Chapter 3

The Models

3.1 Introduction

In this section, three models have been formulated and analyzed by giving their mathematical as well as biological explanation. It is also proved that the models are meaningful in the sense that the solutions starting with positive initial conditions stay positive for all time, and there exists a compact positively invariant set which is an attracting set for the given systems.

The three models considered here originate from a resource available in the environment. In all the three models the food web originates with the interaction of one prey and two predators living in the environment. However, they differ in the way the predators interact with each other and the prey. Thus, the following three specific models are considered.

3.1.1 Model 1 – Competing Predators Model

In this model, a single prey interacts with two predators where these two predators and competing with each other in order to survive. (see Fig. 3.1)

In the figure 3.1, x is the prey and y and z are predators. This model can be referred to as the "Competing Predators Model".

(37)
Fig. 3.1 Competing Predators Model

Mathematically this model is described by a system of the following autonomous ordinary differential equations:

\[
\begin{align*}
    x'(t) &= \alpha x g(x) - y p_1(x) - z p_2(x) \quad \text{(3.1.1)} \\
    y'(t) &= y[-s_1(y) - q_1(z) + c_1 p_1(x)] \quad \text{(3.1.2)} \\
    z'(t) &= z[-s_2(z) - q_2(y) + c_2 p_2(x)] \quad \text{(3.1.3)}
\end{align*}
\]

with the initial conditions given by

\[
\begin{align*}
    x(0) &= x_0 \geq 0, \quad y(0) = y_0 \geq 0, \quad z(0) = z_0 \geq 0
\end{align*}
\]

Here \(x(t)\) denotes the prey density and \(y(t)\) and \(z(t)\), the competing predator density. For the purpose of the analysis in this entire study, all functions are assumed to be smooth so that the existence and uniqueness for all time \(t > 0\), is satisfied by the initial value problem (3.1).
The following assumptions are made:

(H1.1) The function $g(x)$ denotes the specific growth rate of prey in the absence of predation. It is assumed that $g(0) > 0$, $g'(x) < 0$, and there exists a $k > 0$ such that $g(k) = 0$. The first two conditions imply that $g(x)$ is a decreasing function. It implies that growth rate decreases as prey density increases. Secondly, in the absence of predators, the prey can grow from rare to even and so the growth rate is density dependent. The constant $k$ is the carrying capacity of the prey population. Further $\alpha$ is taken as a non-negative parameter.

(H1.2) $p_i(x)$ is the predator response function with the following properties:

$$p_i(0) = 0, \quad p_i'(x) > 0, \quad \text{for } i = 1, 2.$$ The former condition states that in the absence of prey, there can be no predation and the latter implies that the predation increases as prey density increases. The non-negative constants $c_1$ and $c_2$ denote the rate of conversion of prey biomass to predator biomass.

(H1.3) The function $q_1(z)$ and $q_2(y)$ represent competition between the predators $y$ and $z$ with the properties $q_i(0) = 0$, for $i = 1, 2$. Furthermore, $q_1'(z) > 0$ and $q_2'(y) > 0$. These conditions may be interpreted as that in the absence of competing predators there is no competition and the competition increases as the rival species density increases. Further if $\frac{\partial q_{ij}}{\partial q_i} = 0 \ \forall \ i,j$ then there is no competition between the predators $y$ and $z$. 

(39)
(H1.4) $s_i(0) > 0$, for $i = 1, 2$ and $s_1'(y) > 0$, $s_2'(z) > 0$ where $s_i(y)$ and $s_i(z)$ are the specific death rates of the predators $y$ and $z$ respectively in the absence of predation. These conditions specify that the death rates remain positive.

### 3.1.2 Model 2 – Cooperating Predators Model

In this model, a single prey interacts with two predators which are cooperating with each other in order to survive. (see Fig. 3.2)

![Fig. 3.2 Cooperating Predators Model](image)

In the figure 3.2, $x$ is the prey and $y$ and $z$ are predators. This model can be referred to as the "Cooperating Predators Model".

Mathematically this model is described by a system of the following autonomous ordinary differential equations:

\[
\begin{align*}
    x'(t) &= \alpha xg(x) - yp_1(x) - zp_2(x) \quad \text{... (3.2.1)} \\
    y'(t) &= y[-s_1(y) + q_1(z) + c_1 p_1(x)] \quad \text{... (3.2.2)} \\
    z'(t) &= z[-s_2(z) + q_2(y) + c_2 p_2(x)] \quad \text{... (3.2.3)}
\end{align*}
\]

(40)
with the initial conditions given by

\[ x(0) = x_0 \geq 0, \quad y(0) = y_0 \geq 0, \quad z(0) = z_0 \geq 0 \]

Here \( x(t) \) denotes the prey density and \( y(t) \) and \( z(t) \), the cooperating predator density. For the purpose of the analysis in this entire study, all functions are assumed to be smooth so that the existence and uniqueness for all time \( t > 0 \), is satisfied by the initial value problem (3.2).

The assumptions on the functions described for model 1, (H1.1), (H1.2) and (H1.4) can be applied to this model in the same way and may be called (H2.1), (H2.2) and (H2.4) respectively. Assumption (H2.3) for the functions \( q_i(z) \) and \( q_j(y) \) is as follows:

(H2.3) The function \( q_i(z) \) and \( q_j(y) \) represent cooperation between the predators \( y \) and \( z \), i.e., \( q(z) \) is cooperation to \( y \) from \( z \) and \( q(y) \) is cooperation to \( z \) from \( y \). The properties of \( q \) are \( q_i(0) = 0 \), for \( i = 1, 2 \). Furthermore, \( q_i'(z) > 0 \) and \( q_j'(y) > 0 \). These conditions may be interpreted as that in the absence of cooperating predators there is no cooperation and the cooperation increases as the rival species density increases. Further if \( \frac{\partial q_i}{\partial q_j} = 0 \) \( \forall \ i, j \) then there is no cooperation between the predators \( y \) and \( z \).
3.2.3 Model 3 – Alternative Resource Predators Model

In this model, a single prey interacts with two predators in such a way that first predator, $y$, uses the second predator, $z$, as an alternative resource. (see Fig. 3.3)

\[ \text{Fig. 3.3 Alternative Resource Predators Model} \]

In the figure 3.3, $x$ is the prey and $y$ and $z$ are predators. This model can be referred to as the "Alternative Resource Predators Model".

Mathematically this model is described by a system of the following autonomous ordinary differential equations:

\[
\begin{align*}
    x'(t) &= \alpha xg(x) - yp_1(x) - zp_2(x) \\
    y'(t) &= y[-s_1(y) + q_1(z) + c_1p_1(x)] \\
    z'(t) &= z[-s_2(z) - q_2(y) + c_2p_2(x)]
\end{align*}
\]

(3.3.1) \quad (3.3.2) \quad (3.3.3)

with the initial conditions given by
Here \( x(t) \) denotes the prey density and \( y(t) \) and \( z(t) \), the competing and cooperating predator density. For the purpose of the analysis in this entire study, all functions are assumed to be smooth so that the existence and uniqueness for all time \( t > 0 \), is satisfied by the initial value problem (3.3).

The assumptions on the functions described for model 1, (H1.1), (H1.2) and (H1.4) can be applied to this model in the same way and may be called (H3.1), (H3.2) and (H3.4) respectively. Assumption (H3.3) for the functions \( q_1(z) \) and \( q_2(y) \) is as follows:

(H3.3) The function \( q_1(z) \) and \( q_2(y) \) represent competition and cooperation between the predators \( y \) and \( z \), i.e., \( q_1(z) \) is cooperation to \( y \) from \( z \) whereas \( q_2(y) \) determines the consumption of \( z \) predator by \( y \) predator as an alternative resource. The properties of \( q \) are \( q_1(0) = 0 \) and \( q_2(0) = 0 \). Furthermore, \( q_1'(z) > 0 \) and \( q_2'(y) > 0 \). These conditions may be interpreted as that in the absence of these predators there is no cooperation or suppression of \( z \) predator by \( y \) predator and the competition or suppression increases as the rival species density increases. Further if \( \frac{\partial q_1}{\partial q_1} = 0 \) \( \forall \ i, j \) then there is no interaction between the predators \( y \) and \( z \).

For all the three models, the death rate \( s \) implies that it is a combination of natural death and death by harvesting of one predator by the other predator. Obviously, our model is valid if a predator is harvested by the other predator or they die a natural death.
3.3 Initial Theorems

Theorem 3.1.1: Solution of the systems (3.1),(3.2),(3.3) with positive initial conditions stay positive \( \forall t > 0 \). i.e. if \([x(t), y(t), z(t)]\) is a solution of the systems (3.1),(3.2),(3.3) with positive initial conditions, then \( x(t) > 0, y(t) > 0 \) and \( z(t) > 0 \) for every \( t > 0 \).

Proof:
Consider the Model 1 (3.1)

We can rewrite equation (3.1.1) as

\[
\frac{dx}{x} = \left[ \alpha g(x) - \frac{y}{x} p_1(x) - \frac{z}{x} p_2(x) \right] dt
\]

integrating the above equation from 0 to \( t \), we get

\[
\ln \frac{x}{x_0} = \int_{0}^{t} \left[ \alpha g(x) - \frac{y}{x} p_1(x) - \frac{z}{x} p_2(x) \right] dt
\]

\[
x = x_0 e^{\int_{0}^{t} \left[ \alpha g(x) - \frac{y}{x} p_1(x) - \frac{z}{x} p_2(x) \right] dt}
\]

\[
x > 0, \text{ if } x_0 > 0 \tag{3.4}
\]

Equation (3.1.2) can be rewritten as

\[
\frac{dy}{y} = \left[ -s_1(y) - q_1(z) + c p_1(x) \right] dt
\]
integrating the above equation from 0 to t, we get

\[
\ln \frac{y}{y_0} = \int_0^t \left[ -s_1(y) - q_1(z) + c_1 p_1(x) \right] dt
\]

\[
y = y_0 e^\theta
\]

\[
y > 0, \text{ if } y_0 > 0 \quad \ldots (3.5)
\]

Similarly by taking equation (3.1.3) we can prove that

\[
z > 0, \text{ if } z_0 > 0 \quad \ldots (3.6)
\]

equations (3.4), (3.5) and (3.6) together prove the theorem for model (3.1).

This proof for the model (3.2) and (3.3) can be carried out in exactly similar way, which will prove the theorem for model (3.2) and (3.3).

Note: By L’ Hospital’s Rule \( \frac{p_1(x)}{x} \) and \( \frac{p_2(x)}{x} \) in the limit don’t become indeterminate when \( x = 0 \).

Clearly for our models to be meaningful, the solutions of model (3.1), (3.2) and (3.3) must always remain bounded. Therefore, we can prove the following:
Theorem 3.1.2: There exists a compact positively invariant set $S$, where

$$S = \{(x,y,z) \mid 0 \leq x \leq k + \varepsilon, 0 \leq c_1x + y \leq M, 0 \leq c_2x + z \leq N\}$$  \hspace{1cm} (3.7)

and where $M = \frac{\beta}{s_1(y)}$ and $N = \frac{\gamma}{s_2(z)}$.

For Model 1 (3.1)
$$\beta = c_1[\alpha g(0) + s_1(y)](k + \varepsilon)$$
$$\gamma = c_2[\alpha g(0) + s_2(z)](k + \varepsilon)$$

For Model 2 (3.2)
$$\beta = c_1[\alpha g(0) + s_1(y)](k + \varepsilon) + yq_1(z)$$
$$\gamma = c_2[\alpha g(0) + s_2(z)](k + \varepsilon) + zq_2(y)$$

For Model 3 (3.3)
$$\beta = c_1[\alpha g(0) + s_1(y)](k + \varepsilon) + yq_1(z)$$
$$\gamma = c_2[\alpha g(0) + s_2(z)](k + \varepsilon)$$

Furthermore, this set is an attracting set for the given systems.

Proof:

Case 1: Model 1 (3.1)

The equation (3.1.1) is
$$x'(t) = \alpha xg(x) - yp_1(x) - zp_2(x)$$
\hspace{1cm} (46)
where \( x(t_0) = x_0 \geq 0 \)

Let \( u'(t) = \alpha u g(u) \)
\( u(t_0) = u_0 \geq 0 \) ... (3.8)

Then by standard comparison theorem we get that solution of (3.8) will always dominate the solution of (3.1.1)

Now if \( x_0 \leq k + \varepsilon \), then in case of \( x_0 \leq k \), we get \( x(t) \leq k \leq k + \varepsilon \)

Further if \( k < x_0 \leq k + \varepsilon \) then

\[
\lim \sup_{t \to \infty} x(t) = k
\]

and

\[
x(t) \leq k + \varepsilon \quad \forall t, \text{ as } x' \leq 0
\]

Also, it follows from the existence and uniqueness of solutions to initial value problems that \( x \geq 0 \)

Further, \( x(t) > k + \varepsilon \)

\( \Rightarrow x \) is a decreasing function. Hence eventually, \( x(t) \leq k + \varepsilon \)

Now for (3.1.2) we can write the expression \( (c_1x + y)' \) as

\[
(c_1x + y)' = \alpha c_1x g(x) - c_1y p_1(x) - c_2 z p_2(x) + y[-s_1(y) - q_1(z) + c_1 p_1(x)]
\]

\[
\leq \alpha c_1x g(x) - c_1y p_1(x) - c_2 z p_2(x) + y[-s_1(y) + c_1 p_1(x)]
\]

(47)
\[ \leq \alpha c_1 x \cdot g(x) - y s_1 (y) \]
\[ \leq \alpha \left( c_1 (k + \varepsilon) g(0) - y s_1 (y) \right) + c_1 x s_1 (y) - c_1 x s_1 (y) \]
\[ = c_1 (k + \varepsilon) \left[ \alpha g(0) + s_1 (y) \right] - (c_1 x + y) s_1 (y) \]

Let \( u = c_1 x + y \)

Consider \( u' + s_1 (y) u = c_1 \left[ \alpha g(0) + s_1 (y) \right] (k + \varepsilon) \)

Let \( c_1 \left[ \alpha g(0) + s_1 (y) \right] (k + \varepsilon) = \beta \)

Then \( (ue^{s_1 (y) t})' = \beta e^{s_1 (y) t} \)

\[ \Rightarrow u(t)e^{s_1 (y) t} - u(0) = \frac{\beta}{s_1 (y)} \left( e^{s_1 (y) t} - 1 \right) \]
\[ \Rightarrow u(t) \leq u(0)e^{-s_1 (y) t} + \frac{\beta}{s_1 (y)} \left( 1 - e^{-s_1 (y) t} \right) \]
\[ \Rightarrow u(t) \leq \frac{\beta}{s_1 (y)} + e^{-s_1 (y) t} \left( u(0) - \frac{\beta}{s_1 (y)} \right) \]

if \( u(0) > \frac{\beta}{s_1 (y)} \Rightarrow u(t) \leq \frac{\beta}{s_1 (y)} + \varepsilon \) for large \( t \)

if \( u(0) \leq \frac{\beta}{s_1 (y)} \Rightarrow u(t) \leq \frac{\beta}{s_1 (y)} \) \( \forall t \geq 0 \)

Now for (3.1.3) we can write the expression \( (c_2 x + z)' \) as

\[ (c_2 x + z)' = \alpha c_2 x \cdot g(x) - c_2 y p_1 (x) - c_2 z p_2 (x) + z[-s_2 (z) - q_2 (y) + c_2 p_2 (x)] \]
\[ \leq \alpha c_2 x \cdot g(x) - c_2 y p_1 (x) - c_2 z p_2 (x) + z[-s_2 (z) + c_2 p_2 (x)] \]

(48)
\[ \leq \alpha c_2 x g(x) - z s_2(z) \]
\[ \leq \alpha c_2 (k + \varepsilon) g(0) - z s_2(z) + c_2 x s_2(z) - c_2 x s_2(z) \]
\[ = c_2 (k + \varepsilon) [\alpha g(0) + s_2(z)] - (c_2 x + z) s_2(z) \]

Let \( v = c_2 x + z \)

Consider \( v' + s_2(z)v = c_2 [\alpha g(0) + s_2(z)](k + \varepsilon) \)

Let \( c_2 [\alpha g(0) + s_2(z)](k + \varepsilon) = \gamma \)

Then \( (ve^{s_2(z)t})' = \gamma e^{s_2(z)t} \)

\[ \Rightarrow v(t)e^{s_2(z)t} - v(0) = \frac{\gamma}{s_2(z)}(e^{s_2(z)t} - 1) \]
\[ \Rightarrow v(t) \leq v(0)e^{-s_2(z)t} + \frac{\gamma}{s_2(z)}(1 - e^{-s_2(z)t}) \]
\[ = \frac{\gamma}{s_2(z)} + e^{-s_2(z)t}(v(0) - \frac{\gamma}{s_2(z)}) \]

if \( v(0) > \frac{\gamma}{s_2(z)} \) \( \Rightarrow v(t) \leq \frac{\gamma}{s_2(z)} + \varepsilon \) for large \( t \)

if \( v(0) \leq \frac{\gamma}{s_2(z)} \) \( \Rightarrow v(t) \leq \frac{\gamma}{s_2(z)} \forall t \geq 0 \)

from this the result for the case of Model (3.1) follows.
Case 2: Model 2 (3.2)

The equation (3.2.1) is
\[ x'(t) = axg(x) - y p_1(x) - z p_2(x) \]
where \( x(t_0) = x_0 \geq 0 \)
Let \( u'(t) = \alpha u g(u) \)
\[ u(t_0) = u_0 \geq 0 \]

... (3.9)

Then by standard comparison theorem we get that solution of (3.9) will always dominate the solution of (3.2.1)

Now if \( x_0 \leq k + \epsilon \), then in case of \( x_0 \leq k \), we get \( x(t) \leq k \leq k + \epsilon \)
Further if \( k < x_0 \leq k + \epsilon \) then
\[ \lim_{t \to \infty} \sup x(t) = k \]
and
\[ x(t) \leq k + \epsilon \quad \forall t, \text{ as } x' \leq 0 \]

Also, it follows from the existence and uniqueness of solutions to initial value problems that \( x \geq 0 \)

Further, \( x(t) \geq k + \epsilon \)
\[ \Rightarrow x \text{ is a decreasing function. Hence eventually, } x(t) \leq k + \epsilon \]

Now for (3.2.2) we can write the expression \( (c_1 x + y)' \) as

(50)
Let \( u = c(x + y) \)

Consider \( u' + s_1(y)u = c_1(k + \varepsilon)[\alpha g(0) + s_1(y)] + yq_1(z) \)

Let \( c_1(k + \varepsilon)[\alpha g(0) + s_1(y)] + yq_1(z) = \beta \)

Then \( (ue^{s_1(y)t})' = \beta e^{s_1(y)t} \)

\[
\Rightarrow u(t)e^{s_1(y)t} - u(0) = \frac{\beta}{s_1(y)}(e^{s_1(y)t} - 1)
\]

\[
\Rightarrow u(t) \leq u(0)e^{-s_1(y)t} + \frac{\beta}{s_1(y)}(1 - e^{-s_1(y)t})
\]

\[
= \frac{\beta}{s_1(y)} + e^{-s_1(y)t}(u(0) - \frac{\beta}{s_1(y)})
\]

if \( u(0) > \frac{\beta}{s_1(y)} \Rightarrow u(t) \leq \frac{\beta}{s_1(y)} + \varepsilon \) for large \( t \)

if \( u(0) \leq \frac{\beta}{s_1(y)} \Rightarrow u(t) \leq \frac{\beta}{s_1(y)} \) \( \forall t \geq 0 \)

\( (51) \)
Now for (3.2.3) we can write the expression \((c_2x + z)'\) as

\[
(c_2x + z)' = \alpha c_2x g(x) - c_2 z p_1(x) - c_2 z p_2(x) + z[-s_2(z) + q_2(y) + c_2 p_2(x)]
\]

\[
= \alpha c_2x g(x) - c_2 z p_2(x) + zq_2(y) - zs_2(z) \\
\leq \alpha c_2x g(x) - zs_2(z) + zq_2(y) \\
\leq \alpha c_2(k + \varepsilon)g(0) - zs_2(z) + zq_2(y) + c_2 xs_2(z) - c_2 xs_2(z) \\
= \alpha c_2(k + \varepsilon)g(0) + zq_2(y) + c_2 xs_2(z) - zs_2(z) - c_2 xs_2(z) \\
= c_2(k + \varepsilon)[\alpha g(0) + s_2(z)] + zq_2(y) - s_2(z)[c_2x + z]
\]

Let \(v = c_2x + z\)

Consider \(v' + s_2(z)v = c_2[\alpha g(0) + s_2(z)](k + \varepsilon) + zq_2(y)\)

Let \(c_2[\alpha g(0) + s_2(z)](k + \varepsilon) + zq_2(y) = \gamma\)

Then \((ve^{s_2(x)t})') = \gamma e^{s_2(x)t}\)

\[
\Rightarrow v(t)e^{s_2(x)t} - v(0) = \frac{\gamma}{s_2(z)}(e^{s_2(x)t} - 1)
\]

\[
\Rightarrow v(t) \leq v(0)e^{-s_2(x)t} + \frac{\gamma}{s_2(z)}(1 - e^{-s_2(x)t})
\]

\[
= \frac{\gamma}{s_2(z)} + e^{-s_2(x)t}(v(0) - \frac{\gamma}{s_2(z)})
\]

if \(v(0) > \frac{\gamma}{s_2(z)} \Rightarrow v(t) \leq \frac{\gamma}{s_2(z)} + \varepsilon\) for large \(t\)

\(\text{(52)}\)
if \( v(0) \leq \frac{\gamma}{s_2(z)} \Rightarrow v(t) \leq \frac{\gamma}{s_2(z)} \quad \forall \ t \geq 0 \)

from this the result for the case of Model (3.2) follows.

**Case 3 : Model 3 (3.3)**

The equation (3.3.1) is
\[
x'(t) = \alpha x g(x) - y p_1(x) - z p_2(x)
\]
where \( x(t_0) = x_0 \geq 0 \)

Let \( u'(t) = \alpha u g(u) \)
\[
u(t_0) = u_0 \geq 0 \tag{3.10}
\]

Then by standard comparison theorem we get that solution of (3.10) will always dominate the solution of (3.3.1)

Now if \( x_0 \leq k + \varepsilon \), then in case of \( x_0 \leq k \), we get \( x(t) \leq k \leq k + \varepsilon \)

Further if \( k < x_0 \leq k + \varepsilon \) then

\[
\lim_{t \to \infty} \sup x(t) = k
\]

and

\[
x(t) \leq k + \varepsilon \quad \forall t, \text{ as } x' \leq 0
\]

Also, it follows from the existence and uniqueness of solutions to initial value problems that \( x \geq 0 \)
Further, $x(t) > k + \varepsilon$

$\Rightarrow x$ is a decreasing function. Hence eventually, $x(t) \leq k + \varepsilon$

Now for (3.3.2) we can write the expression $(c_1x + y)'$ as

$$(c_1x + y)' = \alpha c_1xg(x) - c_1y_p_1(x) - c_2z_p_2(x) + y[-s_1(y) + q_1(z) + c_1p_1(x)]$$

$$= \alpha c_1xg(x) - c_2z_p_2(x) + yq_1(z) - ys_1(y)$$

$$\leq \alpha c_1xg(x) - ys_1(y) + yq_1(z)$$

$$\leq \alpha c_1(k + \varepsilon)g(0) - ys_1(y) + yq_1(z) + c_1xs_1(y) - c_1xs_1(y)$$

$$= \alpha c_1(k + \varepsilon)g(0) + yq_1(z) + c_1xs_1(y) - ys_1(y) - c_1xs_1(y)$$

$$= c_1(k + \varepsilon)[\alpha g(0) + s_1(y)] + yq_1(z) - s_1(y)[c_1x + y]$$

Let $u = c_1x + y$

Consider $u' + s_1(y)u = c_1(k + \varepsilon)[\alpha g(0) + s_1(y)] + yq_1(z)$

Let $c_1(k + \varepsilon)[\alpha g(0) + s_1(y)] + yq_1(z) = \beta$

Then $(ue^{s_1(y)t})' = \beta e^{s_1(y)t}$

$\Rightarrow u(t)e^{s_1(y)t} - u(0) = \frac{\beta}{s_1(y)}(e^{s_1(y)t} - 1)$

$\Rightarrow u(t) \leq u(0)e^{-s_1(y)t} + \frac{\beta}{s_1(y)}(1 - e^{-s_1(y)t})$

$$= \frac{\beta}{s_1(y)} + e^{-s_1(y)t}(u(0) - \frac{\beta}{s_1(y)})$$

(54)
if \( u(0) > \frac{\beta}{s_1(y)} \Rightarrow u(t) \leq \frac{\beta}{s_1(y)} + \varepsilon \) for large \( t \)

if \( u(0) \leq \frac{\beta}{s_1(y)} \Rightarrow u(t) \leq \frac{\beta}{s_1(y)} \forall t \geq 0 \)

Now for (3.3.3) we can write the expression \( (c_2x + z)' \) as

\[
(c_2x + z)' = \alpha c_2xg(x) - c_2yp_1(x) - c_2zp_2(x) + z[-s_2(z) - q_2(y) + c_2p_2(x)]
\]

\[
\leq \alpha c_2xg(x) - c_2yp_1(x) - c_2zp_2(x) + z[-s_2(z) + c_2p_2(x)]
\]

\[
\leq \alpha c_2xg(x) - zs_2(z)
\]

\[
\leq \alpha c_2(k + \varepsilon)g(0) - zs_2(z) + c_2xs_2(z) - c_2xs_2(z)
\]

\[
= c_2(k + \varepsilon)[\alpha g(0) + s_2(z)] - (c_2x + z)s_2(z)
\]

Let \( v = c_2x + z \)

Consider \( v' + s_2(z)v = c_2[\alpha g(0) + s_2(z)](k + \varepsilon) \)

Let \( c_2[\alpha g(0) + s_2(z)](k + \varepsilon) = \gamma \)

Then \( (ve^{s_2(z)t})' = \gamma e^{s_2(z)t} \)

\[
 \Rightarrow v(t)e^{s_2(z)t} - v(0) = \frac{\gamma}{s_2(z)}(e^{s_2(z)t}) - 1
\]

\[
 \Rightarrow v(t) \leq v(0)e^{-s_2(z)t} + \frac{\gamma}{s_2(z)}(1 - e^{-s_2(z)t})
\]

\[
= \frac{\gamma}{s_2(z)} + e^{-s_2(z)t}(v(0) - \frac{\gamma}{s_2(z)})
\]

(55)
if \( v(0) \geq \frac{\gamma}{s_2(z)} \Rightarrow v(t) \leq \frac{\gamma}{s_2(z)} + \epsilon \) for large \( t \)

if \( v(0) \leq \frac{\gamma}{s_2(z)} \Rightarrow v(t) \leq \frac{\gamma}{s_2(z)} \) \( \forall \ t \geq 0 \)

from this the result for the case of Model (3.3) follows.

From the above results for the models (3.1), (3.2) and (3.3) the proof of the theorem follows.