Chapter 4

Interaction of One Prey and Two Predators: Competing Predators

This chapter is focussed completely on the analysis of the competing predators model. The chapter is divided into four sections in which each section deals with a particular aspect of the analysis. Namely, first section deals with various boundary equilibria and the interior equilibrium point. Second section is devoted to the global stability of the interior equilibrium point and the existence of periodic solutions for the system. In the third section, the aspect of persistence is analyzed and in the fourth section, a variation of this model with a constant time lag, which is also developed in this study, is discussed along with its stability.

4.1 Equilibria and their Stability

In this section, the criteria for the existence and non-existence of various boundary equilibria and the interior equilibrium point for first model, the "Competing Predators Model" defined by the system of equations (3.1) has been established. Their local and asymptotic stability has been studied. The asymptotic stability of an equilibrium point tells us about the behaviour of solutions starting sufficiently close to it. Besides this, these results will later be used to further study model 1 (3.1).
4.1.1 Boundary Equilibrium Points

To work out the boundary equilibrium points let us rewrite our system (3.1) as:

\[ x'(t) = \alpha xg(x) - y p_1(x) - z p_2(x) = 0 \] ... (4.1)

\[ y'(t) = y[-s_1(y) - q_1(z) + c_1 p_1(x)] = 0 \] ... (4.2)

\[ z'(t) = z[-s_2(z) - q_2(y) + c_2 p_2(x)] = 0 \] ... (4.3)

Values \( x = y = z = 0 \) will clearly satisfy the equations (4.1) – (4.3). Therefore it is clear that \( E_0 = E_0(0,0,0) \) is an equilibrium point.

Equations of the system will also be satisfied when \( x = k \), where \( k \in \mathbb{R}^+ \), and \( y = z = 0 \). From that we can say that \( E_1 = E_1(k,0,0) \) is also an equilibrium point.

**Theorem 4.1.1:** A necessary and sufficient condition for an equilibrium of the form \( E_2 = E_2(x_1,y_1,0) \) to exist in \( \mathbb{R}^+_{xy} \) is that the following two hypotheses are satisfied.

**Hypothesis 1:** \( \exists \hat{x} \) such that \( 0 < \hat{x} < k \), where \( k \in \mathbb{R}^+ \)

**Hypothesis 2:** \( \frac{s_1(0)}{c_1} = p_1(\hat{x}) \)

**Proof:**

Necessity:

Since \( x(t) \leq k + \varepsilon \), eventually we get that the right hand side of the equation...
w' = w[-s, (0) + c, p, (k + ε)] dominates the equation y' = y[-s, (y) + c, p, (x)]

Further, we choose an ε so small that α = -s, (0) + c, p, (k + ε) < 0. Hence, by standard comparison theorem, \( \lim_{y \to 0} y(t) = 0 \).

Sufficiency:

Since \( E_0 \) and \( E_1 \) are saddle points, by Poincaré–Bendixon theorem, either every solution initially in the positive XY quadrant contains one or more equilibrium points lying in the interior of positive XY quadrant or the \( ω \)-limit set is a periodic orbit. In either case, there exists an equilibrium point in the interior of XY quadrant. (See note)

Hence the proof.

Note: Under the hypothesis of theorem given below, \( Γ \) contains at least one critical point of equation (4.4) on its interior, and, assuming that there are only a finite number of critical points of equation (4.4) on the interior of \( Γ \), the sum of the indices at these critical points is equal to one. See that theorem below.

Theorem: Suppose that \( f \in C^1(E) \) where \( E \) is an open subset of \( \mathbb{R}^2 \) and that \( E \) contains a cycle \( Γ \) of the system
\[
\dot{x} = f(x) \tag{4.4}
\]

it follows that \( I_k(Γ) = 1 \).
Theorem 4.1.2: A necessary and sufficient condition for an equilibrium of the form $E_3 = E_3(x_2, 0, z_2)$ to exist in $\mathbb{R}^+$ is that the following two hypotheses are satisfied.

Hypothesis 1: $\exists \bar{x}$ such that $0 < \bar{x} < k$, where $k \in \mathbb{R}^+$

Hypothesis 2: $\frac{s_z(0)}{c_2} = p_z(\bar{x})$

Proof: The proof of this theorem is similar to the proof of the above theorem 4.1.1.

![Geometrical Representation of Equilibrium Points](image)

Figure 4.1: Geometrical Representation of Equilibrium Points

Finally, the last result of this section concerns with the existence of an equilibrium point interior to the positive x-y-z octant.

Theorem 4.1.3: An interior equilibrium point $E^*(x^*, y^*, z^*)$ will exist if the system (3.1) is uniformly persistent.
Proof:
The notion of uniform persistence is explained in the section 4.3 and various criteria are obtained for the uniform persistence of the system in consideration. The proof of the above result follows from Butler et. al. (1986).

The various equilibrium points are illustrated in the figure 4.1.

4.1.2 Local Stability

The stability of an equilibrium point $E$ is determined by the eigenvalues of the variational matrix evaluated at $E$. Thus, the general variational matrix $V(x,y,z)$ is computed. Computation of the variational matrix for the stability analysis using their eigenvalues can be derived as under:

$$V = \begin{bmatrix}
\alpha g(x) + \alpha xg'(x) - yp_1'(x) - zp_2'(x) & -p_1(x) & -p_2(x) \\
yc_1p_1'(x) & -s_1(y) - q_1(z) + c_1p_1(x) & -yq_1'(z) \\
zc_2p_2'(x) & -zq_2'(y) & -s_2(z) - q_2(y) + c_2p_2(x)
\end{bmatrix}$$

$\ldots(4.5)$

Eigenvalues of variational matrix using all the four equilibrium points are then computed. In case of $E_0(0, 0, 0)$ we put $x, y, z$ to be equal to zero and thus find $V(E_0)$ to be:
\[
V(E_0) = \begin{bmatrix}
\alpha g(0) & 0 & 0 \\
0 & -s_1(0) & 0 \\
0 & 0 & -s_2(0)
\end{bmatrix} = \text{diag}[\alpha g(0), -s_1(0), -s_2(0)]
\]

It is clear from this diagonal matrix that the eigenvalues in the x direction is positive whereas they are negative in y and z direction. Therefore, it was concluded that \(E_0\) is stable in y and z direction and unstable in x direction.

In case of \(E_1(k, 0, 0)\) we put \(x = k\) and \(y, z\) to be equal to zero and thus find \(V(E_1)\) to be:

\[
V(E_1) = \begin{bmatrix}
\alpha kg'(k) & -p_1(k) & -p_2(k) \\
0 & -s_1(0) + c_1p_1(k) & 0 \\
0 & 0 & -s_2(0) + c_2p_2(k)
\end{bmatrix}
\]

Eigenvalues of \(V(E_1)\) will also be the diagonal elements since it is an upper triangular matrix. Its eigenvalues are \((\alpha kg'(k), -s_1(0) + c_1p_1(k), -s_2(0) + c_2p_2(k))\). From the hypothesis (H1.1) it follows that \(E_1\) is stable in x direction and unstable in y and z directions respectively. The eigenvalues of \(E_1\) in y and z direction are:

\(-s_1(0) + c_1p_1(k)\) and \(-s_2(0) + c_2p_2(k)\) respectively.

From this the following result may be concluded:

\[(62)\]
Theorem 4.1.4: If $\alpha_2 < 0$ and $\alpha_3 < 0$ then \( \lim_{t \to -\infty} y(t) = \lim_{t \to -\infty} z(t) = 0 \) and $E_1(k,0,0)$ is asymptotically stable for the given system (3.1) where $\alpha_2 = -s_1(0) + c_1p_1(k)$ and $\alpha_3 = -s_2(0) + c_2p_2(k)$.

Further, $E_2(x_1, y_1, 0)$ is given by

\[
V(E_2) = \begin{bmatrix}
\alpha g(x_1) + \alpha x_1g'(x_1) - y_1p_1'(x_1) & -p_1(x_1) & -p_2(x_1) \\
y_1c_1p_1(x_1) & -s_1(y) + c_1p_1(x_1) & -y_1q_1'(0) \\
0 & 0 & -s_2(0) - q_2(y_1) + c_2p_2(x_1)
\end{bmatrix}
\]

Eigenvalues of $V(E_2)$ are the solution of the following determinant

\[
\begin{vmatrix}
\alpha g(x_1) + \alpha x_1g'(x_1) & -p_1(x_1) & -p_2(x_1) \\
y_1p_1'(x_1) - \lambda & -s_1(y) + c_1p_1(x_1) - \lambda & -y_1q_1'(0) \\
0 & 0 & -s_2(0) - q_2(y_1) + c_2p_2(x_1) - \lambda
\end{vmatrix} = 0
\]

All functions are evaluated at $E_2(x_1, y_1, 0)$ in this matrix. Clearly, the eigenvalue of $V(E_2)$ in the $z$ direction given by:

\[
\mu = -s_2(0) - q_2(y_1) + c_2p_2(x_1)
\]

The other two eigenvalues in $x$ and $y$ direction may be obtained from the sub-matrix

(63)
\[ M = \begin{bmatrix} \alpha g(x_1) + \alpha x_1 g'(x_1) - y_1 p'(x_1) & -p_1(x_1) \\ y_1 c_1 p'_1(x_1) & -s_1(y_1) + c_1 p_1(x_1) \end{bmatrix} \]

Clearly, eigenvalues of \( M \) will have negative real parts whenever \( \text{tr}(M) < 0 \) and \( \det(M) > 0 \). This leads to the following result:

**Theorem 4.1.5:** The equilibrium \( E_2(x_1, y_1, 0) \) will be asymptotically stable whenever

(i) \( \mu < 0 \)

(ii) \( a - s_1(y_1) + c_1 p_1(x_1) < 0 \)

(iii) \( a[-s_1(y_1) + c_1 p_1(x_1)] + y_1 c_1 p_1(x_1) p'_1(x_1) > 0 \)

where

\[ a = \alpha g(x_1) + \alpha x_1 g'(x_1) - y_1 p'(x_1) \]

Similarly, \( E_3(x_2, 0, z_2) \) can be computed as

\[ V(E_3) = \begin{bmatrix} \alpha g(x_2) + \alpha x_2 g'(x_2) - z_2 p'_2(x_2) & -p_1(x_2) & -p_2(x_2) \\ 0 & -s_1(0) - q_1(z_2) + c_1 p_1(x_2) & 0 \\ z_2 c_2 p'_2(x_2) & -z_2 q'_2(0) & -s_2(z_2) + c_2 p_2(x_2) \end{bmatrix} \]

All functions are evaluated at \( E_3(x_2, 0, z_2) \) in this matrix. Clearly, the eigenvalues of \( V(E_3) \) in the \( y \) direction is given by:

\[(64)\]
\[ \eta = -s_1(0) - q_1(z_2) + c_1p_1(x_2) \]

It can be easily seen that its other two eigenvalues are the eigenvalues of the sub-matrix

\[
N = \begin{bmatrix}
\alpha g(x_2) + \alpha x_2 g'(x_2) - z_2 p'_2(x_2) & -p_2(x_2) \\
z_2 c_2 p'_2(x_2) & -s_2(z_2) + c_2 p_2(x_2)
\end{bmatrix}
\]

Clearly, eigenvalues of \( N \) will have negative real parts whenever \( \text{tr}(N) < 0 \) and \( \det(N) > 0 \). This leads to the following result:

**Theorem 4.1.6:** The equilibrium \( E_3(x_2, 0, z_2) \) will be asymptotically stable whenever

(i) \( \eta < 0 \)

(ii) \( b - s_2(z_2) + c_2 p_2(x_2) < 0 \)

(iii) \( b[-s_2(z_2) + c_2 p_2(x_2)] + z_2 c_2 p_2(x_2) p'_2(x_2) > 0 \)

where

\[ b = \alpha g(x_2) + \alpha x_2 g'(x_2) - z_2 p'_2(x_2) \]

Finally, the stability of the interior equilibrium \( E^*(x^*, y^*, z^*) \) is examined. For this purpose, \( V(E^*) \) is computed. All functions in \( V(E^*) \) below are assumed to be evaluated at \( E^*(x^*, y^*, z^*) \).

(65)
\[
V(E') = \begin{bmatrix}
\alpha g(x') + \alpha x' g'(x') - y' p'(x') - z' p_2'(x') & -p_1(x') & -p_2(x') \\
y' c_1 p_1'(x') & -s_1(y') - q_1(z') + c_1 p_1(x') & -y' q_1'(z') \\
z' c_2 p_2'(x') & -z' q_2'(y') & -s_2(z') - q_2(y') + c_2 p_2(x')
\end{bmatrix}
\]

Its characteristic polynomial is given by

\[P(\lambda) = \lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 \]  \hspace{1cm} \text{...(4.6)}

Where \(A_1, A_2, A_3\) are defined below

\[
A_1 = -B_1 - B_2 - B_3 \\
A_2 = B_1 B_2 + B_1 B_3 + B_2 B_3 + z q_2'(y) y q_1'(z) + p_1(x) y c_1 p_1'(x) + p_2(x) z c_2 p_2'(x) \\
A_3 = -B_1 B_2 B_3 + B_1 z q_1'(y) y q_1'(z) - p_1(x) y c_1 p_1'(x) B_3 - p_1(x) z c_2 p_2'(x) y q_1'(z) \\
\quad - p_2(x) y c_1 p_1'(x) z q_2'(y) - p_2(x) z c_2 p_2'(x) B_2
\]

\[
B_1 = \alpha g(x) + \alpha x g'(x) - y p_1'(x) - z p_2'(x) \\
B_2 = -s_1(y) - q_1(z) + c_1 p_1(x) \\
B_3 = -s_2(z) - q_2(y) + c_2 p_2(x)
\]

and \(A_i\)'s and \(B_i\)'s are evaluated at \(E'\). The eigenvalues of variational matrix, \(V(E')\) are the roots of the polynomial \(\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3\).

Hence from Routh – Hurwitz criteria the following results follows:

\textbf{Theorem 4.1.7:} If an interior equilibrium point \(E'\) exists and

\(A_i > 0\) for \(i = 1, 2, 3\) and \((A_1 A_2 - A_3) > 0\) then

\(E'\) is asymptotically stable.
A’s are given by the equation (4.6). Thus whenever, $E^*$ exists and the inequalities given by the equation (4.7) hold, solutions starting close to $E^*$ stay close to it for all time and eventually tend to $E^*$.

4.2 Global Stability and Periodic Solutions

This section deals with the global stability of the interior equilibrium point $E'(x^*, y^*, z^*)$. An equilibrium point is called globally stable with respect to the solutions initiating in the positive octant if all the solutions of the system (3.1) with positive initial conditions tend to the equilibrium point as time tends to infinity. Besides this, existence of periodic solutions for the system (3.1) has also been considered.

4.2.1 Global Stability of Interior Equilibrium Point

In this section conditions are derived under which the equilibrium point $E'(x^*, y^*, z^*)$ is globally stable with respect to $\text{Int}(R^3_+)$, the interior of the non-negative octant. For this purpose, a Liapunov function is constructed whose domain of validity is $R^3_+$. We define

$$V(x, y, z) = V_1(x) + V_2(y) + V_3(z)$$

Functions $V_1(x), V_2(y)$ and $V_3(z)$ are taken to be
\[ V_1(x) = x - x^* - x^* \log \frac{x}{x^*} \]

\[ V_2(y) = \int \frac{P(x^*, \xi, z^*)}{q_2(\xi)} \, d\xi \]

\[ V_3(z) = \int \frac{Q(x^*, y^*, \zeta)}{q_1(\zeta)} \, d\zeta \]

where

\[ P(x^*, \xi, z^*) = -s_2(z^*) - q_2(\xi) + c_2p_2(x^*) \]

and

\[ Q(x^*, y^*, \zeta) = -s_1(y^*) - q_1(\zeta) + c_1p_1(x^*) \]

We note that \( V_i(\chi) \), where \( \chi = x, y \) or \( z \), has the following properties:

(i) \( V_i(x^*) = 0 \)

(ii) \( V_i(\chi) > 0 \) for \( 0 < \chi < \infty, \chi \neq x^* \)

(iii) \( \lim_{\chi \to \infty} V_i(\chi) = \lim_{\chi \to 0^+} V_i(\chi) = +\infty \)

also

(iv) \( V_2(y^*) = 0 \) and \( \lim_{y \to 0^+} V_2(y) = +\infty \)

(v) \( V_3(z^*) = 0 \) and \( \lim_{z \to 0^+} V_3(z) = +\infty \)

\( P(x^*, y, z^*) \) is assumed to be such that \( \lim_{y \to \infty} V_2(y) = +\infty \) and \( V_2(y) \) is positive for \( y \leq k + \varepsilon_1 \), where \( \varepsilon_1 \) is any arbitrary small positive number. Also \( Q(x^*, y^*, z) \) is
such that $\lim_{z \to +\infty} V_3(z) = +\infty$ and $V_3(z)$ is positive for $z \leq k + \epsilon_2$, where $\epsilon_2$ is any arbitrary small positive number.

Now $\dot{V}(x, y, z)$ can be computed which is the derivative of $V(x, y, z)$ along the solutions of the system (3.1).

$$\dot{V} = \frac{dV}{dt} = \frac{\partial V_1}{\partial x} \times \dot{x} + \frac{\partial V_2}{\partial y} \times \dot{y} + \frac{\partial V_3}{\partial z} \times \dot{z}$$

The components of $\dot{V}$ are worked out as:

$$\frac{\partial V_1}{\partial x} = \frac{x - x^*}{x}, \quad \frac{\partial V_2}{\partial y} = \frac{P(x^*,y^*,z)}{q_2(y)} \quad \text{and} \quad \frac{\partial V_3}{\partial z} = \frac{Q(x^*,y^*,z)}{q_1(z)}$$

then

$$\dot{V} = (x - x^*) \left[ \alpha g(x) - y \frac{p_1(x)}{x} - z \frac{p_2(x)}{x} \right]$$

$$+ \frac{P(x^*,y^*,z)}{q_2(y)} \left[ -s_1(y) - q_1(z) + c_1 p_1(x) \right] y$$

$$+ \frac{Q(x^*,y^*,z)}{q_1(z)} \left[ -s_2(z) - q_2(y) + c_2 p_2(x) \right] z \quad \text{...}(4.8)$$

Equation (4.8) can also be written as

$$\dot{V} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{12} + a_{13} + a_{22} + a_{23} + a_{33}$$

where

$$\text{(69)}$$
\[
\begin{align*}
    a_{11} &= (x - x^*)(\alpha g(x) - y \frac{p_1(x)}{x} - z \frac{p_2(x)}{x}) \\
    a_{12} &= \frac{P(x^*, y, z^*)}{q_2(y)}(x - x^*)(z - z^*) \\
    a_{13} &= \frac{Q(x^*, y^*, z)}{q_1(z)}(x - x^*)(y - y^*) \\
    a_{22} &= \frac{P(x^*, y^*, z^*)}{q_2(y)}[-s_1(y) - q_1(z) + c_1 p_1(x)]y \\
    a_{23} &= \frac{P(x^*, y, z^*)}{q_2(y)}(x - x^*)(z - z^*) - \frac{Q(x^*, y^*, z)}{q_1(z)}(x - x^*)(y - y^*) \\
    a_{33} &= \frac{Q(x^*, y^*, z)}{q_1(z)}[-s_2(z) - q_2(y) + c_2 p_2(x)]z
\end{align*}
\]

From the above equations, we may deduce that \(a_i\)'s are such that

\[
\begin{align*}
    a_{11} &= -b_{11}(x)(x - x^*)^2 \\
    a_{12} &= -2b_{12}(x, y)(x - x^*)(y - y^*) \\
    a_{13} &= -2b_{13}(x, z)(x - x^*)(z - z^*) \\
    a_{22} &= -b_{22}(y)(y - y^*)^2 \\
    a_{23} &= -2b_{23}(y, z)(y - y^*)(z - z^*) \\
    a_{33} &= -b_{33}(z)(z - z^*)^2
\end{align*}
\]

With these notations, we can also write

\[
\dot{V} = -PBp^r
\]  
\[\text{...} (4.9)\]  
\[(70)\]
Where B is a 3x3 symmetric matrix, B(x,y,z), whose $i^\text{th}$ term is $b_{ii}$

$$
B(x,y,z) = \begin{bmatrix}
    b_{11} & b_{12} & b_{13} \\
    b_{12} & b_{22} & b_{23} \\
    b_{13} & b_{23} & b_{33}
\end{bmatrix}
$$

$$
P = [(x - x^*), (y - y^*), (z - z^*)]
$$

$$
P^T = \begin{bmatrix}
    x - x^* \\
    y - y^* \\
    z - z^*
\end{bmatrix}
$$

Now the following theorem can be concluded.

**Theorem 4.2.1:** Let $B(x,y,z)$ be a positive definite matrix for all points in the set $S \cap R^3_+$, where the set $S$ is given by equation (3.7). Then the interior equilibrium $E^*$ is globally asymptotically stable with respect to the solutions initiating in the positive octant $R^3_+$.

**Proof:** Let $(x,y,z) \in S \cap R^3_+$. Then since B is positive definite, $\dot{V}(x,y,z) \leq 0$. Hence, $V(x,y,z)$ defines a Liapunov function. Further, as the set $(x,y,z) \in R^3_+: \dot{V} = 0$ is precisely $\{E^*\}$, we may conclude that $E^*$ is globally stable (see Hale (1969)).

**Remark:** A necessary condition for $B(x,y,z)$ to be positive definite is that $b_{ii} > 0$ in $S$, $i = 1,2,3$, except at the equilibrium values.
4.2.2 Periodic Solution

According to Freedman (1987), in two-dimensional predator-prey model, stable limit cycles or periodic solutions will exist when the interior equilibrium is unstable. We now state a theorem giving conditions for "small amplitude" periodic solutions of the system (3.1). These solutions will be non-trivial in the sense that they lie in the positive octant. In deriving the results of the following theorem, the technique of Hopf-Bifurcation theorem (Hassard et al. (1981)) has been used.

Theorem 4.2.2: Let us assume that $A_1, A_2, A_3 > 0$, where $A_1, A_2, A_3$ are given by the equation (4.6). Further suppose $b_1 \alpha^2 + b_2 \alpha + b_3 = 0$, has a positive root $\alpha_0$ such that $2b_1 \alpha_0 + b_2 \neq 0$. Then the interior equilibrium, $E^* (x^*, y^*, z^*)$, bifurcates into small amplitude periodic solutions as $\alpha$ passes through $\alpha_0 > 0$, where $b_1, b_2, b_3$ are defined below:

\[
b_1 = -[g(x) + xg'(x)]^2(B_2 + B_3),
\]

\[
b_2 = [g(x) + xg'(x)][2(B_2 + B_3)(yp'_1(x) + zp'_2(x)) - D - (B_2 + B_3)^2 - zq'_2(y)yq'_1(z) + B_2B_3]
\]

\[
b_3 = [yp'_1(x) + zp'_2(x)][B_2B_3 - zq'_2(y)yq'_1(z) - (yp'_1(x) + zp'_2(x))(B_2 + B_3) - D - (B_2 + B_3)^2] - p_1(x)yc_1p'_1(x)B_3 - p_1(x)zc_2p'_2(x)yq'_1(z) - p_2(x)yc_1p'_1(x)zq'_2(y) - p_2(x)zc_2p'_2(x)B_2 - (B_2 + B_3)D
\]

where

\[
(72)
\]
\[ B_1 = \alpha g(x) + \alpha x g'(x) - y p_1(x) - z p_2(x) \]
\[ B_2 = -s_1 - q_1(z) + c_1 p_1(x) \]
\[ B_3 = -s_2 - q_2(y) + c_2 p_2(x) \]

and

\[ D = B_2 B_3 + z q'_2(y) y q'_1(x) + p_1(x) y c_1 p'_1(x) + p_2(x) z c_2 p'_2(x) \]

all the functions above are evaluated at \( E(x', y', z') \).

**Proof:** The theorem shall be proved by showing that the conditions of Hopf-Bifurcation are satisfied. For the purpose, it will be shown that the polynomial \( P(\lambda) \), given by equation (4.6), has a pair of pure imaginary roots and one real negative root.

Since \( \alpha = \alpha_0 > 0 \):
\[ A_1(\alpha_0) A_2(\alpha_0) - A_3(\alpha_0) = b_1 \alpha_0^2 + b_2 \alpha_0 + b_3 = 0, \]

It follows from Orlando formula (see Hahn, 1967) that the polynomial, \( P(\lambda) \), has a pair of equal but opposite roots. Let these roots be \( \lambda \) and \(-\lambda\).

From equation (4.6), we get
\[ \lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0 \]

and
\[ -\lambda^3 + A_1 \lambda^2 - A_2 \lambda + A_3 = 0 \]
Adding these two equations yields

\[ A_1 \lambda^2 + A_3 = 0 \]  

...(4.10)

If \( \lambda \) is real then equation (4.10) gives a contradiction, therefore \( \lambda \) must be complex. It implies that the roots of the polynomial are complex, say,

\[ u + iv \text{ and } -u - iv \]  

...(4.11)

Since pair of complex roots always occur in the form of a conjugate pair, the roots will be

\[ u + iv \text{ and } u - iv \]  

...(4.12)

It is clear from equations (4.11) and (4.12) that the real part, \( u \), is equal to zero. Hence the two roots of the polynomial are purely imaginary roots, viz., \( +iv \) and \( -iv \).

The third root of the polynomial, \( P(\lambda) \), is real and negative, since the sum of roots of \( P(\lambda) \) is \(-A_1\) and \( A_1 > 0\), therefore the third root will be real and negative.

Now, we will show that \[ \frac{du}{d\alpha_{(\alpha=a_0)}} \neq 0 \]

Since \( \lambda = u + iv \) satisfies equation (4.6), therefore substituting its value in this equation and differentiating the equation with respect to \( \alpha \), we obtain
\[ 3(u + iv)^2 \left( \frac{du}{d\alpha} + i \frac{dv}{d\alpha} \right) + 2A_1 \left( \frac{du}{d\alpha} + i \frac{dv}{d\alpha} \right) + A_2 \left( \frac{du}{d\alpha} + i \frac{dv}{d\alpha} \right) + (u + iv)^3 \frac{dA_1}{d\alpha} + (u + iv) \frac{dA_2}{d\alpha} + \frac{dA_3}{d\alpha} = 0 \]

at \( \alpha = \alpha_0, \ u(\alpha_0) = 0 \), therefore the above equation becomes

\[
\left[ -3v^2(\alpha_0) + 2A_1(\alpha_0)iv(\alpha_0) + A_2(\alpha_0) \right] \left[ \frac{du}{d\alpha} + i \frac{dv}{d\alpha} \right]_{\alpha_0} = v^2(\alpha_0) \left( \frac{dA_1}{d\alpha} \right)_{\alpha_0} - iv(\alpha_0) \left( \frac{dA_2}{d\alpha} \right)_{\alpha_0} - \left( \frac{dA_3}{d\alpha} \right)_{\alpha_0}
\]

... (4.13)

Since at \( \alpha = \alpha_0, \ \lambda(\alpha_0) = iv(\alpha_0) \), therefore putting this value in equation (4.6) and simplifying, we get

\[ [A_3 - A_1 v^2(\alpha_0)] + iv(\alpha_0) [A_2 - v^2(\alpha_0)] = 0 \]

The above equation yields the result as \( V^2(\alpha_0) = A_2(\alpha_0) \). Utilizing this in equation (4.13), we obtain

\[
\left[ -3A_2(\alpha_0) + i2A_1(\alpha_0) + A_2(\alpha_0) \right] \left[ \frac{du}{d\alpha} + i \frac{dv}{d\alpha} \right]_{\alpha_0} = A_2(\alpha_0) \left( \frac{dA_1}{d\alpha} \right)_{\alpha_0} - iv(\alpha_0) \left( \frac{dA_2}{d\alpha} \right)_{\alpha_0} - \left( \frac{dA_3}{d\alpha} \right)_{\alpha_0}
\]

... (4.14)

Equating the real part of the above equation and solving it, we obtain

\[
\left( \frac{du}{d\alpha} \right)_{\alpha_0} = \frac{-A_2(\alpha_0) \left( \frac{dA_1}{d\alpha} \right)_{\alpha_0} - A_1(\alpha_0) \left( \frac{dA_2}{d\alpha} \right)_{\alpha_0} + \left( \frac{dA_3}{d\alpha} \right)_{\alpha_0}}{2[A_1^2(\alpha_0) + A_2(\alpha_0)]}
\]

(75)
Since
\[
\frac{dA_3}{d\alpha} - A_2 \frac{dA_1}{d\alpha} - A_1 \frac{dA_2}{d\alpha} - \frac{d}{d\alpha} (A_3 - A_1 A_2) = -\frac{d}{d\alpha} \left[-(b_1 \alpha^2 + b_2 \alpha + b_3)\right] = -[2b_1 \alpha + b_2]
\]

Hence
\[
\left(\frac{du}{d\alpha}\right)_{\alpha_0} = -\frac{(2b_1 \alpha_0 + b_2)}{2[A_1^2(\alpha_0) + A_2(\alpha_0)]} \neq 0
\]

Thus, the theorem is proved.

4.2.3 Illustration on Interior Equilibrium Point and Periodic Solutions

In this illustration the occurrence of the interior equilibrium point for a particular system is shown. Also periodic solutions are demonstrated by the usage of theorem 4.2.2. To get a particular model for illustration, general functions of the model, such as, specific growth rates, predator response etc., are replaced with specific functions. For this illustration, the functions are taken as follows:

\[
g(x) = 1 - \frac{x}{k} \]
\[
p_1(x) = \alpha a_1 x \]
\[
p_2(x) = \alpha a_2 x
\]
\[ q_1(z) = b_1z \]
\[ q_2(y) = b_2y \]

With these functions in place, we have the following system for consideration.

\[
x'(t) = \alpha x(1 - \frac{x}{k}) - \alpha a_1 xy - \alpha a_2 xz \\
y'(t) = y[-s_1 y - b_1 z + c_1 \alpha a_1 x] \\
z'(t) = z[-s_2 z - b_2 y + c_2 \alpha a_2 x]
\] ...(4.15)

If the values of the parameters specified in the system (4.15) are taken as

\[ \alpha = 1.12, \ k = 3, \ c_1 = 0.6, \ c_2 = 0.7, \ s_1 = 0.1, \ s_2 = 0.22, \ a_1 = 0.8, \ a_2 = 0.53, \ b_1 = 0.65 \text{ and } b_2 = 0.55 \]

then the interior equilibrium point \( E^* \) of the system (4.15) comes out to be \( E^* \approx (0.3527, 1.29403, 0.03989) \). When the system is evaluated at the interior equilibrium point then we get \( A_1 \approx 0.86613, \ A_2 \approx 0.39828 \) and \( A_3 \approx 0.10609 \). With this \( b_1, \ b_2 \text{ and } b_3 \) are computed as \( b_1 \approx 0.31571, \ b_2 \approx -1.3985 \text{ and } b_3 \approx 0.3428 \). Further, the root of the equation \( b_1 \alpha^2 + b_2 \alpha + b_3 = 0, \ alpha_0 \) is computed as \( \alpha_0 = 4.17048 \). Now, since \( A_1, \ A_2, \ A_3 > 0 \text{ and } \alpha_0 > 0 \text{ and satisfies } 2b_1\alpha_0 + b_2 \neq 0, \) therefore, the periodic solutions for the system (4.15) will occur as \( \alpha \) passes through \( \alpha_0 \).
4.3 Persistence

In this section, uniform persistence (also called permanence) for the system under consideration is addressed. This is of great importance in mathematical ecology as it gives sufficient conditions under which solutions remain bounded, away from zero for all time, \( t \geq 0 \).

On the basis of persistence, systems may be categorized into three types: weakly persistent, uniformly persistent and strongly persistent or simply persistent (Butler et.al (1986)). These may be understood as the following: The system (3.1) is said to be weakly persistent if all its solutions vectors, \( X(t) = (x, y, z) \), with positive initial conditions are such that

\[
\limsup_{t \to \infty} x(t) > 0, \quad \limsup_{t \to \infty} y(t) > 0 \quad \text{and} \quad \limsup_{t \to \infty} z(t) > 0.
\]

Also, if \( \limsup \) is replaced by \( \liminf \) in the above definition then the system is called persistent (or strongly persistent). Further, we say that the system is uniformly persistent if there exists a \( \delta > 0 \) such that \( \liminf_{t \to \infty} Z(t) \geq \delta \) for each component \( z(t) \) of \( X(t) \), where \( X(t) \) is any solution with positive initial conditions.

In the next section, the conditions under which the system under consideration is uniformly persistent are worked out. Average Liapunov function method is used to work out the conditions of uniform persistence.

The validity of our results has also been illustrated with an example. This example is a constructed one since data based on real-life problems pertaining to the model
considered in the present study were not available. The particular coefficients chosen, therein, are for mathematical convenience and do not have biological meaning or apply to specific populations.

4.3.1 Uniform Persistence

The sufficient condition for permanent co-existence and extinction of species of system (3.1) is obtained from the knowledge of saturated equilibria and by constructing the average Liapunov function.

Let the average Liapunov function for the system (3.1) be

\[ \sigma(x) = x^a y^b z^c \]

Clearly, \( \sigma(X) \) is a non-negative \( C^1 \) function defined in \( \mathbb{R}^+_{xyz} \) and \( a, b, c \) are assumed to be positive. Then

\[ \dot{\sigma}(X) = \frac{\partial \sigma}{\partial t} = ax^{a-1}x'y^b z^c + x^a b'y^b z^{c-1} + x^a y^b cz^{c-1} \]

From this we can compute

\[ \psi(X) = \frac{\dot{\sigma}(X)}{\sigma(X)} = a \frac{\dot{x}}{x} + b \frac{\dot{y}}{y} + c \frac{\dot{z}}{z} \]

or

(79)
\[ \psi(X) = a \left[ \alpha g(x) - y \frac{p_1(x)}{x} - z \frac{p_2(x)}{x} \right] \\
+ b \left[ -s_1(y) - q_1(z) + c_1 p_1(x) \right] \\
+ c \left[ -s_2(z) - q_2(y) + c_2 p_2(x) \right] \]

If \( \psi(X) > 0 \) for \( \omega \)-limit sets of trajectories initiated in \( \delta R^+_{xyz} \), then the trajectories move away from the boundary and hence the system (3.1) is permanent. Since there are four boundary fixed points on \( \delta R^+_{xyz} \), by Theorem 3.1.1, system (3.1) is dissipative and from the local behaviour it is evident that under the given assumptions there is no periodic trajectory in the boundary planes. Hence if there exists positive \( a, b \) and \( c \) such that \( \psi(E_j) > 0 \) for all \( j = 0, 1, \ldots, 3 \) then system (3.1) is permanent.

From Table 4.1, we get the following system of inequalities for at least one positive vector \( p=(a,b,c) \) such that:

\[
\begin{align*}
E_0: \quad & \psi(E_0) = a \alpha g(0) - b s_1(0) - c s_2(0) > 0 \quad \ldots (4.16a) \\
E_1: \quad & \psi(E_1) = b [-s_1(0) + c_1 p_1(k)] + c [-s_2(0) + c_2 p_2(k)] > 0 \quad \ldots (4.16b) \\
E_2: \quad & \psi(E_2) = c [-s_2(0) - q_2(y,1) + c_2 p_2(x,1)] > 0 \quad \ldots (4.16c) \\
E_3: \quad & \psi(E_3) = b [-s_1(0) - q_1(z,2) + c_1 p_1(x,2)] > 0 \quad \ldots (4.16d)
\end{align*}
\]
<table>
<thead>
<tr>
<th>( \dot{x} )</th>
<th>( \dot{y} )</th>
<th>( \dot{z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
<td>( z )</td>
</tr>
<tr>
<td>( E_0 )</td>
<td>( \alpha g(0) )</td>
<td>( -s_1(0) )</td>
</tr>
<tr>
<td>( E_1 )</td>
<td>---</td>
<td>( -s_1(0) + c_1 p_1(k) )</td>
</tr>
<tr>
<td>( E_2 )</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>( E_3 )</td>
<td>---</td>
<td>( -s_1(0) - q_1(z_2) + c_1 p_1(x_2) )</td>
</tr>
</tbody>
</table>

For \( E_1(k,0,0) \), \( \frac{\dot{x}}{x} \) will be obviously 0 because of the hypothesis (H1.1). For \( E_2(x_1,y_1,0) \), \( \frac{\dot{x}}{x} \) and \( \frac{\dot{y}}{y} \) will be 0 because at \( E_2 \), \( x_1 \) and \( y_1 \) are the positive solution of the subsystem

\[
\alpha x g(x) - y p_1(x) = 0 \\
- s_1(y) + c_1 p_1(x) = 0
\]

Similarly, \( \frac{\dot{x}}{x} \) and \( \frac{\dot{z}}{z} \) will be 0 for \( E_3(x_2,0,z_2) \) because at \( E_3 \), \( x_2 \) and \( z_2 \) are the positive solution of the subsystem

\[
\alpha x g(x) - z p_2(x) = 0 \\
- s_2(z) + c_2 p_2(x) = 0
\]

From the local behaviour of different boundary equilibria, it is evident that (4.16a) – (4.16d) hold if the following assumptions are true.
Also (4.16a) is automatically satisfied by increasing $a$ to a sufficiently large value. Since $a$ is a positive constant of our choice, this is possible. From this the following result are derived:

**Theorem 4.3.1:** If the following assumptions hold

(i) $s_i(0) < c_i p_i(k)$ for $i = 1$ and $2$
(ii) $s_1(0) < c_1 p_1(x_2) - q_1(z_2)$
(iii) $s_2(0) < c_2 p_2(x_i) - q_2(y_1)$

then the system (3.1) is uniformly persistent.

From the Theorem 4.3.1, it follows that the two predators-one prey system is permanent (or uniformly persistent) if predator-prey subsystem is globally stable and the death rate of the predators is less than a certain threshold value depending upon the food conversion efficiency of the predators.

**Theorem 4.3.2:** If one of the following conditions

(i) $s_i(0) > c_i p_i(k)$ for $i = 1$ and $2$
(ii) $s_1(0) > c_1 p_1(x_2) - q_1(z_2)$
(iii) $s_2(0) > c_2 p_2(x_i) - q_2(y_1)$

hold then the system (3.1) is not permanent and either (i) the prey, or (ii) first predator or (iii) the second predator will go to extinction for at least one initial condition.
Proof: From the given assumptions we see that inequalities (4.16a) – (4.16d) have no positive solutions. Thus the distance to the boundary decreases along orbits near fixed points $E_1$, $E_2$ and $E_3$. Hence by theorem 3 of Amann and Hofbauer (1985) there is a positive invariant set $M \subset \delta R^+_{sys}$ containing the fixed points $E_1$, $E_2$ and $E_3$. Thus the trajectory initiated in $R^+_{sys}$ must converge to either of these boundary equilibrium points. So the system is not permanent and the prey or either of the predators will go to extinction.

4.3.2 Illustration on Boundary Equilibrium Points and Uniform Persistence

In this illustration the occurrence of all the boundary equilibrium point for a particular system is shown. Also uniform persistence is demonstrated by the usage of theorem 4.3.1. For this purpose consider the following system:

\[
\begin{align*}
  x'(t) &= \frac{\alpha x r(k - x)}{k + \varepsilon x} - \frac{y m_1 x^{a_1}}{b_1 + x^{a_1}} - \frac{z m_2 x^{a_2}}{b_2 + x^{a_2}} \\
  y'(t) &= y(s, y - n_1 z^{d_1} + \frac{c_1 m_1 x^{a_1}}{b_1 + x^{a_1}}) \\
  z'(t) &= z(s, z - n_2 y^{d_2} + \frac{c_2 m_2 x^{a_2}}{b_2 + x^{a_2}}) \\
\end{align*}
\]

... (4.17)

This system is obtained by replacing the following functions into the system (3.1)

\[
g(x) = \frac{r(k - x)}{k + \varepsilon x}
\]

(83)
\[ p_1(x) = \frac{m_1 x^{a_1}}{b_1 + x^{a_1}} \quad a_1 \geq 1 \]
\[ p_2(x) = \frac{m_2 x^{a_2}}{b_2 + x^{a_2}} \quad a_2 \geq 1 \]
\[ q_1(z) = n_1 z^{d_1} \quad 1 \geq d_1 > 0 \]
\[ q_2(y) = n_2 y^{d_2} \quad 1 \geq d_2 > 0 \]

with \( \alpha = 1.12, k = 3, c_1 = 0.6, c_2 = 0.8, s_1 = 0.1, s_2 = 0.15, a_1 = 1.8, a_2 = 1.7, r = 2.75, \varepsilon = 0.35, m_1 = 0.85, m_2 = 0.99, b_1 = 0.35, b_2 = 0.45, d_1 = 0.015, d_2 = 0.01, n_1 = 0.148, n_2 = 0.168. \)

With these parameters \( E_0(0,0,0) \) clearly comes out to be an equilibrium point. One dimensional equilibrium point is \( E_1 \approx (3, 0, 0) \). Two dimensional equilibria are \( E_2 \approx (0.681299, 3.003585, 0) \) and \( E_3 \approx (0.680317, 0, 2.830632) \). It follows from Hsu (1978) that \( E_2 \) and \( E_3 \) are globally stable in their respective planes. Since \( s_1 = 0.1 \) and \( s_2 = 0.15 \) and \( c_1 p_1(k) = 0.486438, c_2 p_2(k) = 0.740525, c_1 p_1(x_2) - q_1(z_2) = 0.149646, \) and \( c_2 p_2(x_1) - q_2(y_1) = 0.255024 \) all the conditions of the theorem 4.3.1 are satisfied and the hence the considered system (4.17) is uniformly persistent.

### 4.4 Model with Time Lag

In this section a one prey and two competing predators model with a constant time lag due to gestation is considered. This is the time lag version of model 1 that is currently under consideration in this chapter. The time lag due to gestation refers to the time lag in conversion of prey biomass to predator biomass. Effects
of such a delay have been considered by Freedman & Rao (1983), Erbe & Freedman (1986) and Freedman & Kumar (1991).

### 4.4.1 The Model

The considered model is given by the following set of three differential equations:

\[
\begin{align*}
x'(t) &= \alpha x g(x) - y p_1(x) - z p_2(x) \quad \cdots (4.18.1) \\
y'(t) &= y[-s_1(y) - q_1(z) + c_1 p_1[x(t - \tau)]] \quad \cdots (4.18.2) \\
z'(t) &= z[-s_2(z) - q_2(y) + c_2 p_2[x(t - \tau)]] \quad \cdots (4.18.3)
\end{align*}
\]

with the initial conditions

\[(x(t), y(t), z(t)) = (x^0(t), y^0(t), z^0(t)) \quad t \in [-\tau, 0]\]

Where \(x^0(t), y^0(t)\) and \(z^0(t)\) are continuous non-negative functions on \([-\tau, 0]\). The parameter \(\tau\) represents the time delay due to gestation. All the above functions are assumed to satisfy conditions described for the model (3.1). In section 4.4.2, we obtain the characteristic equation and discuss stability as the time lag \(\tau\) changes.

### 4.4.2 Stability

Equilibrium points for the system (4.18) and the system (3.1) are the same. Therefore, the interior equilibrium, \(E^*\), for the system (4.18) will exist if the condition for the uniform persistence, as discussed in section 4.3.1 for the system
(3.1), hold. In this section, it is assumed that $E^*(x^*, y^*, z^*)$ exists. Now we will compute the variational system for the model (4.18) at $E^*$, given by

$$\frac{d\psi}{dt} = A\psi(t) + B\psi(t - \tau)$$  \hspace{1cm} \ldots(4.19)$$

where

$$\psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \end{bmatrix}$$

and

$$A = \begin{bmatrix} \alpha g(x) + \alpha g'(x) - yp_1(x) - zp_2(x) & -p_1(x) & -p_2(x) \\ 0 & -s_1(y) - q_1(z) + c_1p_1(x) & -yq_1'(z) \\ 0 & -zq_2'(y) & -s_2(z) - q_2(y) + c_2p_2(x) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ yc_1p_1'(x) & 0 & 0 \\ zc_2p_1'(x) & 0 & 0 \end{bmatrix}$$ \hspace{1cm} \ldots(4.20)$$

where all the functions in $A$ and $B$ are assumed to be evaluated at $E^*$. Putting $\psi(t) = e^{\lambda t}p$ and $0 \neq p \in \mathbb{R}^3$, in equation (4.19), we get

$$\lambda p = Ap + Be^{-\lambda t}p$$

(86)
or

\[(A + Be^{-\lambda t} - \lambda I)p = 0\]

\[\Rightarrow \det(A + Be^{-\lambda t} - \lambda I) = 0\]

or

\[
\begin{vmatrix}
    a_{11} - \lambda & a_{12} & a_{13} \\
    yc_1p'_{1}(x) & a_{22} - \lambda & a_{23} \\
    zc_2p'_{2}(x) & a_{32} & a_{33} - \lambda
\end{vmatrix} = 0
\]

Solving the above determinant yields

\[\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + [a_4 + \lambda a_5]e^{-\tau \lambda} = 0\]

...(4.22)

where

\[a_1 = -a_{11} - a_{22} - a_{33}\]
\[a_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{32}a_{23}\]
\[a_3 = a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33}\]
\[a_4 = yc_1p'_{1}(x)(a_{33}a_{12} - a_{32}a_{13}) + zc_2p'_{2}(x)(a_{22}a_{13} - a_{23}a_{12})\]
\[a_5 = -yc_1p'_{1}(x)a_{12} - zc_2p'_{2}(x)a_{13}\]

and

\[a_{11} = \alpha g(x) + \alpha xg'(x) - yp'_1(x) - zp'_2(x)\]
\[a_{12} = -p_1(x)\]
\[a_{13} = -p_2(x)\]

(87)
\[ a_{22} = -s_1(y) - q_1(z) + c_1p_1(x) \]
\[ a_{23} = -yq_1'(z) \]
\[ a_{22} = -zq_2'(y) \]
\[ a_{23} = -s_2(z) - q_2(y) + c_2p_2(x) \]

The sign of the real parts of roots of the equation (4.22) determine the stability of the interior equilibrium \( E^*(x^*,y^*,z^*) \). Substituting \( \lambda = \mu + iv \) in equation (4.22), we get

\[
(\mu + iv)(\mu^2 - v^2 + 2i\mu v) + a_1(\mu^2 - v^2 + 2i\mu v) + a_2(\mu + iv) + a_3 + [a_4 + a_5(\mu + iv)]e^{-\tau[\mu + iv]} = 0
\]

The above equation yields

\[
(\mu^3 - 3\mu v^2) + a_1(\mu^2 - v^2) + a_2\mu + a_3 + [(a_4 + a_5\mu)\cos(\tau v) + a_5v\sin(\tau v)]e^{-\tau v} = 0
\]

... (4.23)

and

\[
v(3\mu^2 - v^2) + 2a_1\mu v + a_2v + [a_3v\cos(\tau v) - (a_4 + a_5\mu)\sin(\tau v)]e^{-\tau v} = 0
\]

... (4.24)

Conditions will be derived under which stable (unstable) equilibrium in the absence of time delay, \( \tau \), remains stable (unstable) for all \( \tau \geq 0 \). We will restrict our analysis to the case when \( \lambda = 0 \) is not a root of the equation (4.22), i.e., \( a_3 + a_4 \neq 0 \). Now, from the Routh-Hurwitz criteria \( E^*(x^*,y^*,z^*) \) will be stable at \( \tau = 0 \), if and only if
Further, at $\tau = 0$, the sum of all the roots of the equation (4.22) is $-a_1$ and the product of all the roots of the equation (4.22) is $-(a_3 + a_4)$. Therefore, we conclude that $E'$ is unstable, for $\tau = 0$, if

$$a_1 < 0 \text{ or } a_3 + a_4 < 0 \quad \ldots (4.26)$$

whenever the inequalities given by the equation (4.25) and (4.26) hold, change in stability can occur only if for some value of $\tau$, the characteristic equation (4.22) has purely imaginary roots. These roots may either be positive or negative or both. Thus to obtain conditions for no change of stability of $E'$, we put $\mu = 0$ in the equation (4.23) and (4.24). It yields

$$-a_1 \mu^2 + a_3 = -a_4 \cos(\tau v) - a_5 v \sin(\tau v) \quad \ldots (4.27a)$$

and

$$-v^3 + a_2 v = a_4 \sin(\tau v) - a_5 v \cos(\tau v) \quad \ldots (4.27b)$$

Equation (4.27a) and (4.27b) give

$$v^6 + (a_1^2 - 2a_2)v^4 + (a_2^2 - 2a_1a_2 - a_3^2)v^2 + a_3^2 - a_4^2 = 0 \quad \ldots (4.28)$$

It is observed that whenever equation (4.27) has a positive root, so does the equation (4.28). Further, if equation (4.28) has no positive roots, then the equation

\[ \quad \ldots (89) \]
(4.27) has no root at all. Next, substituting \( v^2 = Z \), in the equation (4.28), the following result is obtained:

**Theorem 4.4.1:** Whenever the polynomial

\[
p(Z) = Z^3 + (a_1^2 - 2a_2)Z^2 + (a_2^2 - 2a_1a_3 - a_5^2)Z + a_3^2 - a_4^2
\]  

...(4.29)

has no positive roots, there is no change of stability of the interior equilibrium \( E^* \) for all \( \tau \geq 0 \).

**Corollary 4.4.1:** Whenever the inequalities given by the equation (4.25) hold and if \( p(Z) \) as given by the equation (4.29) has no positive root, then \( E^* \) is locally asymptotically stable for all \( \tau \geq 0 \).

**Corollary 4.4.2:** Whenever the inequalities given by the equation (4.26) hold and if \( p(Z) \) has no positive root, then \( E^* \) is unstable for all \( \tau \geq 0 \).