6

Momentum Fractions carried by quarks and gluons in models of proton structure functions

6.1 Introduction

How the quarks and gluons share their longitudinal momentum in proton is an important topic of study by itself. It has been studied in [94–96, 52, 97, 98] within perturbative QCD and Lattice QCD [99]. It is equally interesting to study the corresponding pattern of such momentum fractions in other phenomenological models of proton [100–105], available in
current literature. Since the physics at small $x$ has not yet been understood completely, it is an worthwhile topic to study phenomenologically.

In chapters 2-5, we have discussed some of such models based on self-similarity. In this chapter, we will consider self-similarity based models (models 1-4, 7), one QCD based and Froissart bound compatible model (model 5), one with DGLAP approach (model 6) and one with Froissart bound compatible based on self-similarity (model 8). Each of the model has its own phenomenological range of validity, while models 1-3 have power law growth in $Q^2$, the other model has a logarithmic growth in $Q^2$.

The aim of the present chapter is to calculate the partial momentum fractions of small $x$ quarks ($\langle \hat{x} \rangle_q$) and the corresponding upper bound for small $x$ gluons ($\langle \hat{x} \rangle_g$). We will then compare the predictions of all the models with perturbative QCD, Lattice QCD and Ads/QCD models.

Since each of the models has phenomenological range of validity only for a limited small $x$ range, its role in calculating the second moments of parton distributions might be minor. Still it will be instructive to calculate quantitatively how much it contributes to the total momentum fractions.

Another aim of this chapter is therefore to compare the models predictions of $\langle \hat{x} \rangle_q$ at fixed $Q^2$ considering a common $x$-range and compare with theory and experimental data. We will also study if by any specific model can be preferred over others from this analysis.

Finally, possible role of high $x$ quarks and ultra small $x$ gluons are also discussed to realize the expected QCD behavior, not included in the small $x$ models under study.

In section 6.2, we outline the models and essential formalism. In section 6.3, we report the results and discussion. Section 6.4 contains the conclusion.

### 6.2 Models

The method of construction of self-similarity based models has been already discussed in chapters 2, 3, 4 and 5. For completeness we will outline six models for the calculation of momentum fractions carried by quarks and gluons:
Model 1: Proton structure function based on self-similarity:

We will consider Eq. 2.6 as model 1 from chapter 2:

$$F_2(x, Q^2) = \frac{e^{D_0} Q_0^2 (\frac{1}{x})^{D_2-1}}{M^2 (1 + D_3 + D_1 \log (\frac{1}{x}))} \left( \frac{1}{x} \right)^{D_1 \log \left( \frac{1 + Q^2}{Q_0^2} \right)} \left( 1 + \frac{Q^2}{Q_0^2} \right)^{D_3+1} - 1 \right) \tag{6.1}$$

with the validity range:

$$6.2 \times 10^{-7} \leq x \leq 10^{-2}$$

$$0.045 \leq Q^2 \leq 120 \text{ GeV}^2$$

Model 2: Phenomenological analysis of model 1 with more recent data

From the same chapter i.e. chapter 2, we take Eq. 2.14 is for model 2.

$$F_2''(x, Q^2) = \frac{e^{D_0'} Q_0'^2 (\frac{1}{x})^{D_2'^{-1}}}{M^2 (1 + D_3' + D_1' \log (\frac{1}{x}))} \left( \frac{1}{x} \right)^{D_1' \log \left( \frac{1 + Q^2}{Q_0'^2} \right)} \left( 1 + \frac{Q^2}{Q_0'^2} \right)^{D_3'+1} - 1 \right) \tag{6.2}$$

with the validity range:

$$6.62 \times 10^{-6} \leq x \leq 10^{-2}$$

$$0.35 \leq Q^2 \leq 150 \text{ GeV}^2$$

Model 3: Singularity free self-similarity based structure function at small $x$

Model 3 represents Eq. 3.8 of chapter 3.

$$F_2'(x, Q^2) = \frac{e^{D_0'} Q_0'^2 (\frac{1}{x})^{D_2'-1}}{M^2 (1 + D_3' + D_1' \log (\frac{1}{x}))} \left( \frac{1}{x} \right)^{D_1' \log \left( \frac{1 + Q^2}{Q_0'^2} \right)} \left( 1 + \frac{Q^2}{Q_0'^2} \right)^{D_3'+1} - 1 \right) \tag{6.3}$$
with the validity range:

\[ 2 \times 10^{-5} \leq x \leq 0.02 \]
\[ 0.85 \leq Q^2 \leq 10 \text{ GeV}^2 \]

**Model 4: An improved singularity free self-similarity based model of proton structure function at small \( x \)**

Eq. 4.15 is taken from chapter 4 as model 4.

\[
\tilde{F}_2(x, Q^2) = e^{D_0 \tilde{Q}_0^2} \left( \frac{1}{x} \right)^{D_2-1} \tilde{B}_1 \left\{ \log \left( 1 + \frac{Q^2}{\tilde{Q}_0^2} \right) - \frac{\tilde{B}_2}{\tilde{B}_1} \left( \frac{1}{1 + \frac{Q^2}{\tilde{Q}_0^2}} - 1 \right) \right\} 
\]

(6.4)

with the validity range:

\[ 2 \times 10^{-5} \leq x \leq 0.4 \]
\[ 1.2 \leq Q^2 \leq 800 \text{ GeV}^2 \]

**Model 5: Froissart bound compatible model of Block, Durand, Ha and McKay**

For model 5, we choose Block et. al model as described in chapter 5; Eq. 5.33

\[
F_p^B(x, Q^2) = (1-x) \left\{ \frac{F_p}{1-x_p} + A(Q^2) \ln \frac{x_p(1-x)}{x(1-x_p)} + B(Q^2) \ln^2 \frac{x_p(1-x)}{x(1-x_p)} \right\} 
\]

(6.5)

Where,

\[
A(Q^2) = a_0 + a_1 \ln Q^2 + a_2 \ln^2 Q^2 \\
B(Q^2) = b_0 + b_1 \ln Q^2 + b_2 \ln^2 Q^2 
\]

(6.6)
with the validity range

\[ x \leq 0.11 \]
\[ 0.11 \leq Q^2 \leq 1200 \text{ GeV}^2 \]

The numerical values of the model parameters are already given in chapter 5 of Eq. 5.36.

**Model 6: The model of structure function based on approximate solution of DGLAP equation of small \( x \)**

Model 6 represents the model of structure function based on approximate solution of DGLAP equation of small \( x \). Here, we will consider the \( t \)-evolution of singlet structure function [107]

\[ F_S^2(x,t) = F_S^2(x,t_0) \left( \frac{t}{t_0} \right) \]  \hspace{1cm} (6.7)

where, \( t = \log \frac{Q^2}{\Lambda^2} \) and \( t_0 = \log \frac{Q_0^2}{\Lambda^2} \) and \( \Lambda = 0.22 \text{ GeV} \).

The above Eq. 6.7 is based on small \( x \) approximation of DGLAP equation [108] and obtained their solution with Lagrange method [109].

Using the inputs provided by HERAPDF2.0 [92] at \( Q^2 = Q_0^2 = 1.9 \text{ GeV}^2 \) in the definition of

\[ F_S^2 = \sum_i x(q_i + \bar{q}_i) \]  \hspace{1cm} (6.8)

we can get the form of \( F_S^2(x,t_0) \) as:

\[ F_S^2(x,t_0) = 4.07x^{0.714}(1 - x)^{4.84}(1 + 13.4x^2) + 3.15x^{0.806}(1 - x)^{4.08} \]
\[ + 0.105x^{-0.172}(1 - x)^{8.06}(1 + 11.9x) + 0.1056x^{-0.172}(1 - x)^{4.88} \]  \hspace{1cm} (6.9)

and use this in Eq. 6.7 for further calculation.
Model 7: An improved singularity free self-similarity based model of proton structure function extrapolated to large $x$

Eq. 4.25 is the model 7, taken from chapter 4.

$$\bar{F}_2'(x,Q^2) = \frac{e^{\bar{D}_0 Q_0^2}}{M^2} \left( \frac{1}{x} \right)^{\bar{D}_2 - 1} (1-x)\bar{B}_1' \left[ \log \left( 1 + \frac{Q^2}{Q_0^2} \right) - \frac{\bar{B}_2'}{\bar{B}_1'} \left( \frac{1}{1 + \frac{Q^2}{Q_0^2}} - 1 \right) \right]$$

with the validity range:

$$2 \times 10^{-5} \leq x \leq 0.4$$
$$1.2 \leq Q^2 \leq 1200 \text{ GeV}^2$$

Model 8: Froissart Saturated structure function of proton based on self-similarity

Model 8 is Eq. 5.32 taken from chapter 5, which is more closer to the model of [53].

$$\bar{F}_2'(x,Q^2) = e^{\bar{D}_0 Q_0^2} (1-x) \ln \frac{1-x}{x} \bar{B}_1' \left[ \log \left( 1 + \frac{Q^2}{Q_0^2} \right) - \frac{\bar{B}_2'}{\bar{B}_1'} \left( \frac{1}{1 + \frac{Q^2}{Q_0^2}} - 1 \right) \right]$$

with the validity range:

$$1.3 \times 10^{-4} \leq x \leq 0.02$$
$$6.5 \leq Q^2 \leq 120 \text{ GeV}^2$$

6.2.1 Momentum Sum Rule and partial momentum fractions

The momentum sum rule is given as [78, 79, 110]

$$\int_0^1 x \sum (q_i(x,Q^2) + \bar{q}_i(x,Q^2)) \, dx + \int_0^1 G(x,Q^2) \, dx = 1$$  \hspace{1cm} (6.12)
where

\[ G(x, Q^2) = xg(x, Q^2) \]  \hspace{1cm} (6.13)

\( g(x, Q^2) \) is the gluon number density. It can be converted \cite{79} into an inequality if the information about quarks and gluons is available only in a limited range of \( x \), say \( x_a \leq x \leq x_b \)

i.e.

\[ \int_{x_a}^{x_b} x \sum (q_i(x, Q^2) + \bar{q}_i(x, Q^2)) \, dx + \int_{x_a}^{x_b} G(x, Q^2) \, dx < 1 \]  \hspace{1cm} (6.14)

We have omitted the equality sign in Eq. 6.14 because it will correspond to a nucleon, populated by small quarks and gluons (parton) only within the range \( x_a < x < x_b \), which makes no sense physically. This yields the respective information when the momentum fractions carried by small \( x \) quarks and gluons in \( x_a < x < x_b \) to be

\[ \langle \hat{x} \rangle_q = \int_{x_a}^{x_b} x \sum (q_i(x, Q^2) + \bar{q}_i(x, Q^2)) \, dx \]  \hspace{1cm} (6.15)

Using Eq. 2.5, we can write

\[ \langle \hat{x} \rangle_q = \left( \sum_{i=1}^{N_f} e_i^2 \right)^{-1} \int_{x_a}^{x_b} F_2(x, Q^2) \, dx \]  \hspace{1cm} (6.16)

\( e_i \) is the fractional electric charges of quarks and anti quarks. If we assume their flavored dependence and take number of flavors \( N_f = 4 \), we obtain

\[ \sum_{i=1}^{4} e_i^2 = \frac{10}{9} \]  \hspace{1cm} (6.17)

for \( u, d, s \) and \( c \) quarks leading to

\[ \langle \hat{x} \rangle_q = \frac{9}{10} \int_{x_a}^{x_b} F_2(x, Q^2) \, dx \]  \hspace{1cm} (6.18)

Similarly, for \( N_f = 5 \) i.e. for \( u, d, s, c \) and \( b \) quarks, we will have
\[ \sum_{i=1}^{5} e_i^2 = \frac{11}{9} \] (6.19)

and

\[ \langle \hat{x} \rangle_q = \frac{9}{11} \int_{x_a}^{x_b} F_2(x, Q^2) dx \] (6.20)

and

\[ \langle \hat{x} \rangle_g < \int_{x_a}^{x_b} G(x, Q^2) dx < 1 - \langle \hat{x} \rangle_q \] (6.21)

Note that Eq. 2.5 yields only the upper limit of the fractional momentum carried by the gluons in the regime \(x_a < x < x_b\).

In terms of structure function, the momentum sum rule inequality is

\[ \int_{x_a}^{x_b} \left\{ a F_2(x, Q^2) + G(x, Q^2) \right\} dx < 1 \] (6.22)

where \(a = \frac{e^{-D_0}}{e^{D_0}}\) is \(Q^2\)-independent parameter, determined from data [111], \(a = 3.1418\) [78], using the fractionally charged quarks.

We note that the structure function defined in Eq. 2.5 and used subsequently in our calculation is only a singlet nature, the valence-quark being assumed to be negligible at small \(x\).

### 6.3 Results and Discussion

#### 6.3.1 Numerical results of self-similarity based models with linear rise in \(Q^2\): Models 1, 2, 3

We take recourse to numerical method i.e. we evaluate \(\langle \hat{x} \rangle_q\) numerically by using Eq. 6.16 for a few representative values of \(Q^2\) (GeV\(^2\)). For comparison we choose a particular range of \(x\): \(x_a \leq x \leq x_b\) i.e. \(6.2 \times 10^{-7} \leq x \leq 10^{-2}\) as in Ref. [1]. The choice is made because it will be suitable for all the models (models 1-4).
Table 6.1 Results of $\langle \hat{x}\rangle_q$ for $N_f = 4$ of Model 1, 2 and 3 for different $Q^2$

<table>
<thead>
<tr>
<th>$Q^2$ (GeV$^2$)</th>
<th>$\langle \hat{x}\rangle_q$ (Model 1)</th>
<th>$\langle \hat{x}\rangle_q$ (Model 2)</th>
<th>$\langle \hat{x}\rangle_q$ (Model 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q^2 = Q_0^2$</td>
<td>$6.063 \times 10^{-4}$</td>
<td>$2.008 \times 10^{-3}$</td>
<td>$1.232 \times 10^{-3}$</td>
</tr>
<tr>
<td>2</td>
<td>$3.740 \times 10^{-3}$</td>
<td>$4.021 \times 10^{-3}$</td>
<td>$3.539 \times 10^{-3}$</td>
</tr>
<tr>
<td>6</td>
<td>$5.328 \times 10^{-3}$</td>
<td>$5.593 \times 10^{-3}$</td>
<td>$1.455 \times 10^{-2}$</td>
</tr>
<tr>
<td>10</td>
<td>$6.179 \times 10^{-3}$</td>
<td>$6.377 \times 10^{-3}$</td>
<td>$2.833 \times 10^{-2}$</td>
</tr>
<tr>
<td>60</td>
<td>$9.791 \times 10^{-3}$</td>
<td>$9.714 \times 10^{-3}$</td>
<td>-</td>
</tr>
<tr>
<td>80</td>
<td>$1.050 \times 10^{-2}$</td>
<td>$1.033 \times 10^{-2}$</td>
<td>-</td>
</tr>
<tr>
<td>120</td>
<td>$1.152 \times 10^{-2}$</td>
<td>$1.123 \times 10^{-2}$</td>
<td>-</td>
</tr>
<tr>
<td>150</td>
<td>-</td>
<td>$1.172 \times 10^{-2}$</td>
<td>-</td>
</tr>
</tbody>
</table>

In Table 6.1, column 2, 3 and 4 represents the numerical values of $\langle \hat{x}\rangle_q$ for models 1, 2 and 3 taking $N_f = 4$. Here, $\langle \hat{x}\rangle_q$ is recorded up to 150 GeV$^2$, the maximum phenomenological limit for model 2. From the same Table, we observe that the $\langle \hat{x}\rangle_q$ in model 3 is much more than that of model 1 and 2 within its valid range $Q^2 \leq 10$ GeV$^2$. However, the models 1 and 2 have nearly equal $\langle \hat{x}\rangle_q$.

### 6.3.2 Numerical results of self-similarity based model with linear rise in log $Q^2$: Model 4

In Table 6.2, we have recorded the numerical results of $\langle \hat{x}\rangle_q$ of model 4 in column 2 up to its valid range: $Q^2 = 800$ GeV$^2$, considering number of flavor $N_f = 4$ within the same range of $x$ as that of used in earlier calculation (Table 6.1). One can see the pattern of $\langle \hat{x}\rangle_q$ is increasing with increasing $Q^2$ as that of models 1, 2 and 3 above. In column 3, we have put the corresponding results of upper limit of $\langle \hat{x}\rangle_g$ and this is decreasing with increasing $Q^2$. As an illustration for model 4, the ratio of $\langle \hat{x}\rangle_g$ vs $\langle \hat{x}\rangle_q$ are nearly equal to 120, 100 and 91 for $Q^2 = 60, 300$ and 800 GeV$^2$ respectively, far above unity. It indicates that the $\langle \hat{x}\rangle_q$ will never exceed the corresponding upper bound of $\langle \hat{x}\rangle_g$ within their phenomenological ranges of validity where the models make sense.
Fig. 6.1 $\langle \hat{x} \rangle_q$ vs $Q^2$ (GeV$^2$) for $n_f = 4$ of Model 2 (dots), Model 3 (squares) and Model 4 (diamonds) respectively.

6.3.3 Comparison of models 2, 3 and 4:

In Fig. 6.1, we have shown the pattern of $\langle \hat{x} \rangle_q$ for the models 2, 3 and 4 graphically. The faster linear growth in model 3 can be prominently seen from Fig. 6.1.

6.3.4 Numerical results of models 5 and 6

In Table 6.3, we have recorded the numerical values of $\langle \hat{x} \rangle_q$ as well as the upper limit of $\langle \hat{x} \rangle_g$ (using Eq. 6.21) for $Q^2$ upto 1200 GeV$^2$ for models 5 and 6. For comparison, the $Q^2$-range is taken as that of Ref. [53]: $0.85 \leq Q^2 \leq 1200$ GeV$^2$ for both the models and also the $x$-range.

Table 6.2 Results of $\langle \hat{x} \rangle_q$ and upper limit of $\langle \hat{x} \rangle_g$ for $N_f = 4$ of Model 4 for different $Q^2$

<table>
<thead>
<tr>
<th>$Q^2$ (GeV$^2$)</th>
<th>$\langle \hat{x} \rangle_q$</th>
<th>$\langle \hat{x} \rangle_g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q^2 = Q_0^2$</td>
<td>$4.563 \times 10^{-3}$</td>
<td>$9.954 \times 10^{-1}$</td>
</tr>
<tr>
<td>10</td>
<td>$6.574 \times 10^{-3}$</td>
<td>$9.934 \times 10^{-1}$</td>
</tr>
<tr>
<td>60</td>
<td>$8.288 \times 10^{-3}$</td>
<td>$9.917 \times 10^{-1}$</td>
</tr>
<tr>
<td>80</td>
<td>$8.591 \times 10^{-3}$</td>
<td>$9.914 \times 10^{-1}$</td>
</tr>
<tr>
<td>150</td>
<td>$9.161 \times 10^{-3}$</td>
<td>$9.908 \times 10^{-1}$</td>
</tr>
<tr>
<td>300</td>
<td>$9.812 \times 10^{-3}$</td>
<td>$9.901 \times 10^{-1}$</td>
</tr>
<tr>
<td>500</td>
<td>$1.031 \times 10^{-2}$</td>
<td>$9.896 \times 10^{-1}$</td>
</tr>
<tr>
<td>800</td>
<td>$1.076 \times 10^{-2}$</td>
<td>$9.892 \times 10^{-1}$</td>
</tr>
</tbody>
</table>
6.3 Results and Discussion

Table 6.3 Results of $\langle \hat{x} \rangle_q$ and upper limit of $\langle \hat{x} \rangle_g$ for $N_f = 4$ of Model 5 and 6 for different $Q^2$

<table>
<thead>
<tr>
<th>$Q^2$ (GeV$^2$)</th>
<th>$\langle \hat{x} \rangle_q$ (Model 5)</th>
<th>$\langle \hat{x} \rangle_g$ (Model 5)</th>
<th>$\langle \hat{x} \rangle_q$ (Model 6)</th>
<th>$\langle \hat{x} \rangle_g$ (Model 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.85</td>
<td>$2.051 \times 10^{-3}$</td>
<td>$9.979 \times 10^{-1}$</td>
<td>$1.553 \times 10^{-2}$</td>
<td>$9.844 \times 10^{-1}$</td>
</tr>
<tr>
<td>10</td>
<td>$5.882 \times 10^{-3}$</td>
<td>$9.941 \times 10^{-1}$</td>
<td>$4.873 \times 10^{-2}$</td>
<td>$9.512 \times 10^{-1}$</td>
</tr>
<tr>
<td>80</td>
<td>$9.114 \times 10^{-3}$</td>
<td>$9.908 \times 10^{-1}$</td>
<td>$7.673 \times 10^{-2}$</td>
<td>$9.232 \times 10^{-1}$</td>
</tr>
<tr>
<td>150</td>
<td>$1.009 \times 10^{-2}$</td>
<td>$9.899 \times 10^{-1}$</td>
<td>$8.520 \times 10^{-2}$</td>
<td>$9.148 \times 10^{-1}$</td>
</tr>
<tr>
<td>500</td>
<td>$1.196 \times 10^{-2}$</td>
<td>$9.880 \times 10^{-1}$</td>
<td>$1.014 \times 10^{-1}$</td>
<td>$8.986 \times 10^{-1}$</td>
</tr>
<tr>
<td>1200</td>
<td>$1.367 \times 10^{-2}$</td>
<td>$9.863 \times 10^{-1}$</td>
<td>$1.162 \times 10^{-1}$</td>
<td>$8.838 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

is $x_a \leq x \leq x_b$, where $x_a = 6.2 \times 10^{-7}$, the lower limit of $x$ taken from Ref. [1] and $x_b = 0.11$, the extreme limit of $x$ from Ref. [53]. The calculation is done for $N_f = 4$. Column 2 and 4 represents the numerical values of $\langle \hat{x} \rangle_q$ while column 3 and 5 is for the upper limit of $\langle \hat{x} \rangle_g$ of models 5 and 6 respectively. From Table 6.3, at $Q^2 = 500$ GeV$^2$ the ratio of $\langle \hat{x} \rangle_g$ vs $\langle \hat{x} \rangle_q$ of model 5 is 82 and that of for 1200 GeV$^2$ of model 6 is 72 which are again far above unity. A comparison of models 5 and 6 indicates that at any $Q^2$ under study, $\langle \hat{x} \rangle_q$ of model 5 remains around $(\frac{1}{10})^{th}$ of model 6.

It indicates that within the experimental range of validity of each model, upper limit of $\langle \hat{x} \rangle_g$ allowed by the momentum sum rule far exceeds the corresponding value of $\langle \hat{x} \rangle_q$. As noted earlier that the possibility of $\langle \hat{x} \rangle_q$ exceeding the upper bound of $\langle \hat{x} \rangle_g$ cannot be realized in the phenomenological ranges of validity of these two models as well.

6.3.5 Comparison of models having linear growth in $\log Q^2$: Models 4, 5 and 6

In Fig. 6.2, we compare the pattern of $\langle \hat{x} \rangle_q$ for models 4, 5 and 6 by taking $Q^2$ upto 800 GeV$^2$. We observe, all the three pattern of $\langle \hat{x} \rangle_q$ increase on increasing $Q^2$. But the growth for model 6 is larger than that of models 4 and 5. However, the growth can be made closer to models 4 and 5 by decreasing the exponent of $\left( \frac{t}{t_0} \right)$ of Eq. 6.7) as has been noted in Refs. [107, 112] by obtaining more generalized solution of small $x$ DGLAP equation [108] using.
Momentum Fractions carried by quarks and gluons in models of proton structure functions

Fig. 6.2 $\langle \hat{x} \rangle_q$ vs $Q^2 (\text{GeV}^2)$ for $n_f = 4$ of Model 4 (dots), Model 5 (squares) and Model 6 with $\left( \frac{t}{t_0} \right)$ (diamonds) and Model 6$'$ with $\left( \frac{t}{t_0} \right)^{0.2}$ (triangles) respectively.

the Lagrange method [109]. An evolution of the form of $\sim \left( \frac{t}{t_0} \right)^{0.2}$ which makes the model closer to models 4 and 5 is shown in the same Fig. 6.2. Here after this will be defined as model 6$'$.

A common feature of all the models (1-8) is that the $\langle \hat{x} \rangle_q$ increases with $Q^2$ while $\langle \hat{x} \rangle_g$ decreases. However, the upper bound of $\langle \hat{x} \rangle_g$ is always far above the corresponding value of $\langle \hat{x} \rangle_q$ within the phenomenological range of validity of each model as noted earlier. In models 1-3, the rise is faster than that of models 4, 5 and 6 as due to the power law growth in structure function with $Q^2$ of the three models.

6.3.6 Momentum fraction calculation in Froissart bound compatible

Proton structure function (model 8) and its comparison with models 5 and 7

Here, we will compare the models 5, 7 and 8 within the $Q^2$-range taken as $4.5 \leq Q^2 \leq 120 \text{ GeV}^2$ which is common for the three models to obtain the values of $\langle \hat{x} \rangle_q$ within the $x$-range:
Fig. 6.3 $\langle \hat{x} \rangle_q$ vs $Q^2$ (GeV$^2$) for $n_f = 4$ of model 7 (dots), model 8 (squares) and model 5 (diamonds) respectively.

$6.62 \times 10^{-7} \leq x \leq 0.02$. Note that while the model 7 has got power law growth in $\frac{1}{x}$, models 5 and 8 have slower growth of $\log^2 \frac{1}{x}$.

In Table 6.4, we have listed the values of $\langle \hat{x} \rangle_q$ for the models 7 and 8 and shown their pattern w.r.t $Q^2$ in Fig. 6.3. From the Fig. 6.3, we can observe that model 7 and 8 has the faster rise than the model 5. Results of model 5 are taken from Table 6.3 column 2.

So from the above analysis, we can conclude that in all the models of small $x$ partons, partial momentum fraction carried by quarks $\langle \hat{x} \rangle_q$ rises with $Q^2$ in various degrees. However, invariably fall short of allow upper limit of corresponding upper bound partial momentum fractions of gluons $\langle \hat{x} \rangle_g$.

Table 6.4 Results of $\langle \hat{x} \rangle_q$ for $N_f = 4$ of models 7 and 8 for different $Q^2$

<table>
<thead>
<tr>
<th>$Q^2$ (GeV$^2$)</th>
<th>$\langle \hat{x} \rangle_q$ (model 7)</th>
<th>$\langle \hat{x} \rangle_q$ (model 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$8.500 \times 10^{-3}$</td>
<td>$1.008 \times 10^{-2}$</td>
</tr>
<tr>
<td>6</td>
<td>$9.270 \times 10^{-3}$</td>
<td>$1.045 \times 10^{-2}$</td>
</tr>
<tr>
<td>10</td>
<td>$1.023 \times 10^{-2}$</td>
<td>$1.089 \times 10^{-2}$</td>
</tr>
<tr>
<td>60</td>
<td>$1.367 \times 10^{-2}$</td>
<td>$1.251 \times 10^{-2}$</td>
</tr>
<tr>
<td>80</td>
<td>$1.050 \times 10^{-2}$</td>
<td>$1.276 \times 10^{-2}$</td>
</tr>
<tr>
<td>100</td>
<td>$1.467 \times 10^{-2}$</td>
<td>$1.294 \times 10^{-2}$</td>
</tr>
<tr>
<td>120</td>
<td>$1.501 \times 10^{-2}$</td>
<td>$1.311 \times 10^{-2}$</td>
</tr>
</tbody>
</table>
The above feature of the analysis appears to be in apparent conflict with the QCD expectation that the total momentum fractions of quarks in a Proton should decrease with $Q^2$ while that of gluons should increase. However, in a subsequent subsection 6.3.8, we will indicate this feature is not incompatible with QCD, by considering the dominance of sea quarks at small $x$ and valence quarks at large $x$ together with a faster rise of gluon at ultra small $x$ compared with sea quarks.

### 6.3.7 Comparison with perturbative QCD, Lattice QCD and Ads/QCD models

The predictions of perturbative QCD are:

$$\lim_{Q^2 \to \infty} \langle x \rangle_q = \frac{3N_f}{2N_g + 3N_f}, \quad (6.23)$$

$$\lim_{Q^2 \to \infty} \langle x \rangle_g = \frac{2N_g}{2N_g + 3N_f}, \quad (6.24)$$

Here, $N_f$ and $N_g$ represent the number of active flavors and number of gluons respectively.

For SU(3)$_c$, $N_g = 8$. For $N_f = 5$, Eqs. 6.23 and 6.24 yield $\langle x \rangle_g = \frac{1}{2} (\langle x \rangle_q + \langle x \rangle_g)$: 50% of the momentum of proton is carried by gluons, as noted in [95] and claimed to be experimentally tested in [111].

In Ref. [97], it has alternative asymptotic prediction:

$$\lim_{Q^2 \to \infty} \langle x \rangle_q = \frac{6N_f}{N_g + 6N_f}, \quad (6.25)$$

$$\lim_{Q^2 \to \infty} \langle x \rangle_g = \frac{N_g}{N_g + 6N_f}, \quad (6.26)$$

Where Eqs. 6.23 and 6.24 imply that except for $N_f = 6$, $\langle x \rangle_q < \langle x \rangle_g$. Specifically, for $N_f = 5$, Eqs. 6.23-6.24 yield $\langle x \rangle_g = \frac{1}{2} (\langle x \rangle_q + \langle x \rangle_g)$ and Eqs. 6.25-6.26 give $\langle x \rangle_g = \frac{1}{5} (\langle x \rangle_q + \langle x \rangle_g)$. 
In the above equations, $\langle x \rangle_q$ and $\langle x \rangle_g$ denote the momentum fractions carried by quarks and gluons respectively for the entire $x$-range.

The difference between Eqs. 6.23-6.24 and Eqs. 6.25-6.26 is attributed in Ref. [97] to the proper gauge invariant definition of gluon momentum density; its definition in earlier works [94–96, 52] includes a quark - gluon interaction term and hence resulted in an inflated value of gluon momentum fraction in proton.

However, later Ji [98] refutes the claim of Chen et al [97], underlying the correctness of the QCD prediction, Eqs. 6.23-6.24 [94–96, 52].

However, none of the Refs. [94–96, 52, 97, 98] specifically states about the behavior of partial momentum fractions $\langle \hat{x} \rangle_q$ and $\langle \hat{x} \rangle_g$, relevant for phenomenological study in limited small $x$ regimes and finite $Q^2$, as in the present analysis.

This is also true for Lattice QCD and Ads/QCD models. With this limitation in mind, we outline the prediction of Lattice QCD [99] as well as Ads/QCD [114] models.

In Lattice QCD, its predictions for total momentum fractions for individual flavor are $\langle x \rangle_u = 34\%$, $\langle x \rangle_d = 16\%$, $\langle x \rangle_s = 4\%$ leading to total $\langle x \rangle_q = 54\%$. The lattice analysis also yields $\langle x \rangle_g = 36\%$, while remaining 10% proton momentum fraction remained unaccounted. The analysis was carried out at momentum scale $\mu^2 = 4 \text{ GeV}^2$. Thus, the analysis does not yet rule out the possibility of $\langle x \rangle_q$ that exceeds $\langle x \rangle_g$ at low momentum scale of lattice QCD, where perturbative QCD is not applicable.

Ads/QCD [114–116] based models of proton structure function on the other hand predicts that the proton momentum fraction carried by valence quarks decreases with $Q^2$ consistence with perturbative QCD [112, 107] and is reported in Table 6.5.

### Table 6.5 Proton momentum fraction carried by valence quarks $\langle \hat{x} \rangle_{qv}$ with $Q^2$.

<table>
<thead>
<tr>
<th>$Q^2 \text{ GeV}^2$</th>
<th>0.2</th>
<th>0.6</th>
<th>1.0</th>
<th>7.0</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle x \rangle_{qv}$</td>
<td>0.60</td>
<td>0.48</td>
<td>0.45</td>
<td>0.38[0.55]</td>
<td>0.37</td>
<td>0.35</td>
</tr>
</tbody>
</table>
Table 6.6 $\langle \hat{x} \rangle_q$ of various models at a fixed $Q^2 (= 7 \text{ GeV}^2)$ and the same $x$-range

<table>
<thead>
<tr>
<th>Models</th>
<th>2</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \hat{x} \rangle_q$</td>
<td>$5.826 \times 10^{-3}$</td>
<td>$5.328 \times 10^{-3}$</td>
<td>$2.671 \times 10^{-3}$</td>
<td>$5.487 \times 10^{-3}$</td>
<td>$6.830 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 6.7 $\langle x \rangle_q$ of Lattice QCD ($= 4 \text{ GeV}^2$) and Ads/QCD, ZEUS data ($= 7 \text{ GeV}^2$)

<table>
<thead>
<tr>
<th>Models</th>
<th>Lattice QCD</th>
<th>Ads/QCD</th>
<th>ZEUS data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle x \rangle_q$</td>
<td>0.54</td>
<td>0.38</td>
<td>0.55</td>
</tr>
</tbody>
</table>

For comparison, we note that the recent ZEUS data [117] yields the momentum fraction due to valence quarks at $Q^2 = 7 \text{ GeV}^2$ to be $\langle x \rangle_q \approx 0.55$, which is also included in Table 6.5.

We therefore finally compare our predictions of partial momentum fractions of quarks $\langle \hat{x} \rangle_q$ of models 2, 5, 7 and 8 with the predictions of total momentum fractions carried by quarks $\langle x \rangle_q$ in Lattice QCD and Ads/QCD at $Q^2 = 7 \text{ GeV}^2$ and $x$-range: $6.62 \times 10^{-6} \leq x \leq 10^{-2}$ within which each model has got its validity. In Table 6.6 results are given for $\langle \hat{x} \rangle_q$. We note that the model predictions of Lattice QCD is at 4 GeV$^2$ and not at 7 GeV$^2$. In Table 6.7 we show the results of Lattice QCD and Ads/QCD together with ZEUS data.

A comparison of Table 6.6 with that of 6.7 indicates that the partial momentum fractions of small $x$ quarks calculated in all the models are only a very small fraction of the predicted total momentum fraction in Lattice QCD, Ads/QCD or data which is however not unexpected. The range of small $x$ is merely $6.62 \times 10^{-6} \leq x \leq 10^{-2}$ to be compared with $0 < x < 1$. Therefore it is not possible to infer which model is closer to theory. However, in general a phenomenological model with a larger applicability range should be preferred, unless that comes at the price of making unjustified assumption of the model itself. In this sense, model 5 ($x \leq 0.11$ and $0.11 \leq Q^2 \leq 1200 \text{ GeV}^2$) should be preferred followed by model 7 ($2 \times 10^{-5} \leq x \leq 0.4$ and $1.2 \leq Q^2 \leq 1200 \text{ GeV}^2$). For the models 5 and 7, the momentum fractions carried by the small $x$ quarks at $Q^2 = 7 \text{ GeV}^2$ carries merely 0.2% of the total momentum of proton and 0.1% of the experimentally determined valence quarks momentum fraction.
6.3.8 Possible role of ultra small $x$ gluons and large $x$ partons

As noted above, the partial momentum fraction of quarks discussed above is a very small fraction of the total momentum fractions of proton.

For completeness, we therefore discuss the plausible role of valence quarks and ultra small $x$ gluons to account for the remaining part of momentum fraction of proton.

It is to be noted that the rise of partial momentum fraction of small $x$ quarks with $Q^2$ (specifically, the logarithmic rise with $Q^2$ in models 4, 5 and 6) can be accommodated within the overall predictions of total momentum fractions as predicted in perturbative QCD [94–96, 52]. At large $x$ ($x \geq 0.2$), valence quarks dominate with the fall of structure function as $F_2^p(x, Q^2) \sim \frac{1}{t^n}$ where $n > 0$ [112, 107] while at small $x$, ($x \leq 0.2$) sea quarks dominate and the rise is power law in $F_2^p(x, Q^2) \sim t^m$, where $m > 0$ [112, 107]. As a result, the parton momentum fraction carried by dominantly sea quarks at small $x$ is expected to rise while the corresponding momentum fraction carried by dominantly valence quarks at large $x$ is expected to fall. However, as the corresponding rise in $Q^2$ for the gluons is faster [118, 81]

$$xG(x) = k(t)^\sigma F_2^g(x, Q^2); \quad k \geq 0, \sigma \geq 0$$

(6.27)

than the quarks, the total gluon momentum will rise faster than the quarks.

We also recall the well known result that the behavior of quarks and gluons at very small and large $x$ limit are [89]:

when $x \to 0$, for small $x$ [85]

$$xf_i(x, Q^2) \rightarrow x^{a_i}(Q^2)$$

(6.28)

for gluon

$$xf_g(x, Q^2) \rightarrow x^{a_g}(Q^2)$$

(6.29)

and for large $x$, when $x \to 1$ [90]

$$xf_i(x, Q^2) \rightarrow (1-x)^{b_i}(Q^2)$$

(6.30)
Momentum Fractions carried by quarks and gluons in models of proton structure functions

also for gluon

\[ x f_g(x, Q^2) \rightarrow (1 - x)^{h_g(Q^2)} \]  \hspace{1cm} (6.31)

Here \( a_{f_g} \) is -ve and others are +ve.

At intermediate \( x \) scale, one generally uses an interpolating function as polynomial \([92]\) in \( x \sim \sum_{j=0}^{n} A_j x^j \).

Taking into account all these aspects, it is therefore reasonable to realize the expected QCD behavior. The analysis, done in the present chapter only yields a phenomenological evidence that the partial momentum fractions carried by sea quarks increase with \( Q^2 \) but the rise is not inconsistent with QCD expectation that the total momentum fraction carried by quarks (valence quarks) will fall while that of gluons will rise.

6.4 Summary

In this chapter, we have made analysis of small \( x \) partial momentum fraction carried by quark \( \langle \hat{x} \rangle_q \) and gluon \( \langle \hat{x} \rangle_g \) in nine alternative phenomenological models of proton structure function valid in limited small \( x \) regions: \( x_a \leq x \leq x_b \); the limits being determined by phenomenological range of validity in each model. Since the physics of small \( x \) is not completely understood at this point, we have considered both self-similarity based as well as QCD based models. We find that while the self-similarity based models with linear rise in \( Q^2 \) has limited phenomenological ranges of validity, an improved version with liner rise in \( \log Q^2 \) has an wider phenomenological range. We have also considered phenomenological models with Froissart saturation as well. We then compare the partial momentum fractions in all the small \( x \) models and compare with perturbative QCD, Lattice QCD and Ads/QCD.

Our analysis shows that small \( x \) quarks under study contribute merely 0.2% of the total momentum fractions of the proton and plays a minor role in accounting for the predicted and experimentally observed feature of second moments quark distributions. Therefore it
is not possible to find which of the models is closest to the theory. However, if the range of phenomenological validity is taken as the only criteria for choice of a phenomenological model, the model 5 with leading $\log^2 Q^2$ and $\log^2 \frac{1}{x}$ behavior is the most favorable one followed by the model 7 with $\log Q^2$ and power law growth in $\left( \frac{1}{x} \right)$.