Chapter 6

POWER SERIES SOLUTION OF HEAT TRANSPORT EQUATION FOR A NONLINEAR NON-DARCY FLOW THROUGH RECTANGULAR POROUS CHANNEL AND CYLINDRICAL POROUS ANNULUS

6.1 Introduction

The proceedings of the previous three chapters very clearly indicate that the finite difference method along with the homotopy continuation method is an excellent means of solving boundary value problems arising in the Darcy-Brinkman-Forchheimer model concerning rectangular channel, cylindrical tube and annulus flows. The study of heat transfer in such flows requires the use of interpolation polynomials to effect numerical integration. Analysing the three methods we found the finite difference method along with the homotopy continuation method to be the best combination to obtain the flow equation and using the velocity at the discretized points to subsequently estimate the heat transfer. Earlier reported results are only on heat transfer in rectangular channel flows. In this chapter we not only extend the range of applicability of the heat transfer results for rectangular channel flows, we estimate reasonably well the heat transfer in cylindrical tube and annulus problems. The results for most part are new in this chapter.

6.2 Heat transport in a Brinkman Forchheimer flow through a cylindrical annulus

Local thermal equilibrium and homogeneity are assumed and hence the steady-state thermal energy equation in the absence of heat source terms, axial conduction, and
thermal dispersion is:
\[
\rho c_p U \frac{\partial T}{\partial z} = k \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right),
\]  
(6.2.1)

where \( U \) is the velocity of the Brinkman Forchheimer flow (see Fig 3.2) obtained by HCM in chapter III. It follows from the first law of thermodynamics that
\[
\frac{\partial T}{\partial z} = \frac{2q''}{\rho c_p U_m}. 
\]  
(6.2.2)

We now define the mean velocity \( U_m \) and the bulk mean temperature \( T_m \) in the following form:
\[
U_m = \frac{1}{1-\epsilon} \int_\epsilon^1 U Y dY, \quad T_m = \frac{2}{(1-\epsilon)U} \int_\epsilon^1 U TY dY.
\]

Further, dimensionless variables will now be introduced as (see Hooman and Gurgenci (2007))
\[
\hat{u} = \frac{U}{U_m}, \quad \theta = \frac{T - T_w}{T_m - T_w}. 
\]  
(6.2.3)

The Nusselt number \( Nu \) is defined by:
\[
Nu = \frac{2q''}{k(T_w - T_m)}. 
\]  
(6.2.4)

As noted by Nield (2006), though the local temperature \( T \) is a function of both axial and radial coordinates, dimensionless temperature distribution in the fully developed region, \( \theta \), is a function of the radial coordinates \( (Y) \) only, while the bulk mean temperature is a function of the axial coordinate \( (z) \) only. In non-dimensional form (6.2.1) becomes (when (3.3.3), (6.2.3) and (6.2.4) are used):
\[
\frac{d^2 \theta}{dY^2} + 1 \frac{d \theta}{Y dY} + \hat{u} Nu = 0, 
\]  
(6.2.5)

where the boundary conditions are as follows
\[
\left. \frac{d \theta}{dY} \right|_{Y=\epsilon} = 0 \text{ and } \theta |_{Y=1} = 0. 
\]  
(6.2.6)

Equation (6.2.5) can be solved, using boundary conditions (6.2.6), to obtain the following expression for \( \theta(Y) \):
\[
\theta(Y) = \frac{Nu}{2} \int_\epsilon^1 \frac{1}{Y} (\eta \hat{u} d\eta) dY. 
\]  
(6.2.7)
Using the compatibility condition,

\[ \int_{\epsilon}^{1} Y \dot{u} \theta dY = \frac{1}{2} (1 - \epsilon) \]  \hspace{1cm} (6.2.8)

The Nusselt number can be obtained in the form:

\[ Nu = \frac{2 \left( \int_{\epsilon}^{1} Y \dot{u}(Y) dY \right)^2}{\int_{\epsilon}^{1} Y \dot{u}(Y) \left[ \int_{\epsilon}^{1} \frac{1}{\xi} \left( \int_{\epsilon}^{\xi} \eta \dot{u}(\eta) d\eta \right) d\xi \right] dY}. \]  \hspace{1cm} (6.2.9)

The results obtained by solving equations (3.4.10), subject to conditions (3.4.4), by the Runge-Kutta method are documented in figures 3.8 - 3.16. Figures 6.1-6.4 show the variation of Nusselt number with different parameters.

### 6.3 Results and discussion

As made known quite explicitly in the introduction it is the intention of the paper to propose the computer-assisted continuation method for solving a non-linear, non-Darcy equation with quadratic drag. Before we embark on a discussion of the solution we note here that the definition of Brinkman and Darcy numbers as used in the paper is inverse of the classical definitions. We now move on to discuss the results obtained in the paper.

Figure (6.1) depicts the variation of Nusselt number with \( Da \). It can easily be observed that for small values of \( Da \) in the range \( 0.1 \leq Da < 1 \), Nusselt number varies insignificantly with \( Da \). A similar observation can be made on effect of \( Da \) in the range \( 100 \leq Da < \infty \). From the above observations we may thus infer that Nusselt number variation in porous media is significant only in the range \( 1 \leq Da < 100 \). As a consequence we may also conclude that the inclusion of the Forchheimer term is justified in a porous medium whose \( Da \) value lies in the range 1 to 100.

Figure (6.3) illustrates the fact that \( Nu \) varies linearly with \( \Lambda \). This is in keeping with the observation that can made on the linear variation of \( U(Y) \) with \( \Lambda \).

Figure (6.3) reinforces the observation on the Nusselt number variation with \( Da \). Significant changes in \( Nu \) with change in \( F \) is seen only in the case when \( 1 \leq F < 1000 \). From the Figure we observe that the effect of increasing form drag is to increase the heat transport. Another important result from the present study is on the variation of
$Nu$ with scaled radius of the concentric cylindrical insert. It is apparent that there is less heat transport in a cylinder with no insert in comparison with the corresponding result on Nusselt number when an insert is present. The observation on the plots (6.1) - (6.3) are also applicable to cylindrical flows and rectangular channel flow.

Tables 1 and 2 show close qualitative agreement between the present results on Nusselt number in the limiting cases of Hooman (2008), Nield (2006) and Hooman and Gurgenci (2007).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\Lambda = F = Da = 1$</th>
<th>$L = 0.1, F = 1, Da = 1$</th>
<th>$\Lambda = 0.1, F = 1, Da = 31.6227766$</th>
<th>$\Lambda = F = 1, Da = 100$</th>
</tr>
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<tr>
<td>Nu Present paper</td>
<td>4.146</td>
<td>4.115</td>
<td>5.12886</td>
<td>5.84798</td>
</tr>
<tr>
<td>Nield</td>
<td>4.159</td>
<td>4.122</td>
<td>5.129</td>
<td>—</td>
</tr>
<tr>
<td>Hooman</td>
<td>4.181</td>
<td>4.131</td>
<td>5.139</td>
<td>5.8935</td>
</tr>
</tbody>
</table>

Table 6.1: Rectangular Problem-Comparison between present results on Nusselt number with those of Hooman (2008) and Nield (2006).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\Lambda = 1, F = 0, Da = 0$</th>
<th>$\Lambda = 1, F = 0, Da = 0.5$</th>
<th>$\Lambda = 1, F = 2, Da = 0.5$</th>
<th>$\Lambda = 1, F = 0, Da = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nu Present paper</td>
<td>4.33108</td>
<td>4.39733</td>
<td>4.33804</td>
<td>5.98708</td>
</tr>
<tr>
<td>Hooman &amp; Gurgenci</td>
<td>4.36</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 6.2: Cylindrical Problem-Comparison between present results on Nusselt number with those of Hooman and Gurgenci (2007).
Figure 6.1: Plot of Nusselt number $Nu$, vs. $Da$ for $\Lambda = 1$ and $F = 1$, for a cylindrical annulus ($\epsilon = 0.2$, $\delta = 1$).
Figure 6.2: Plot of $Nu$ vs. $\Lambda$ for $Da = 1$ and $F=1$, for a cylindrical annulus ($\epsilon=0.2$, $\delta=1$).
Figure 6.3: Plot of $Nu$ vs. $F$ for $Da =1$ and $\Lambda=1$ for a cylindrical annulus ($\epsilon=0.2$, $\delta=1$).

Figure 6.4: Plot of $Nu$ vs. $\epsilon$ for fixed values of $Da =1$, $F=2$, and $\Lambda=1$ for a cylindrical annulus ($\delta=1$).