Chapter 8

SHOOTING METHOD OF FINDING CRITICAL EIGENVALUE FOR STUDY OF RAYLEIGH-BÉNARD-BRINKMAN CONVECTION WITH GENERAL BOUNDARY CONDITIONS

8.1 Introduction

The previous chapter presented a new computer based algorithm involving the successive linearization (SLM) method to find the critical eigenvalue in the Rayleigh-Bénard-Brinkman convection problem with general boundary conditions. The results obtained by the method though satisfactory was found to be very tedious in implementation and thus did not seem quite encouraging as a method for estimating eigenvalue in eigen boundary value problem. The method to be discussed in this chapter is a combination of the weighted residual method (for guessing unknown initial values and eigenvalue) and the shooting method that involves a combination of the Runge-Kutta-Fehlberg45 (RKF45) method and Newton-Raphson method. The shooting method also involves solutions of five initial value problems simultaneously. The proposal of a procedure for scientific choice of initial values is the crucial aspect of the method that ensures convergence of the solution. Twelve different eigen boundary value problems are studied in a unified way using the new algorithm presented in this chapter and suggests that there is no need to study the individual problems in isolation. The eigenvalues, further, can be obtained to any desired accuracy.
8.2 Mathematical formulation

We consider a layer of porous medium with depth d which is heated from below and cooled from above, as depicted in Figure (8.2). The upper surface is held at a temperature $T_0$ while the lower one is at $T_0 + \Delta T$. It is assumed that form-drag is insignificant, that the porous medium is isotropic but that local thermal non-equilibrium applies.

![Schematic diagram of the physical configuration.](image)

Thus the governing equations, i.e., the continuity equation, Brinkman equation and the energy equation, subject to the Boussinesq approximation, take the forms:

1. \[ \nabla \cdot \vec{q} = 0, \]  
2. \[ -\nabla p + \mu^1 \nabla^2 \vec{q} - \frac{\mu_f}{K} \vec{q} + \rho_0 \bar{g} \beta (T - T_0) y = 0, \]  
3. \[ (\rho c)_f \bar{q} \cdot \nabla T_f = \epsilon k_f \nabla^2 T_f + h (T_s - T_f), \]  
4. \[ (1 - \epsilon)(\rho c)_s \frac{\partial T_s}{\partial t} = (1 - \epsilon) k_s \nabla^2 T_s - h (T_s - T_f). \]

The constants and variables used in these equations are defined in chapter II. The boundary conditions are given in the following table:

Equations (8.2.1)-(8.2.4) are non-dimensionalised using the transformations:

\[
\begin{align*}
X &= \frac{1}{d} x, \quad Y = \frac{1}{d} y, \quad \vec{q} = \frac{\epsilon k_f}{(\rho c)_f d} \vec{q}, \\
P &= \frac{\mu k_f}{(\rho c)_f K} P, \quad \theta = \frac{T_f - T_0}{\delta T}, \quad \phi = \frac{T_s - T_0}{\delta T}.
\end{align*}
\]  

(8.2.5)
and the governing equations become

\[ \nabla \cdot \overrightarrow{q} = 0, \quad (8.2.6) \]
\[ e^2 F_1 \nabla P + Da \nabla^2 \overrightarrow{q} - \overrightarrow{q} + R \overrightarrow{y} = 0, \quad (8.2.7) \]
\[ \overrightarrow{q} \cdot \nabla \theta = \nabla^2 \theta - \overrightarrow{q} + H(\phi - \theta), \quad (8.2.8) \]
\[ \overrightarrow{q} \cdot \nabla \phi = \nabla^2 \phi + \gamma H(\theta - \phi). \quad (8.2.9) \]

In Equations (8.2.8)-(8.2.9), the following constants were introduced:

\[ F_1 = \frac{\rho_f kK}{e^2 d^2 \mu_f}, \quad F_2 = \frac{\rho_f \kappa K^2}{d^2 \mu_f}, \]
\[ \alpha = \frac{(\rho c)_s k_f}{(\rho c)_f k_s}, \quad \text{the diffusivity ratio}. \]

We note that the usual Rayleigh number, which is based on the mean properties of the porous medium is given by \( \frac{R \gamma}{(1 + \gamma)}. \) The boundary conditions are as in table (8.1).

The basic conduction profile, whose stability is the subject of this chapter, is given by

\[ \overrightarrow{q} = 0, \quad \theta = \phi = 1 - Y, \quad (8.2.10) \]

We focus our attention to the 2D case and we introduce the stream-function \( \psi \) according to

\[ U = -\frac{\partial \psi}{\partial Y}, \quad V = \frac{\partial \psi}{\partial X} \quad (8.2.11) \]

The basic conduction profile given by (8.2.9) is perturbed by setting:

\[ \psi = \overrightarrow{\psi}, \quad \theta = 1 - y + \overrightarrow{\Theta}, \quad \phi = 1 - y + \overrightarrow{\Phi}. \quad (8.2.12) \]

In what follows we present the method of solving the non-dimensional governing equations for the problem formulated in earlier section.

### 8.3 Method of solution

We now discuss the following important matters pertaining to the implementation of the shooting method used for solving the eigen boundary value problem.

\[ -Da \left( \frac{\partial^4 \overrightarrow{\psi}}{\partial X^4} + 2 \frac{\partial^2 \overrightarrow{\psi}}{\partial X^2 \partial Y^2} + \frac{\partial^4 \overrightarrow{\psi}}{\partial Y^4} \right) + \frac{\partial^2 \overrightarrow{\psi}}{\partial X^2} + \frac{\partial^2 \overrightarrow{\psi}}{\partial Y^2} = R \frac{\partial \overrightarrow{\Theta}}{\partial X}, \quad (8.3.1) \]
\[
\frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2} + \frac{\partial \psi}{\partial X} + H(\Phi - \Theta) = 0, \quad (8.3.2)
\]
\[
\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} + \gamma H(\Theta - \Phi) = 0. \quad (8.3.3)
\]

We now assume periodic cells in the $x$-direction and hence we may assume

\[
\begin{aligned}
\psi(X,Y) &= \psi(Y) \cos(kX) \\
\Theta(X,Y) &= \Theta(Y) \sin(kX) \\
\Phi(X,Y) &= \Phi(Y) \sin(kX)
\end{aligned}
\]  
\[
(8.3.4)
\]

Substituting equation (8.3.4) in equation (8.3.1)-(8.3.3), we get

\[
- Da\left(D^4 - 2k^2 D^2 + k^4\right) \Psi + (D^2 - K^2) \Psi = Rk \Theta, \quad (8.3.5)
\]
\[
(D^2 - k^2) \Theta - k \Psi + H(\Phi - \Theta) = 0, \quad (8.3.6)
\]
\[
(D^2 - k^2) \Phi - k \Psi + \gamma H(\Theta - \Phi) = 0. \quad (8.3.7)
\]

The boundary and trial functions to solve equation (8.3.5)-(8.3.7) and the boundary conditions given in table 1.

**Application of Galerkin variant of weighted residual method**

On reason of using the single-term Galerkin technique we multiply equation (8.3.5) by $\Psi$, equation (8.3.6) by $\Theta$ and equation (8.3.7) by $\Phi$ and integrating the resulting equations by parts with respect to $z$ from 0 to 1. Then taking $\Psi = A \Psi_1$, $\Theta = B \Theta_1$, $\Phi = C \Phi_1$ in which $A$, $B$ and $C$ are constants and $\Psi_1$, $\Theta_1$, $\Phi_1$ are trial functions, yields the following equation for the eigen values:

\[
R = \frac{(y_2 - Da y_1)}{K^2 y_4} \left( -y_3 - \gamma H \frac{(\psi_1 \Theta_1)(\Phi_1 \Theta_1)}{\Phi_1 D^2 \Phi_1 - (K^2 + \gamma H)(\Phi_1^2)} \right), \quad (8.3.8)
\]
\[
y_1 = \langle \psi_1 D^4 \psi_1 \rangle - 2K^2 \langle \psi_1 D^2 \psi_1 \rangle + K^4 \langle \psi_1^2 \rangle,
\]
\[
y_2 = \langle \psi_1 D^2 \psi_1 \rangle - K^2 \langle \psi_1^2 \rangle,
\]
\[
y_3 = \langle K^2 + H \rangle \langle \Theta_1^2 \rangle - \langle \Theta_1 D^2 \Theta_1 \rangle,
\]
\[
y_4 = \langle \psi_1 \Theta_1 \rangle,
\]
\[
y_5 = \frac{\langle \psi_1 \Phi_1 \rangle \langle \Phi_1 \Theta_1 \rangle}{\langle \Phi_1 D^2 \Phi_1 \rangle - (K^2 + \gamma H)(\Phi_1^2)}. \]
In the above expression the angular bracket $\langle \quad \rangle$ denotes integration with respect to $z$ from 0 to 1. We have adopted the single-term Galerkin procedure to obtain initial estimate of the eigen value as a function of the parameters of the problem for different boundary combinations. In a later section we improve upon the Galerkin values by the shooting method based on Runge-Kutta-Fehlberg45 (RKF45) and Newton-Raphson Methods.

The value of the critical Rayleigh number obviously depends on the boundary conditions. In this paper we consider various boundary conditions and these are documented along with the appropriate trial functions in table (8.2). The results of computations using Galerkin method are presented in table (8.3) within parenthesis. In the next section we discuss the shooting method for improving the Galerkin values for each boundary combination.

**Application of shooting method for improving the single-term Galerkin values**

We consider the FIFI boundary combination (listed in table (8.1)) to explain the shooting method adopted in the paper. For the other boundary combinations also the shooting method is similarly implemented. To implement the shooting method we write down equations (8.3.5)-(8.3.7), with $\omega = 0$ as a system of 8 first order ordinary differential equations. Using the nomenclature $y_1 = \psi, \quad y_5 = \theta, \quad y_7 = \phi, \quad y_9 = R$, equations (8.3.5)-(8.3.7), for boundary condition of the boundary combination FIFI may now be written as:
Observe that in writing IVP1, four conditions of the boundary combination FIFI that are specified at \( z=0 \) are retained and the other four at \( z=1 \) are replaced by 3 assumed conditions at \( z=0 \) thereby resulting in three unknowns \( \beta, \eta \) and \( \delta \). Since it is an eigen boundary value problem that we are solving and hence \( R \) is also an unknown which is to be determined. We modify the shooting procedure of boundary value problems by adding one more differential equation for \( R \). This added one is the last differential equation in equation (21). The term \( \mu \) is the unknown eigenvalue to be determined. In effect it means that an initial value problem with a few known and a few assumed initial conditions will have to be solved and the solution depends on the choice of \( \mu, \beta, \eta \) and \( \delta \). A suggestive notation is now introduced to drive home this point. We denote \( y_i \) by \( y_i(\mu, \beta, \eta, \delta) \). The assumed conditions can at best be good estimates and will have to be refined in order to obtain a fairly accurate critical value of \( R \), viz., \( R_c \).

Towards refinement of the assumed initial conditions, IVP1 is solved using the constant step-size Runge-Kutta-Fehlberg45 (RKF45) method and the values of \( y_i \) at \( z=1 \) are obtained. As per the notation introduced earlier, this means \( y_i(\mu, \beta, \eta, \delta; 1) \)'s are to be obtained by RKF45. We now devise a means of refining the values of \( \mu, \beta, \eta \) and \( \delta \). In the given eigen boundary value problem (EBVP), in the notation of \( y_i \), the values \( y_1(1), y_3(1), y_5(1), y_8(1) \) are known, for a fixed value of \( Da, \alpha, \gamma \) and \( H \). We thus have 4 knowns of \( y \), i.e. \( y_1(1), y_3(1), y_5(1), y_8(1) \) to determine 5 unknowns \( \mu, \beta, \eta \) and \( \delta \). To reduce the unknowns by one, we scale all the quantities by \( \beta \) and solve the IVP1.
In what follows all quantities are scaled quantities. Having obtained $y_i(\mu, \beta, \eta, \delta; 1)$ by RKF45 we may now seek to minimize the following quantities:

\[
\begin{align*}
F_1(\mu, \beta, \eta, \delta; 1) &= y_1(\mu, \beta, \eta, \delta; 1) - [y_1(1)], \\
F_2(\mu, \beta, \eta, \delta; 1) &= y_3(\mu, \beta, \eta, \delta; 1) - [y_3(1)], \\
F_3(\mu, \beta, \eta, \delta; 1) &= y_5(\mu, \beta, \eta, \delta; 1) - [y_5(1)], \\
F_4(\mu, \beta, \eta, \delta; 1) &= y_8(\mu, \beta, \eta, \delta; 1) - [y_8(1)], \\
\end{align*}
\]

(8.3.10)

In view of boundary conditions of FIFI at $z=1$, we note that the quantities within square brackets in equation[22] are zero. The minimization of the $F_i$'s is done with the help of the Newton-Raphson method, for system of non-linear equations, given by

\[
\begin{bmatrix}
\mu_{n+1} \\
\beta_{n+1} \\
\eta_{n+1} \\
\delta_{n+1}
\end{bmatrix} =
\begin{bmatrix}
\mu_n \\
\beta_n \\
\eta_n \\
\delta_n
\end{bmatrix} 
+ 
\begin{bmatrix}
\frac{\partial F_1}{\partial \mu} & \frac{\partial F_1}{\partial \beta} & \frac{\partial F_1}{\partial \eta} & \frac{\partial F_1}{\partial \delta} \\
\frac{\partial F_2}{\partial \mu} & \frac{\partial F_2}{\partial \beta} & \frac{\partial F_2}{\partial \eta} & \frac{\partial F_2}{\partial \delta} \\
\frac{\partial F_3}{\partial \mu} & \frac{\partial F_3}{\partial \beta} & \frac{\partial F_3}{\partial \eta} & \frac{\partial F_3}{\partial \delta} \\
\frac{\partial F_4}{\partial \mu} & \frac{\partial F_4}{\partial \beta} & \frac{\partial F_4}{\partial \eta} & \frac{\partial F_4}{\partial \delta}
\end{bmatrix}^{-1} 
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4
\end{bmatrix}
\]

(8.3.11)

From equations (22) and (23), it is obvious that the derivatives of $y_i$'s with respect to $\mu, \beta, \eta$ and $\delta$ are required for implementing the method equation (24) and these may be obtained from equation (21) by differentiating the same with respect to $\mu, \beta, \eta$ and $\delta$ to obtain the following four additional initial value problems, known as “variational equations” in literature:

Initial value problem 2 (IVP2):

\[
\begin{align*}
\frac{dY_2}{dz} &= Y_4 \\
\frac{dY_3}{dz} &= Y_6 \\
\frac{dY_4}{dz} &= -\frac{1}{D_1} \left( K(y_3 Y_6 + y_5 y_9) + K^2 Y_1 - Y_3 + D_1 K^4 y_1 - 2 D_2 K^2 y_5 \right) \\
\frac{dY_6}{dz} &= K^2 Y_5 + KY_1 - HY_7 + HY_5 \\
\frac{dY_7}{dz} &= K^2 R_7 - \gamma HY_7 + \gamma HY_7 \\
\frac{dY_8}{dz} &= 0
\end{align*}
\]

(8.3.12)
Initial value problem 3 (IVP3):

\[
\begin{align*}
\frac{dU_1}{dz} &= U_2, & U_1(0) &= 0, \\
\frac{dU_2}{dz} &= U_3, & U_2(0) &= 0, \\
\frac{dU_3}{dz} &= U_4, & U_3(0) &= 0, \\
\frac{dU_4}{dz} &= \frac{1}{D_1} \left( K(y_5 U_7 + U_5 y_9) + K^2 U_1 - U_3 + D_1 K^4 U_1 - 2D_1 K^2 U_3 \right), & U_4(0) &= 0, \\
\frac{dU_5}{dz} &= U_6, & U_5(0) &= 1, \\
\frac{dU_6}{dz} &= K^2 U_5 + K U_1 - U_7 + U_5, & U_6(0) &= -L, \\
\frac{dU_7}{dz} &= K^2 U_2 - U_3 + D_1 K^4 U_1 - 2D_1 K^2 U_3, & U_7(0) &= 0, \tag{8.3.13}
\end{align*}
\]

Initial value problem 4 (IVP4):

\[
\begin{align*}
\frac{dV_1}{dz} &= V_2, & V_1(0) &= 0, \\
\frac{dV_2}{dz} &= V_3, & V_2(0) &= 0, \\
\frac{dV_3}{dz} &= V_4, & V_3(0)(0) &= 0, \\
\frac{dV_4}{dz} &= \frac{1}{D_1} \left( K(y_5 V_7 + V_5 y_9) + K^2 V_1 - V_3 + D_1 K^4 V_1 - 2D_1 K^2 V_3 \right), & V_4(0) &= 0, \\
\frac{dV_5}{dz} &= V_6, & V_5(0) &= 0, \\
\frac{dV_6}{dz} &= K^2 V_5 + K V_1 - V_7 + V_5, & V_6(0) &= 0, \\
\frac{dV_7}{dz} &= V_8, & V_7(0) &= 0, \\
\frac{dV_8}{dz} &= K^2 V_2 - V_3 + D_1 K^4 V_1 - 2D_1 K^2 V_3, & V_8(0) &= 0, \tag{8.3.14}
\end{align*}
\]

Initial value problem 5 (IVP5):

\[
\begin{align*}
\frac{dS_1}{dz} &= S_2, & S_1(0) &= 0, \\
\frac{dS_2}{dz} &= S_3, & S_2(0) &= 0, \\
\frac{dS_3}{dz} &= S_4, & S_3(0)(0) &= 0, \\
\frac{dS_4}{dz} &= \frac{1}{D_1} \left( K(y_5 S_7 + S_5 y_9) + K^2 S_1 - S_3 + D_1 K^4 S_1 - 2D_1 K^2 S_3 \right), & S_4(0) &= 0, \\
\frac{dS_5}{dz} &= S_6, & S_5(0) &= 0, \\
\frac{dS_6}{dz} &= S_7, & S_6(0) &= 0, \\
\frac{dS_7}{dz} &= S_8, & S_7(0) &= 0, \\
\frac{dS_8}{dz} &= S_9, & S_8(0) &= 0, \\
\frac{dS_9}{dz} &= K^2 S_2 - \gamma V S_3 + \gamma V S_7, & S_9(0) &= 1. \tag{8.3.15}
\end{align*}
\]
The five IVPs are solved by the RKF45 method, for an assumed guess value of $\mu, \beta, \eta$ and $\delta$, to obtain the solution $y_i, Y_i, U_i, V_i$ and $S_i$ at $z=1$. Using these in equation[23] we can obtain a more refined value of $\mu, \beta, \eta$ and $\delta$. The above procedure will have to be followed for a range of values of the wave number that includes the estimated value obtained from the single-term Galerkin method. Having obtained the eigen value $R$ for each value of $k$ in the considered range of values of $k$ we find the minimum value of $R$ and the corresponding value of $k$. These are respectively called the critical value of $R$ and $k$, and are denoted by $R_c$ and $k_c$ respectively. For each combination of the two parameter values, $Da, \alpha, \gamma$ and $H$, the above procedure is followed to obtain $R_c$ and $k_c$. The success of the shooting technique depends on a scientific choice of the 4 unknown initial values $\mu, \beta, \eta$ and $\delta$. This aspect is discussed in the next section.

**Scientific choice of $\mu, \beta, \eta$ and $\delta$.**

In section 3 a crude estimate of the critical eigen value $\mu$ was obtained to start with. In what follows the procedure to obtain an estimate of $\beta, \eta$ and $\delta$ is presented. We use the trial functions (refer Table 1) to obtain $D^3W(0), DT(0)$ and $\Phi(0)$ and these can be taken to be the estimates of $\beta, \eta$ and $\delta$ respectively. This scientific choice of $\mu, \beta, \eta$ and $\delta$ ensures the convergence of the Newton-Raphson method (23) and helps one to shoot (iterate) to the desired accurate value of $\mu, \beta, \eta$ and $\delta$. If the guess value of $\mu, \beta, \eta$ and $\delta$ is not close to the actual value, then the method (23) as we know, may diverge. The convergence criterion that is used in the Newton-Raphson method (23) is the following:

$$|\mu_{n+1} - \mu_n| < \xi, |\beta_{n+1} - \beta_n| < \xi, |\eta_{n+1} - \eta_n| < \xi, |\delta_{n+1} - \delta_n| < \xi, \quad (n=0,1,2...)$$

for some $n$. In the problem we have chosen $\xi = 10^{-4}$.
8.4 Results and discussion

The 8th order two point eigen boundary value problem with third type boundary conditions discussed in chapter 7 does not possess an analytical solution as indicated earlier. The most general formulation of the problem facilitates the unified handling of twelve limiting case problems which at the present time are being tackled as twelve independent problem. The integrated approach to the eigen boundary value problem give a new direction to the investigation of such problems. The main difficulty in solving such eigen boundary value problems involves convergence of the schemes, generally iterative. This is overcome in the present chapter by using the shooting method as detailed out in the chapter.

The single term Galerkin method is used to guess the unknown initial values and also the eigenvalue. It also provides useful information on the possible range of wave number to consider for investigation on the eigenvalue for each combination of the parameters. A combination of RKF45 and Newton-Raphson method together with the scientific choice of required guess values ensures convergence of the method. As mentioned earlier the results of twelve limiting cases can be extracted from this general study. Out of the twelve problems namely three combination FIFI, RIRI and RAFA(outlined in table (8.1)) have been investigated. Tables (8.2) and (8.3) provide confidence in the results obtained by the present study in the above three limiting cases. Tables (8.4)–(8.6) present all the results not reported by the graphs on critical values of the wave number and Rayleigh number for all the parameters’ combination.

Figures (8.3) to (8.8) gives all the corresponding results for all other boundary conditions as that obtained by Banu and Rees and Postelico and Rees for free-free isothermal boundary combination.
Figure 8.2: Plot of (a) $Ra_c$ versus $Bi_L$, $Bi_U$ and (b) $Ra_c$ versus $Bi^*_L$, $Bi^*_U$ for different boundary combinations.
Figure 8.3: Plot of (a) $Ra_c$ versus $Bi_L, Bi_U$ and (b) $Ra_c$ versus $Bi^*_L, Bi^*_U$ for different values of $Da$ for the boundary combination FIFI.
Figure 8.4: Plot of (a) $Ra_c$ versus $Bi_L$, $Bi_U$ and (b) $Ra_c$ versus $Bi_L^*$, $Bi_U^*$ for different values of $Da$ for the boundary combination RIRI.
Figure 8.5: Plot of (a) $Ra_c$ versus $Bi_L$, $Bi_U$ and (b) $Ra_c$ versus $Bi^*_L$, $Bi^*_U$ for different values of $H$ for the boundary combination FIFI.
Figure 8.6: Plot of (a) $Ra_c$ versus $Bi_L$, $Bi_U$ and (b) $Ra_c$ versus $Bi_L^*$, $Bi_U^*$ for different values of $H$ for the boundary combination RIRI.
Figure 8.7: Plot of (a) $Ra_c$ versus $Bi_L, Bi_U$ and (b) $Ra_c$ versus $Bi_L^*, Bi_U^*$ for different values of $\gamma$ for the boundary combination FIFI.
Figure 8.8: Plot of (a) $Ra_c$ versus $Bi_L$, $Bi_U$ and (b) $Ra_c$ versus $Bi_L^*, Bi_U^*$ for different values of $\gamma$ for the boundary combination RIRI.
Table 8.1: Different boundary combinations and corresponding trial functions for Rayleigh-Bénard-Brinkman convection

<table>
<thead>
<tr>
<th>Case</th>
<th>Boundary condition(BC)</th>
<th>Acronym for boundary condition</th>
<th>Trial functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$z=0$ $W = D^2 W = 0$ Free $T = 0$ $D^2 W = 0$ Isothermal $T = 0$ $D^2 W = 0$ Isothermal</td>
<td>FIFI</td>
<td>$W_1 = z^4 - 2z^3 + z$ $T_1 = z^2 - z$ $\Phi_1 = \cos{z}$</td>
</tr>
<tr>
<td></td>
<td>$z=1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$z=0$ $W = DW = 0$ Rigid $T = 0$ $D^2 W = 0$ Isothermal $W = D^2 W = 0$ Free $T = 0$ $D^2 W = 0$ Isothermal</td>
<td>RIFI</td>
<td>$W_1 = 2z^4 - 5z^3 + 3z^2$ $T_1 = z^2 - z$ $\Phi_1 = \cos{z}$</td>
</tr>
<tr>
<td></td>
<td>$z=1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$z=0$ $W = DW = 0$ Rigid $T = 0$ $D^2 W = 0$ Isothermal $W = DW = 0$ Rigid $T = 0$ $D^2 W = 0$ Isothermal</td>
<td>RIFI</td>
<td>$W_1 = z^4 - 2z^3 + z^2$ $T_1 = z^2 - z$ $\Phi_1 = \cos{z}$</td>
</tr>
<tr>
<td></td>
<td>$z=1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$z=0$ $W = DW = 0$ Rigid $DT = 0$ $D^2 W = 0$ Free $T = 0$ $D^2 W = 0$ Isothermal</td>
<td>RIFI</td>
<td>$W_1 = -2z^4 + 5z^3 - 3z^2$ $T_1 = z^2 - 1$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
</tr>
<tr>
<td></td>
<td>$z=1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$z=0$ $W = DW = 0$ Free $DT = 0$ $D^2 W = 0$ Isothermal $W = DW = 0$ Free $T = 0$ $D^2 W = 0$ Isothermal</td>
<td>FAIFI</td>
<td>$W_1 = z^4 - 2z^3 + z$ $T_1 = z^2 - z$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
</tr>
<tr>
<td></td>
<td>$z=1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$z=0$ $W = DW = 0$ Rigid $DT = 0$ $D^2 W = 0$ Free $T = 0$ $D^2 W = 0$ Isothermal</td>
<td>RARI</td>
<td>$W_1 = z^4 - 2z^3 + z^2$ $T_1 = z^2 - 1$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
</tr>
<tr>
<td></td>
<td>$z=1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$z=0$ $W = DW = 0$ Rigid $T = 0$ $D^2 W = 0$ Isothermal $W = DW = 0$ Free $DT = 0$ $D^2 W = 0$ Isothermal</td>
<td>RIFI</td>
<td>$W_1 = z^4 - 2z^3 + 1.5z^2$ $T_1 = z^2 - 2z$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
</tr>
<tr>
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<td>$z=1$</td>
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<td></td>
</tr>
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<td>8</td>
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<td>FAIFI</td>
<td>$W_1 = z^4 - 2z^3 + z^2$ $T_1 = z^2 - z$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
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<tr>
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<td>9</td>
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<td>$W_1 = z^4 - 2z^3 + z^2$ $T_1 = \cos{\frac{z}{2}}$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
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<td>RARI</td>
<td>$W_1 = z^4 - 2z^3 + z^2$ $T_1 = \cos{\frac{z}{2}}$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
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<td>$W_1 = z^4 - 2z^3 + z^2$ $T_1 = z^2 - 2z$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
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<td>$W_1 = z^4 - 2z^3 + z^2$ $T_1 = z^2 - 2z$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
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<td>$z=0$ $W = DW = 0$ Rigid $T = 0$ $D^2 W = 0$ Isothermal $W = DW = 0$ Rigid $DT = 0$ $D^2 W = 0$ Isothermal</td>
<td>RIFI</td>
<td>$W_1 = z^4 - 2z^3 + z^2$ $T_1 = z^2 - z$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
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<td>FIFI</td>
<td>$W_1 = z^4 - 1.5z^3 + 0.5z^2$ $T_1 = z^2 - 2z$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
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<td>$z=0$ $W = DW = 0$ Free $DT = 0$ $D^2 W = 0$ Isothermal $W = DW = 0$ Rigid $DT = 0$ $D^2 W = 0$ Isothermal</td>
<td>FIFI</td>
<td>$W_1 = z^4 - 1.5z^3 + 0.5z^2$ $T_1 = z^2 - 2z$ $\Phi_1 = \frac{\sin{z}}{z}$</td>
</tr>
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</table>
Table 8.2: Comparison of $R_c$ and $K_c$ obtained from the present study with those obtained by Platten and Legros (1984) and Siddheshwar et al. (2011)

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Platten and Legros (1984) $R_c$</th>
<th>$K_c$</th>
<th>Siddheshwar et al. (2011) $R_c$</th>
<th>$K_c$</th>
<th>Present problem $R_c$</th>
<th>$K_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RIRI</td>
<td>1707.762</td>
<td>3.12</td>
<td>1706.76</td>
<td>3.12</td>
<td>1707.673</td>
<td>3.12</td>
</tr>
<tr>
<td>FIFI</td>
<td>657.511</td>
<td>2.22</td>
<td>657.592</td>
<td>2.22</td>
<td>657.513</td>
<td>2.22</td>
</tr>
<tr>
<td>RAFA</td>
<td>384.693</td>
<td>1.76</td>
<td>387.558</td>
<td>1.76</td>
<td>384.693</td>
<td>1.76</td>
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</table>

Table 8.3: Comparison of $R_c$ and $K_c$ obtained from the present study with those obtained by A. Postelnicu and D. A. S. Rees (2003) for the limiting case ($10^3, \gamma = 1$)

<table>
<thead>
<tr>
<th>Boundary Combination</th>
<th>LogH</th>
<th>$K_c$</th>
<th>$R_c$</th>
<th>$K_c$</th>
<th>$R_c$</th>
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</thead>
<tbody>
<tr>
<td>FIFI</td>
<td>1.5</td>
<td>3.36726</td>
<td>64.75793</td>
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<td>73.81888</td>
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<td>73.82699</td>
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<td>3.15609</td>
<td>78.14846</td>
<td>3.15609</td>
<td>78.14846</td>
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<td>3.12642</td>
<td>79.73422</td>
<td>3.12642</td>
<td>79.74002</td>
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<tr>
<td></td>
<td>3.5</td>
<td>3.11654</td>
<td>80.26042</td>
<td>3.11654</td>
<td>80.26273</td>
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<tr>
<td></td>
<td>4.0</td>
<td>3.11337</td>
<td>80.42940</td>
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<td>80.43092</td>
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<tr>
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<td>4.5</td>
<td>3.11236</td>
<td>80.48310</td>
<td>3.11236</td>
<td>80.48440</td>
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<tr>
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<td>5.0</td>
<td>3.11204</td>
<td>80.50011</td>
<td>3.11204</td>
<td>80.50134</td>
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Table 8.4: Values of $R_c$ and $K_c$ obtained from the present study ($Da = 10^{-2}$, $\gamma = 1$) for different boundary combinations and for different values of $H$

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Log $H$</th>
<th>( R_{IRI} ) (same as RIRA)</th>
<th>( R_{RAI} ) (same as FIRA)</th>
<th>( R_{API} ) (same as FARI)</th>
<th>( R_{ARA} ) (same as FARI)</th>
<th>( R_{RFA} ) (same as FARI)</th>
<th>( R_{FAI} ) (same as FIFA)</th>
<th>( R_{FPA} ) (same as FIFA)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-2</td>
<td>3.23174</td>
<td>60.42749</td>
<td>2.51257</td>
<td>53.15413</td>
<td>2.40667</td>
<td>39.90406</td>
<td>4.68987</td>
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<tr>
<td></td>
<td>-1</td>
<td>3.23811</td>
<td>60.69288</td>
<td>2.53233</td>
<td>53.39958</td>
<td>2.42552</td>
<td>40.29201</td>
<td>4.69845</td>
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<tr>
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<td>0</td>
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<td>63.20008</td>
<td>2.67594</td>
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<td>43.61406</td>
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<td>3.49454</td>
<td>79.74521</td>
<td>2.91002</td>
<td>70.77283</td>
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<td>59.01706</td>
<td>5.15698</td>
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<tr>
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<td>2</td>
<td>3.35197</td>
<td>110.3805</td>
<td>2.61038</td>
<td>97.49414</td>
<td>2.91007</td>
<td>76.80306</td>
<td>5.41112</td>
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<tr>
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<td>3</td>
<td>3.24962</td>
<td>119.3794</td>
<td>2.52118</td>
<td>105.2378</td>
<td>2.4139</td>
<td>79.29078</td>
<td>4.75120</td>
</tr>
</tbody>
</table>

Note: $Da = 10^{-2}$, $\gamma = 1$
Table 8.5: Plot of $R_c$ and $K_c$ obtained from the present study ($Da = 10^{-2}, LogH = 1$) for different boundary combinations and for different values of $\gamma$
Table 8.6: Plot of $R_c$ and $K_c$ obtained from the present study ($\gamma = 1$, $\log H = 2$) for different boundary combinations and for different values of $D_a$. 

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>$R_c$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>RI (same as RIRA)</td>
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<tr>
<td>RARI (same as RIRA)</td>
<td>3.35197</td>
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</tr>
<tr>
<td>RAFI (same as FIRA)</td>
<td>2.61039</td>
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</tr>
<tr>
<td>RARA (same as FARI)</td>
<td>2.28160</td>
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<tr>
<td>FAFI (same as FIFA)</td>
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</tr>
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<td>FAFA</td>
<td>4.38744</td>
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</table>

<table>
<thead>
<tr>
<th>$D_a$</th>
<th>$R_c$</th>
<th>$K_c$</th>
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<td>10^-2</td>
<td>2.40908</td>
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<tr>
<td>10^-1</td>
<td>1.02694</td>
<td>1.09854</td>
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