Chapter 2

Hydromagnetic Flow of a Nanofluid in a Porous Channel with Expanding or Contracting Walls

2.1 INTRODUCTION

The problems related to laminar flow in channels/pipes with permeable walls is having greater impact in technological and biological models. The major activities can be found in respiratory systems, circulatory systems, engineering and industrial applications (Uchida and Aoki, 1977; Goto and Uchida, 1991; Dauenhauer and Majdalani, 1999; Majdalani and Zhou, 2003; Srinivasacharya et al., 2009; Reddy et al., 2013). Majdalani et al. (2002) discussed the viscous flow between two weakly moving porous walls, driven by small wall expansions and contractions by using similarity transformations. Boutros et al. (2007) presented the similarity solutions by using Lie-group method, for two-dimensional viscous flow in a rectangular domain bounded by two weakly moving porous walls. Si et al. (2010) investigated an unsteady flow in a semi infinite channel with expanding or contracting porous walls in presence of a transverse magnetic field. Asghar et al. (2010) reported both the numerical and analytical solutions of Navier-Stokes equations for the flow in a slowly deforming channel with weak permeability. The author’s concluded that their results are in excellent agreement with numerical results. Si et al. (2011a) presented the analytical solutions for different permeabilities in a porous channel with expanding or contracting walls by employing HAM. Si et al. (2011b) investigated the unsteady micropolar fluid flow in a semi porous channel with expanding or contracting walls of different permeabilities with both uni-
form and non-uniform wall regression. The Soret and Dufour effects on the flow of viscous fluid in a channel along with uniformly expanding or contracting porous walls have been examined by Srinivas et al. (2012). Si et al. (2013) studied the influence of flow and heat transfer of an incompressible micropolar fluid in a channel between two expanding/contracting permeable walls.

As pointed out in Chapter 1, nanoparticle is currently an area of intense scientific interest because of its variety of potential applications in biomedical, optical and electronic field (Khanafer et al., 2003; Choi, 2008; Wang and Mujumdar, 2008; Wen et al., 2009; Khanafer and Vafai, 2011; Dalkilic et al., 2012; Ibrahim and Shankar, 2012; Taylor et al., 2013; Tham et al., 2013; Yadav et al., 2013; Dinarvand et al., 2014). A nanofluid is a fluid containing particles having diameter less than 100nm. Due to the importance of nanofluids so many investigations have been carried out several researchers in this direction. Buongiorno (2006) examined the convective heat transport in nanofluids. In this study the author developed a two-component four equation non-homogeneous equilibrium models for mass momentum and heat transport in nanofluids. The natural convective boundary layer flow of nanofluid over a flat vertical plate was investigated by Kuznetsov and Nield (2010). Khan and Pop (2010) addressed the fundamental two-dimensional boundary layer flow of nanofluid over a stretching sheet. Makinde and Aziz (2011) studied about a boundary layer flow past a stretching sheet with convective boundary conditions. Xu and Pop (2012) examined the fully developed mixed convection flow in vertical channel filled with nanofluids. Analytical and numerical solutions for the influence of wall properties on the peristaltic flow of a nanofluid were first developed by Mustafa et al. (2012). Hashmi et al. (2012b) employed HAM to obtain the analytical solutions for the problem of magnetohydrodynamic squeezing flow of nanofluid between parallel disks. Nadeem et al. (2013) have studied non-orthogonal stagnation point flow of a non-Newtonian fluid towards a stretching surface with heat transfer. Kameswaran et al. (2013) have investigated the effects of homogeneous-
heterogenous reaction in nanofluid flow over a porous stretching sheet. Sheikholeslami et al. (2013) applied Least squares and Galerkin methods to solve the problem of laminar nanofluid flow in a semi-porous channel in presence of magnetic field. Matin and Pop (2013) have investigated the fully developed forced convection heat and mass transfer in a horizontal porous channel filled with a nanofluid. El-Kabeir et al. (2014) have investigated theoretically the effects of thermal radiation and the nonlinear Forchheimer terms on boundary-layer flow and heat transfer by non-Darcy natural convection from a vertical cylinder embedded in a porous medium saturated with nanofluids. Recently, Chamkha et al. (2014) have studied the influence of viscous dissipation and magnetic field on natural convection from a vertical plate in a non-Darcy porous medium saturated with nanofluid under convective boundary condition.

In light of such motivation, we have made an investigation regarding hydromagnetic flow of nanofluid in a porous channel with expanding or contracting walls. Such a study is of great value in biological and engineering research. The governing flow equations are transformed into a system of coupled nonlinear ordinary differential equations by using similarity transformations and then solved analytically by using HAM. The HAM solutions of the present problem have been compared with the numerical solutions obtained by the shooting method coupled with a Runge-Kutta scheme and the results are in very good agreement. The features of the flow characteristics have been discussed in detail.

2.2 FORMULATION OF THE PROBLEM

Consider the laminar, isothermal and incompressible electrically conducting nanofluid in a channel bounded by two permeable surfaces that enable the fluid to enter or exit during successive expansions or contractions. A magnetic field of uniform strength $B_0$ is applied perpendicular to the walls. A planar section of the simulated domain is shown in Figure 2.1. One side of the cross section, representing the distance $2a(t)$ between the
walls is much smaller than the width and length of the channel. Both walls are assumed to have equal permeability and expand or contract uniformly at a time dependent rate \( \dot{a}(t) \). As shown in Figure 2.1, a coordinate system may be chosen with the origin \( \hat{x} = 0 \) at the center of the channel. This enables us to assume flow symmetry about \( \hat{x} = 0 \).

Further, the lower and upper walls of the channel are maintained at temperature \( T_1 \) and \( T_0 \) respectively. The nanoparticle concentration at the lower and upper walls is maintained at \( C_1 \) and \( C_0 \) respectively. Under these assumptions the governing equations are (Buongiorno, 2006; Khan and Pop, 2010)

\[
\begin{align*}
\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} &= 0, \\
\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} &= -\frac{1}{\rho_f} \frac{\partial \hat{p}}{\partial \hat{x}} + \nu_f \nabla^2 \hat{u} - \frac{\sigma B_0^2 \hat{u}}{\rho_f}, \\
\frac{\partial \hat{v}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} &= -\frac{1}{\rho_f} \frac{\partial \hat{p}}{\partial \hat{y}} + \nu_f \nabla^2 \hat{v},
\end{align*}
\]

\[
\begin{align*}
\frac{\partial T}{\partial \hat{t}} + \hat{u} \frac{\partial T}{\partial \hat{x}} + \hat{v} \frac{\partial T}{\partial \hat{y}} &= \beta \nabla^2 T + \tau \left(D_B \left( \frac{\partial C}{\partial \hat{x}} \frac{\partial T}{\partial \hat{x}} + \frac{\partial C}{\partial \hat{y}} \frac{\partial T}{\partial \hat{y}} \right) + \frac{D_T}{T_m} \left( \left( \frac{\partial T}{\partial \hat{x}} \right)^2 + \left( \frac{\partial T}{\partial \hat{y}} \right)^2 \right) \right),
\end{align*}
\]

\[
\begin{align*}
\frac{\partial C}{\partial \hat{t}} + \hat{u} \frac{\partial C}{\partial \hat{x}} + \hat{v} \frac{\partial C}{\partial \hat{y}} &= D_B \left( \frac{\partial^2 C}{\partial \hat{x}^2} + \frac{\partial^2 C}{\partial \hat{y}^2} \right) + \frac{D_T}{T_m} \left( \frac{\partial^2 T}{\partial \hat{x}^2} + \frac{\partial^2 T}{\partial \hat{y}^2} \right)
\end{align*}
\]

where \( \hat{u}, \hat{v} \) are the components of velocity along \( \hat{x} \) and \( \hat{y} \) directions respectively, \( \rho_f \) is the density of the base fluid, \( \hat{p} \) is dimensional pressure, \( t \) is time, \( \nu_f \) is kinematic viscosity, \( \sigma_f \) is electrical conductivity, \( B_0 \) is the strength of applied magnetic field, \( \beta \) is thermal diffusivity, \( D_B \) is Brownian-diffusion coefficient, \( D_T \) is the thermophoretic-diffusion coefficient, \( T_m \) is mean temperature, \( T \) and \( C \) are temperature and nanoparticles concentration (nanoparticle volume fraction) and \( \nabla^2 = \frac{\partial^2}{\partial \hat{x}^2} + \frac{\partial^2}{\partial \hat{y}^2} \), \( \tau = \frac{(\rho c)_p}{(\rho c)_f} \), \( (\rho c)_p \) is the effective heat capacity of nanoparticle, \( (\rho c)_f \) is the heat capacity of the fluid.

The corresponding boundary conditions on velocity distributions are (Majdalani and Zhou, 2003; Boutros et al., 2007; Si et al., 2010; Asghar et al., 2010)

\[
\begin{align*}
\hat{u} (\hat{x}, a) &= 0; \hat{v} (\hat{x}, a) = -v_w = -A^a \dot{a}
\end{align*}
\]
\[ \frac{\partial \hat{u}}{\partial \hat{y}} (\hat{x}, 0) = 0; \hat{v}(\hat{x}, 0) = 0; \hat{u} (0, \hat{y}) = 0. \]  

(2.7)

The corresponding boundary conditions on temperature and nanoparticles concentration are

\[ T (\hat{x}, \hat{y}) = T_0; C (\hat{x}, \hat{y}) = C_0 \text{ at } \hat{y} = a \]  

(2.8)

\[ T (\hat{x}, \hat{y}) = T_1; C (\hat{x}, \hat{y}) = C_1 \text{ at } \hat{y} = -a. \]  

(2.9)

The injection or suction coefficient \( A^* \) that appears in Eq. (2.6) is a measure of wall porosity. Introducing the stream function and mean flow velocity by putting

\[ \hat{u} = \frac{\partial \hat{\psi}}{\partial \hat{y}}; \hat{v} = -\frac{\partial \hat{\psi}}{\partial \hat{x}}. \]  

(2.10)

Due to mass conservation, a similar solution can be developed with respect to \( \hat{x} \), since the variation of the channel height is uniform along the axial direction. Following Berman’s classic approach, and in view of the boundary conditions represented by Eqs. (2.6) and (2.7), a similar solution can be assumed to be of the form

\[ \hat{\psi} = \nu_f \hat{x} \hat{F} (\eta, t)/a; \hat{u} = \nu_f \hat{x} a^{-2} \hat{F}_\eta (\eta, t); \hat{v} = -\nu_f a^{-1} \hat{F} (\eta, t) \]  

(2.11)

where \( \eta = \hat{y}/a \); \( \hat{F}_\eta = \frac{\partial \hat{F}}{\partial \eta} \) and \( \hat{F} (\eta, t) \) is independent of the axial coordinate.

Eliminating the pressure from Eqs. (2.2) and (2.3), and then by substituting Eq. (2.11), we get

\[ \hat{F}_{\eta\eta\eta} + \alpha (\hat{F}_{\eta\eta} + 3 \hat{F}_\eta) + \hat{F} \hat{F}_{\eta\eta} - \hat{F}_\eta \hat{F}_{\eta\eta} - M^2 \hat{F}_\eta - a^2 \nu_f^{-1} \hat{F}_{\eta\eta} t = 0 \]  

(2.12)

where, \( \alpha (t) = a \dot{a}/\nu_f \) is non-dimensional wall expansion ratio, is defined as positive for expansion and negative for contraction, \( M = \frac{\sqrt{\sigma_f B_0 a}}{\sqrt{\mu}} \) is Hartmann number.

The boundary conditions given by Eqs. (2.6) and (2.7) transform into

\[ \hat{F}_\eta (0, t) = 0; \hat{F} (0, t) = 0; \hat{F}_\eta (1, t) = 0; \hat{F} (1, t) = R \]  

(2.13)

where \( R \) is permeation Reynolds number defined by \( R = \alpha v_w/\nu_f = A^* \alpha \), which is positive for injection and negative for suction.
A similar solution with respect to both space and time can be developed by following the transformation described by Uchida and Aoki (1977), Dauenhauer and Majdalani (1999) and Majdalani and Zhou (2003) independently. This can be accomplished by considering the case for which the non-dimensional parameter $\alpha(t) = a_0 \dot{a}_0 / \nu_f$ where $a_0$ and $\dot{a}_0$ denote the initial channel height and expansion rate remains constant and $\hat{F} = \hat{F}(\eta)$. This leads to $\hat{F}_{\eta \eta t} = 0$.

Eqs. (2.11), (2.12) and (2.13) can be normalized by putting
\[ \psi = \frac{\hat{\psi}}{a \dot{a}}; u = \frac{\hat{u}}{\dot{a}}; v = \frac{\hat{v}}{a}; x = \frac{\hat{x}}{a}; f = \frac{\hat{F}}{R} \] (2.14)
and so
\[ \psi = xF/c; u = x A^* f; v = -A^* f; \]
\[ f^{IV} + \alpha (\eta f'''' + 3 f'') + R f f'''' - R f' f'' - M^2 f'' = 0 \] (2.16)
\[ f''(0) = 0; f(0) = 0; f'(1) = 0; f(1) = 1 \] (2.17)
where prime denotes differentiation with respect to $\eta$.

The temperature and nanoparticles concentration in the channel can be expressed as
\[ T = T_0 + (T_1 - T_0) \theta(\eta), C = C_0 + (C_1 - C_0) \phi(\eta). \] (2.18)
The dimensionless forms of temperature and nanoparticles concentration from Eq. (2.18) are
\[ \theta = \frac{T - T_0}{T_1 - T_0}, \phi = \frac{C - C_0}{C_1 - C_0}. \] (2.19)
Substituting the Eqs. (2.11) and (2.18) into the Eqs. (2.4) and (2.5), one obtains
\[ \theta'' + \text{Pr} \alpha \eta \theta' + \text{Pr} f \theta' + \text{Pr} Nb \theta' \phi' + \text{Pr} Nt \theta'^2 = 0 \] (2.20)
\[ \phi'' + \text{Le} \alpha \eta \phi' + \text{Re} f \phi' + \frac{Nt}{Nb} \theta'' = 0. \] (2.21)
The corresponding boundary conditions are
\[ \theta(-1) = 1; \theta(1) = 0; \phi(-1) = 1; \phi(1) = 0 \] (2.22)
where $\text{Pr} = \nu_f / \beta$ is Prandtl number, $Nb = \frac{\tau D_B (C_1 - C_0)}{\nu_f}$ is Brownian motion parameter, $Nt = \frac{\tau D_T (T_1 - T_0)}{T_m \nu_f}$ is thermophoresis parameter, $Le = \frac{\beta}{D_B}$ is Lewis number.
2.3 SOLUTION OF THE PROBLEM

In this section, we give the analytical approximation to Eqs. (2.16), (2.20) and (2.21) with the boundary conditions (2.17) and (2.22). For HAM solutions of Eqs. (2.16), (2.20) and (2.21), the initial approximations $f_0$, $\theta_0$ and $\phi_0$ and auxiliary linear operators $L_1$, $L_2$ and $L_3$ are as follows

$$
f_0(\eta) = \frac{3\eta - \eta^3}{2}, \theta_0 = \frac{1 - \eta}{2}, \phi_0 = \frac{1 - \eta}{2}$$

(2.23)

$$
L_1(f) = \frac{d^4f}{d\eta^4}, L_2(\theta) = \frac{d^2\theta}{d\eta^2}, L_3(\phi) = \frac{d^2\phi}{d\eta^2}
$$

(2.24)

with

$$
L_1(c_1\eta^3 + c_2\eta^2 + c_3\eta + c_4) = 0, L_2(c_5\eta + c_6) = 0, L_3(c_7\eta + c_8) = 0
$$

(2.25)

where $c_i (i = 1 - 8)$ are constants.

2.3.1 ZERO-ORDER DEFORMATION EQUATIONS

Let $q \in [0, 1]$ be an embedding parameter and $h$ be the auxiliary non-zero parameter. The deformation equations at zero-order can be written as follows:

$$(1 - q) L_1[\hat{f}(\eta; q) - f_0(\eta)] = qh_N_1[\hat{f}(\eta; q)]$$

(2.26)

$$\hat{f}(0; q) = 0, \hat{f}''(0; q) = 0, \hat{f}'(1; q) = 0, \hat{f}(1; q) = 1$$

(2.27)

$$(1 - q) L_2[\hat{\theta}(\eta; q) - \theta_0(\eta)] = qh_\theta N_2[\hat{\theta}(\eta; q), \hat{f}(\eta; q), \hat{\phi}(\eta; q)]$$

(2.28)

$$\hat{\theta}(-1; q) = 1, \hat{\theta}(1; q) = 0$$

(2.29)

$$(1 - q) L_3[\hat{\phi}(\eta; q) - \phi_0(\eta)] = qh_\phi N_3[\hat{\phi}(\eta; q), \hat{f}(\eta; q), \hat{\theta}(\eta; q)]$$

(2.30)

$$\hat{\phi}(-1; q) = 1, \hat{\phi}(1; q) = 0$$

(2.31)

where

$$N_1[\hat{f}(\eta; q)] = \frac{\partial^4\hat{f}(\eta; q)}{\partial\eta^4} + \alpha \left[ \frac{\partial^2\hat{f}(\eta; q)}{\partial\eta^2} + 3 \frac{\partial^2\hat{f}(\eta; q)}{\partial\eta^2} \right] + R \hat{f}(\eta; q) \frac{\partial^2\hat{f}(\eta; q)}{\partial\eta^2}$$

$$- R \frac{\partial^2\hat{f}(\eta; q)}{\partial\eta^2} - M^2 \frac{\partial^2\hat{f}(\eta; q)}{\partial\eta^2}$$

(2.32)
\[ N_2[\theta (\eta; q), \hat{f}(\eta; q), \hat{\phi}(\eta; q)] = \frac{\partial^2 \hat{\theta}(\eta; q)}{\partial \eta^2} + \alpha Pr \frac{\partial \hat{\theta}(\eta; q)}{\partial \eta} + R Pr \hat{f}(\eta; q) \frac{\partial \hat{\theta}(\eta; q)}{\partial \eta} \\
+ Pr Nb \frac{\partial \hat{\theta}(\eta; q)}{\partial \eta} \frac{\partial \hat{\phi}(\eta; q)}{\partial \eta} + Pr Nt \left( \frac{\partial \hat{\theta}(\eta; q)}{\partial \eta} \right)^2 \]

(2.33)

\[ N_3[\hat{\phi}(\eta; q), \hat{f}(\eta; q), \hat{\theta}(\eta; q),] = \frac{\partial^2 \hat{\phi}(\eta; q)}{\partial \eta^2} + Le \alpha \eta \frac{\partial \hat{\phi}(\eta; q)}{\partial \eta} \]

\[ + RLe \hat{f}(\eta; q) \frac{\partial \hat{\phi}(\eta; q)}{\partial \eta} \]

\[ + \frac{Nt}{Nb} \left( \frac{\partial \hat{\theta}(\eta; q)}{\partial \eta} \right)^2. \]

(2.34)

For \( q = 0 \) and \( q = 1 \), we have

\[ \hat{f}(\eta; 0) = f_0(\eta), \hat{f}(\eta; 1) = f(\eta), \hat{\theta}(\eta; 0) = \theta_0(\eta), \]

\[ \hat{\theta}(\eta; 1) = \theta(\eta), \hat{\phi}(\eta; 0) = \phi_0(\eta), \hat{\phi}(\eta; 1) = \phi(\eta). \]

(2.35)

Further, by Taylor’s series expansion one obtains

\[ \hat{f}(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) q^m \text{ where } f_m(\eta) = \frac{1}{m!} \frac{\partial^m \hat{f}(\eta; q)}{\partial q^m} \bigg|_{q=0} \]

(2.36)

\[ \hat{\theta}(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) q^m \text{ where } \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \hat{\theta}(\eta; q)}{\partial q^m} \bigg|_{q=0} \]

(2.37)

\[ \hat{\phi}(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta) q^m \text{ where } \phi_m(\eta) = \frac{1}{m!} \frac{\partial^m \hat{\phi}(\eta; q)}{\partial q^m} \bigg|_{q=0}. \]

(2.38)

We choose \( h_f, h_\theta \) and \( h_\phi \), properly in such a way that these series are convergent at \( q = 1 \), therefore we have the solution expressions from Eqs. (2.36)-(2.38) as follows:

\[ f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) \]

(2.39)

\[ \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) \]

(2.40)

\[ \phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \]

(2.41)
2.3.2 THE HIGH-ORDER DEFORMATION EQUATIONS

Differentiating the zero-order deformation Eqs. (2.26)-(2.31) \(m\) times with respect to \(q\), then dividing by \(m!\), and finally setting \(q = 0\), one obtains the following \(m^{th}\) order deformation equations:

\[
L_1 \left[ f_m (\eta) - \chi_m f_{m-1} (\eta) \right] = h_f R_{1,m} (\eta) \tag{2.42}
\]

\[
L_2 \left[ \theta_m (\eta) - \chi_m \theta_{m-1} (\eta) \right] = h_\theta R_{2,m} (\eta) \tag{2.43}
\]

\[
L_3 \left[ \phi_m (\eta) - \chi_m \phi_{m-1} (\eta) \right] = h_\phi R_{3,m} (\eta) \tag{2.44}
\]

where

\[
R_{1,m} = f_{IV} - 1 + \alpha \left[ \eta f'''_{m-1} + 3 f''_{m-1} \right] + R \sum_{k=0}^{m-1} f_k f'''_{m-1-k}
\]

\[
- R \sum_{k=0}^{m-1} f'_k f''_{m-1-k} - M^2 f''_{m-1} \tag{2.45}
\]

\[
R_{2,m} = \theta''_{m-1} + \alpha Pr \eta \theta'_{m-1} + R Pr \sum_{k=0}^{m-1} f_k \theta'_{m-1} + Pr Nb \sum_{k=0}^{m-1} \theta'_{k} \phi'_{m-1-k}
\]

\[
+ Pr Nt \sum_{k=0}^{m-1} \theta'_{k} \theta'_{m-1-k} \tag{2.46}
\]

\[
R_{3,m} = \phi''_{m-1} + \alpha Le \eta \phi'_{m-1} + R Le \sum_{k=0}^{m-1} f_k \phi'_{m-1} + \frac{Nt}{Nb} \theta''_{m-1} \tag{2.47}
\]

and

\[
\chi_m = \begin{cases} 
1, & m \neq 1 \\
0, & m = 1.
\end{cases} \tag{2.48}
\]

The corresponding boundary conditions are

\[
f'''_{m} (0) = 0, f_m (0) = 0, f'_{m} (1) = 0, f_{m} (1) = 0, \tag{2.49}
\]

\[
\theta_m (-1) = 0, \phi_m (-1) = 0, \theta_m (1) = 0, \phi_m (1) = 0. \tag{2.50}
\]

Series solutions Eqs.(2.39)-(2.41) are obtained by solving Eqs. (2.42)-(2.44) under the conditions Eqs.(2.48)-(2.50).
2.3.3 CONVERGENCE OF HAM SOLUTION

We can adjust and control the convergence of HAM solutions with the help of the non-zero auxiliary parameters $h_f$, $h_\theta$ and $h_\phi$. If $h_f$, $h_\theta$ and $h_\phi$ are properly chosen, the homotopy series solutions may converge fast. Hence to compute the range of admissible values of $h_f$, $h_\theta$ and $h_\phi$, we plot $h$- curves in Figure 2.2. The range of admissible values of $h_f$, $h_\theta$ and $h_\phi$ are $-1.01 \leq h_f \leq -0.3$, $-0.94 \leq h_\theta \leq -0.4$ and $-0.92 \leq h_\phi \leq -0.35$ respectively. Further, we define the average square residual error as a measure of the accuracy of the obtained analytical solutions by HAM. Substituting the approximate solutions of $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ into equations (2.16), (2.20) and (2.21) yields the residual error as follows:

$$E_1 = f^{IV} + \alpha (\eta f''' + 3f'') + R f f'' - R f' f'' - M^2 f''$$  

(2.51)

$$E_2 = \theta'' + \alpha Pr \eta \theta' + R Pr f \theta' + Pr Nb \theta' \phi' + Pr Nt \theta'^2$$  

(2.52)

$$E_3 = \phi'' + \alpha Le \eta \phi' + R Le f \phi' + \frac{Nt}{Nb} \theta''$$  

(2.53)

where $E_1$, $E_2$ and $E_3$ correspond to the residual error for $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ respectively. The square residual errors for $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ and the averaged square residual error are defined as follows respectively:

$$\Delta_i = \int_{-1}^{1} E_i^2 d\eta, \text{ (where } i = 1, 2, 3); \Delta = \frac{1}{3} \sum_{i=1}^{3} \Delta_i.$$  

(2.54)

Table 2.1 shows the average square residual error ($\Delta$) for HAM solution. From this table it is clear that the average square residual error ($\Delta$) is evaluated by varying the convergence control parameter $h$ to obtain the optimal value of $h$ which leads to a minimum $\Delta$. Further, we show the square residual error for $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ in Figure 2.3. In these figures we select $h = -0.74$. In order to check the analytical solution, we compare the results corresponding to the axial velocity with that of Asghar et al. (2010) in Table 2.2 for the case of hydrodynamic viscous fluid. It is clear that the present results are in good agreement with the previous studies.
Further, the heat and mass transfer rates in terms of the dimensionless form of Nusselt number and Sherwood number respectively at the walls are defined as

\[ \text{Nu} = -\theta'(\eta)|_{\eta = 1}, \quad \text{Sh} = -\phi'(\eta)|_{\eta = 1}. \]  

(2.55)

2.4 RESULTS AND DISCUSSION

In this section, in order to get the physical insight of the problem, dimensionless temperature, nanoparticle concentration, Nusselt number and Sherwood number distributions have been discussed by assigning numerical values to various parameters that have been emerged in the mathematical formulation and the results are shown graphically in Figures 2.4-2.13. It may be noted that the parameters \( Nb \) and \( Nt \) characterize the strengths of Brownian motion and thermophoresis effects. The larger values of \( Nb \) and \( Nt \), the larger will be the strength of the corresponding effects. Thus \( Nb \) and \( Nt \) can take any value in the range of \( 0 \leq Nb, Nt < 1 \). Further, the liquid metals and oils are characterized by the Prandtl number. The small values of \( Pr \) characterize the liquid metals which have high thermal conductivity and low viscosity, while larger values \( Pr \) correspond to high viscosity oils. Prandtl number \( Pr = 0.015, 0.71, 7, 11.4 \) for mercury, air, water and water at \( 4^\circ C \) respectively. To understand the physics of the problem, we choose \( Pr = 6.2, Nb = Nt = 0.2 \) and \( M = 2 \) unless otherwise stated.

The effects of Brownian motion parameter \( Nb \), thermophoresis parameter \( Nt \), Prandtl number \( Pr \), Hartmann number \( M \), the wall expansion ratio \( \alpha \) and the permeation Reynolds number \( R \) on the dimensionless temperature distribution \( \theta \) are shown in Figures 2.4-2.8. Figure 2.4 plots the temperature profiles for different values of \( Nb \) for both the cases of injection combined with wall contraction and expansion. As the Brownian motion strengthens, this leads to effective movement of nanoparticles from the walls to the fluid. Because of this reason, the dimensionless temperature \( \theta \) increases with an increase in \( Nb \) (Mustafa et al., 2012). A similar conclusion can be drawn when \( Nb \) is replaced by \( Nt \) (see Figure 2.5). Figure 2.6 shows the effect of the Prandtl \( Pr \) on \( \theta \).
One can observe that $\theta$ increases for a given increases in $Pr$ for the case of injection combined with wall contraction (see Figure 2.6a). From Figure 2.6b it is clear that for the case of wall expansion, $\theta$ changes rapidly and exhibits an oscillating character with an increase in $Pr$, for the wall expansion combined with injection. Figure 2.7 illustrates the temperature profiles for different values of Hartmann number $M$. It is observed that increasing Hartmann number $M$ increases the temperature for the case of wall contraction combined with injection (see Figure 2.7a). The opposite observation can be drawn for the case of wall expansion combined with suction (see Figure 2.7b). Figure 2.8 depicts the effects of the wall expansion ratio $\alpha$ and the permeation Reynolds number $R$ on $\theta$. From Figure 2.8a it is noticed that for the case of wall expansion, the maximum temperature distribution is near the lower wall and the minimum is near the upper wall. The behavior is reversed for wall contraction. From the same figure one can conclude that the temperature profile is almost parabolic for $\alpha < 0$ but oscillates more for $\alpha \geq 0$ and the maximum temperature is shifted to the boundary layers near the walls. From Figure 2.8b one can observe that for the case of injection, the maximum temperature distribution is near the lower wall and the minimum is near the upper wall. The behavior is reversed for the case of suction. Further, from the same figure one can conclude that the temperature profile is almost parabolic for $R < 0$ but oscillates more for $R \geq 0$.

Figures 2.9-2.11 illustrates the effects of Brownian motion parameter $Nb$, Lewis number $Le$ and thermophoresis parameter $Nt$ on the dimensionless nanoparticles concentration $\phi$. Figure 2.9 elucidates the effect of Brownian motion parameter $Nb$ on nanoparticles concentration. From Figure 2.9a it is clear that $\phi$ is an increasing function of $Nb$ for the case of injection combined wall contraction. This increase is due to the effective movement of nanoparticles from the lower wall to the fluid. From Figure 2.9b one can observe that for the case of injection combined with wall expansion, $\phi$ exhibits oscillating character for a given increase in $Nb$. Further, the amplitude of oscillation decreases with increase of $Nb$. A similar conclusion can be drawn when $Nb$ is replaced.
by $Le$ (see Figure 2.10). Figure 2.11 reveals the effect of thermophoresis parameter $Nt$ on $\phi$. From Figure 2.11a it is noticed that the nanoparticles concentration $\phi$ decreases for given increases in thermophoresis parameter $Nt$ (Mustafa et al., 2012; Akbar et al., 2013). From Figure 2.11b one can observe that $\phi$ exhibits oscillating character with an increase in $Nt$. The amplitude of oscillation increases with an increase of $Nt$.

The effects of Brownian motion parameter $Nb$ and Hartmann number $M$ on Nusselt number $Nu$ are shown in Figures 2.12a and 2.12b respectively against thermophoresis parameter $Nt$. It is observed that the Nusselt number decreases with an increase in $Nt$ at the wall $\eta = -1$, while it is in proportion to $Nt$ at the wall $\eta = 1$. From Figure 2.12a it is clear that $Nu$ decreases as $Nb$ increases at the wall $\eta = -1$, while it increases at the wall $\eta = 1$. From Figure 2.12b it is noticed that $Nu$ decreases for a given increase in $M$ at both the walls. The influence of Brownian motion parameter $Nb$ and Lewis number $Le$ on Sherwood number $Sh$ is shown in Figures 2.13a and 2.13b against thermophoresis parameter $Nt$. It is observed that the $Sh$ increases with an increase in $Nt$ at the wall $\eta = -1$, while it decreases at the wall $\eta = 1$. From Figure 2.13a it is clear that $Sh$ decreases with an increase in $Nb$ at the wall $\eta = -1$, while it increases at the wall $\eta = 1$. From Figure 2.13b it is noticed that $Sh$ decreases for a given increase in $Le$ at both the walls.
Figure 2.1: Two-dimensional domain with expanding or contracting porous walls.
Figure 2.2: $h$-curves on the $20^{th}$ order approximations for the functions $f$, $\theta$, $\phi$ for $M = 2$, $R = 1$, $\alpha = -1$, $Pr = 6.2$, $Nb = Nt = 0.2$, $Le = 1.25$. 
Figure 2.3: Square residual error of $f$, $\theta$, $\phi$ for $M = 2$, $R = 1$, $\alpha = -1$, $\Pr = 6.2$, $Nb = Nt = 0.2$, $Le = 1.25$, $h = -0.74$. 
Figure 2.4: Effect of $Nb$ on temperature distribution.
Figure 2.5: Effect of $Nt$ on temperature distribution.
Figure 2.6: Effect of $Pr$ on temperature distribution.
Figure 2.7: Effect of $M$ on temperature distribution.
Figure 2.8: Temperature distribution (a) Effect of $\alpha$ (b) Effect of $R$. 

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Figure 2.9: Effect of $Nb$ on concentration distribution.
Le / Equal 1, 3, 5, 7, 8
R = 1, \( \alpha \) = 1
Analytical
Numerical
(b)

\[
\begin{array}{cccccc}
-1.0 & -0.5 & 0.0 & 0.5 & 1.0 \\
0.0 & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 \\
\end{array}
\]

Figure 2.10: Effect of \( Le \) on concentration distribution.
Figure 2.11: Effect of $Nt$ on concentration distribution.
Figure 2.12: Nusselt number distribution.
Figure 2.13: Sherwood number distribution.
Table 2.1: The averaged square residual error with varying $h$ at $20^{th}$ order of approximation of HAM for $R = 1$, $\alpha = -1$, $M = 2$, $Pr = 6.2$, $Nb = Nt = 0.2$, $Le = 1.25$.

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Table 2.2: Comparison of present study solutions with that of Majdalani et al. (2002), Boutros et al. (2007), Asghar et al. (2010) for axial velocity of viscous base fluid when $R = 5$, $\phi = 0$, $\alpha = 0.5$ and $M = 0$.

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