.3 and 7.4.

From Fig. 7.1, it can be seen, both the TMDs decrease in both the versions with increasing $k^2_t$.

In several TMD models [125–127], $x$ and $k^2_t$ are parameters in factorisable form:

$$f_i(x, k^2_t; Q^2) = q_i(x, Q^2)h(k^2_t) \quad (7.6)$$
Fig. 7.1 TMD vs $k_t^2$ for two representative values of (a) $x = 10^{-4}$ and (b) $x = 0.01$ for Models 1 and 2. Here, M1(u/d) (line) and M2(u/d) (dotted) represents the TMD for u and d quarks for Models 1 and 2 respectively. Similarly, M1(s/c) (dot-dashed) and M2(s/c) (dashed) represents the TMD for s and c quarks for Models 1 and 2 respectively.
Fig. 7.2 TMD vs $x$ for two representative values of (a) $k_t^2 = 0.01$ GeV$^2$ and (b) $k_t^2 = 0.25$ GeV$^2$ for Models 1 and 2. Here, M1(u/d) (line) and M2(u/d) (dotted) represents the TMD for u and d quarks for Models 1 and 2 respectively. Similarly, M1(s/c) (dot-dashed) and M2(s/c) (dashed) represents the TMD for s and c quarks for Models 1 and 2 respectively.
where $h(k_t^2)$ is the Gaussian of the form of

$$h(k_t^2) = \frac{1}{\langle k_t^2 \rangle} e^{-\frac{k_t^2}{\langle k_t^2 \rangle}} \quad (7.7)$$

with normalization constant

$$\int h(k_t^2) dk_t^2 = 1 \quad (7.8)$$

We note, the assumed factorisable parametrization of TMD in $x$ and $k_t^2$ does not correspond to the $k_t^2$-factorization theorem [16–18, 128–130, 30, 21, 131, 132] which implies integration over $k_t^2$ of the product of the Gaussian function and $f_i(x, k_t^2; Q^2)$. In this case, the result of the integration does not depend on $k_t^2$ but depends on both $<k_t^2>$ and $Q^2$. Eq. 7.6 is only a convenient form of parametrization of TMDs and doesn’t contradict the form of $k_t^2$ factorization theorem.

Such factorization property of TMD is not present in the Models 1 and 2 (Eqs. 7.3-7.4) nor the Gaussian form (Eq. 7.7) [123]. In this sense, the present models are close to the corresponding non-factorisable models of Refs. [122, 133–136]. Only in the absence of correlation term $D_1$ (Eq. 2.1) such factorization property emerges. Specifically, in a model of Ref. [122], it has been shown that this factorization assumption breaks down if one imposes the correct Lorentz structure in the parton model. In this factorisable limit, the $k_t^2$ dependent functional form of TMD (Eqs. 7.3-7.4) are given by

$$h_i(k_t^2) = \frac{1}{M^2} \left( 1 + \frac{k_t^2}{k_0^2} \right)^{D_3} \quad (7.9)$$

and

$$h_i''(k_t^2) = \frac{1}{M^2} \left( 1 + \frac{k_t^2}{k_0''^2} \right)^{D_3'} \quad (7.10)$$

respectively (which are not Gaussian) in contrast to a Gaussian function (Eq. 7.7).
Fig. 7.3 Gaussian TMD vs $k_t^2$ for $h(k_t^2)$ (line) Eq. 7.7, $h_i(k_t^2)$ (dotted) (Model 1), and $h''_i(k_t^2)$ (dashed) (Model 2) respectively.

In Fig. 7.3, we compare the Gaussian TMD Eq. 7.7 with the model TMDs (Eqs. 7.9-7.10) in the absence of the correlation term, taking $\langle k_t^2 \rangle = 0.25$ GeV$^2$ from the Ref. [123] which is used in Eq. 7.7. We note that here the $k_t^2$-dependence is flavor independent. The qualitative feature of Fig. 7.3 is identical to that of Fig. 7.1.

## 7.2.2 TMD ansatz for models having power law growth in log$Q^2$

Let us now discuss the TMDs corresponding to models 4 and 6 of chapter 4. The TMD ansatz for the PDFs of Model 4 and 6 will be:

**Model 4**

TMD

$$\log \hat{f}_i(x, k_t^2) = \hat{D}_1 \log \frac{1}{x} \log \hat{M}_1 + \hat{D}_2 \log \frac{1}{x} + \hat{D}_3 \log \hat{M}_1 + \hat{D}'_0 - \log M^2$$  \hspace{1cm} (7.11)

with

$$\hat{M}_1 = \frac{B_1}{\left(1 + \frac{k_t^2}{k_0^2}\right)} + \frac{B_2}{\left(1 + \frac{k_t^2}{k_0^2}\right)^2}$$  \hspace{1cm} (7.12)
leads to

$$f_i(x, k_t^2) = e^{\hat{D}_0} \left( \frac{1}{x} \right)^{\hat{D}_2} \left( B_1 \left( 1 + \frac{k_t^2}{k_0^2} \right) \right)^{\hat{D}_3 + \hat{D}_1 \log \frac{1}{x}} \left( 1 + \frac{B_2}{B_1} \left( 1 + \frac{k_t^2}{k_0^2} \right) \right)^{\hat{D}_3 + \hat{D}_1 \log \frac{1}{x}} \tag{7.13}$$

Assuming the convergence of the polynomials as occurred in Eq. 7.13 we obtain

$$f_i(x, k_t^2) = e^{\hat{D}_0} \left( \frac{1}{x} \right)^{\hat{D}_2} \left( B_1 \left( 1 + \frac{k_t^2}{k_0^2} \right) \right)^{\hat{D}_3 + \hat{D}_1 \log \frac{1}{x}} \left( 1 + \frac{B_2}{B_1} \left( \hat{D}_3 + \hat{D}_1 \log \frac{1}{x} \right) \right)^{\hat{D}_3 + \hat{D}_1 \log \frac{1}{x}} \tag{7.14}$$

If the parameters $\hat{D}_3$ and $\hat{D}_1$ satisfy the additional condition at

$$\hat{D}_3 + \hat{D}_1 \log \frac{1}{x_0} = 1 \tag{7.15}$$

then the resultant TMD will be

$$f_i(x, k_t^2) = e^{\hat{D}_0} \left( \frac{1}{x} \right)^{\hat{D}_2} \left( \hat{B}_1 \left( 1 + \frac{k_t^2}{k_0^2} \right) \right)^{\hat{D}_3 + \hat{D}_1 \log \frac{1}{x}} \left( 1 + \hat{B}_2 \left( \hat{D}_3 + \hat{D}_1 \log \frac{1}{x} \right) \right)^{\hat{D}_3 + \hat{D}_1 \log \frac{1}{x}} \tag{7.16}$$

**Model 6**

To get the larger $x$ behavior in PDF as well as structure function, magnification factor $\left( \frac{1}{x} \right)$ is changed to $\left( \frac{1}{x} - 1 \right)$ and so the TMD becomes

$$\log f_i(x, k_t^2) = \hat{D}_1 \cdot \log \left( \frac{1}{x} - 1 \right) \cdot \log \left( 1 + \frac{k_t^2}{k_0^2} \right) + \hat{D}_2 \cdot \log \left( \frac{1}{x} - 1 \right)$$

$$+ \hat{D}_3 \cdot \log \left( 1 + \frac{k_t^2}{k_0^2} \right) + \hat{D}_0 - \log M^2 \tag{7.17}$$
Table 7.3 Mean values taken from Table 4.2 for the parameters of model 4

<table>
<thead>
<tr>
<th>$\bar{D}_0$</th>
<th>$\bar{D}_2$</th>
<th>$\bar{B}_1$</th>
<th>$\bar{B}_2$</th>
<th>$\bar{k}_0^2$ (GeV$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.294</td>
<td>1.237</td>
<td>0.438</td>
<td>0.687</td>
<td>0.046</td>
</tr>
</tbody>
</table>

and hence

$$\bar{f}_i(x,k^2_t) = \frac{e^{\bar{D}_0}}{M^2} \left( \frac{1}{x} \right)^{\bar{D}_2} (1-x)^{\bar{B}_2} \left( \frac{\bar{B}_1}{1 + \frac{k^2}{\bar{k}_0^2}} \right)^{\bar{D}_3 + \bar{D}_1 \log \frac{1}{x} + \bar{D}_1 \log (1-x)} \left( 1 + \frac{\bar{B}_2}{\bar{B}_1} \left( \frac{\bar{D}_3 + \bar{D}_1 \log \frac{1}{x} + \bar{D}_1 \log (1-x)}{1 + \frac{k^2}{\bar{k}_0^2}} \right) \right)$$

(7.18)

Imposing the condition

$$\bar{D}_3 + \bar{D}_1 \log \frac{1}{\bar{x}_0} + \bar{D}_1 \log (1 - \bar{x}_0) = 1$$

(7.19)

will lead to corresponding TMD

$$\bar{f}_i'(x,k^2_t) = \frac{e^{\bar{D}'_0}}{M^2} \left( \frac{1}{x} \right)^{\bar{D}'_2} (1-x)^{\bar{B}'_2} \left( \frac{\bar{B}'_1}{1 + \frac{k^2}{\bar{k}'_0^2}} \right)^{\bar{D}'_3 + \bar{D}'_1 \log \frac{1}{x} + \bar{D}'_1 \log (1-x)} \left( 1 + \frac{\bar{B}'_2}{\bar{B}'_1} \left( \frac{\bar{D}'_3 + \bar{D}'_1 \log \frac{1}{x} + \bar{D}'_1 \log (1-x)}{1 + \frac{k^2}{\bar{k}'_0^2}} \right) \right)$$

(7.20)

7.2.3 Graphical representation of TMDs for models having power law growth in $\log Q^2$

To compare the TMDs for Models 4 and 6, we will use Eq. 7.16 for Model 4 with $e^{\bar{D}_0} = 0.964 = e^{\bar{D}'_0}$ and Eq. 7.20 for model 6 with $e^{\bar{D}_0} = 0.241 = e^{\bar{D}_0}$ with $e^{\bar{D}'_0} = 1.004 = e^{\bar{D}'_0}$ and $e^{\bar{D}_0} = 0.251 = e^{\bar{D}_0}$.

The mean values of the parameters for their respective models can be taken from Tables 4.2 and 4.3 and given in Tables 7.3 and 7.4.
Table 7.4 Mean values taken from Table 4.3 for the parameters of model 6

<table>
<thead>
<tr>
<th>$\bar{D}_0'$</th>
<th>$\bar{D}_2'$</th>
<th>$\bar{B}_1'$</th>
<th>$\bar{B}_2'$</th>
<th>$k_0^2$ (GeV$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.335</td>
<td>1.194</td>
<td>0.519</td>
<td>0.082</td>
<td>0.056</td>
</tr>
</tbody>
</table>

Graphical representation of TMDs of Model 4 and 6 are given in Figs. 7.4 and 7.5 within the ranges of $x$: $10^{-4} \leq x \leq 0.01$ and $k_t^2$: $0.01 \leq k_t^2 \leq 0.25$ GeV$^2$ for convenient. It shows both the form of TMDs have got desired $k_t^2$ fall without the burden of singularities as expected. The steep rise of TMDs at small $x$ is due to their growth as power law in $\left(\frac{1}{x}\right)$ as evidence from Eqs. 7.16 and 7.20.

In this case, the models 4 and 6 have got inbuilt factorization in $x$ and $k_t^2$ unlike in the previous case (Models 1 and 2) where TMDs are not factorisable in $x$ and $k_t^2$. The $k_t^2$-dependent functional form of TMD (Eqs. 7.16-7.20) are given by:

$$\bar{h}_i = \frac{1}{M^2} \left( \frac{\bar{B}_1}{1 + \frac{k_t^2}{k_0^2}} \right) \left( 1 + \frac{\bar{B}_2}{\bar{B}_1} \left( \frac{1}{1 + \frac{k_t^2}{k_0^2}} \right) \right)$$  \hspace{1cm} (7.21)

and

$$\bar{h}_i = \frac{1}{M^2} \left( \frac{\bar{B}_1'}{1 + \frac{k_t^2}{k_0^2}} \right) \left( 1 + \frac{\bar{B}_2'}{\bar{B}_1'} \left( \frac{1}{1 + \frac{k_t^2}{k_0^2}} \right) \right)$$  \hspace{1cm} (7.22)

which will be compared with the standard Gaussian $h(k_t^2)$ (Eq. 7.7) in Fig. 7.6.

The above analysis shows whereas the self-similarity based TMDs are in general not factorisable in $x$ and $k_t^2$, the additional constraints resulting in models 4 and 6 indicate such factorization property.

### 7.3 Summary

In this chapter, the self-similarity based model of proton structure function as suggested in Ref. [1] and introduced in chapter 2, is then extended to take into account the transverse structure of the proton by making simple plausible assumptions about defining Transverse
Fig. 7.4 TMD vs $k_t^2$ for two representative values of (a) $x = 10^{-4}$ and (b) $x = 0.4$ for Models 4 and 6. Here, M4(u/d) (line) and M6(u/d) (dotted) represents the TMD for u and d quarks for Models 4 and 6 respectively. Similarly, M4(s/c) (dot-dashed) and M6(s/c) (dashed) represents the TMD for s and c quarks for Models 4 and 6 respectively.
Fig. 7.5 TMD vs $x$ for two representative values of (a) $k_t^2 = 0.01$ GeV$^2$ and (b) $k_t^2 = 0.25$ GeV$^2$ for Models 4 and 6. Here, M4(u/d) (line) and M6(u/d) (dotted) represents the TMD for $u$ and $d$ quarks for Models 4 and 6 respectively. Similarly, M4(s/c) (dot-dashed) and M6(s/c) (dashed) represents the TMD for $s$ and $b$ quarks for Models 4 and 6 respectively.
Momentum Dependent Parton Distribution (TMD). The model ansatzs for TMD however, have general theoretical limitations which are explicitly discussed. Similarities of such model ansatzs for TMD are also compared with several phenomenological models available in current literature. We then obtain graphical representation of TMDs w.r.t. $x$ and $k_t^2$ which have expected $k_t$-behavior. The TMDs are in general non-factorisable in $x$ and $k_t^2$. However, in the models having power law rise in $\log Q^2$ have factorisable property which are similar to a few models, available in the literature.

In this chapter, we however cannot study the TMDs in the entire $x$-range as the model parameters have phenomenological validity only in the limited ranges of $x$ as discussed in chapters from 2 to 4.