CHAPTER 3

CONCEPTUAL FRAMEWORK

It is common that all the production processes are subject to a certain amount of inherent or natural variability regardless of how well designed or carefully maintained it is. This natural variability may be result of many small but unavoidable causes. In the framework of statistical quality control, this natural variability is often called a “stable system of chance causes”. A process that is operating with only chance causes of variation present is said to be in statistical control. In other words, the chance causes are an inherent part of the process.

There are also other kinds of variability that may occasionally be present in the output of a process. This variability in key quality characteristics usually arises from three sources: operator errors, defective raw material or improperly adjusted or controlled machines. Sometimes, variability in quality characteristics due to these sources may be large when compared to the background noise, and it usually represents in unacceptable level of process performance. These sources of variability that are not part of the chance cause pattern are referred to as “assignable causes”. A process that is operating in the presence of assignable causes is said to be out of control. (Douglas C Montgomery 2003)

Under normal conditions, all the production processes will be operating in the in-control state, producing acceptable product for relatively long periods of time. Eventually, however, assignable causes will occur,
seemingly at random, resulting in a “shift” to an out-of-control state where a larger proportion of the process output does not conform to requirements. When the process is out of statistical control, a higher proportion of the process lies outside of the specification limits.

The importance of quality has been long recognized as vital to both competition and survival in the business world. As such, more organizations have adopted the use of statistical process control (SPC) as a mean for obtaining higher product quality. Although SPC is only an element of total quality management (TQM), it is nevertheless a major one. SPC is the use of statistically-based methods to evaluate and monitor a process or its output in order to achieve or maintain a state of control. Many articles have addressed the implementation of SPC in different industries.

The major objective of statistical process control is to quickly detect the occurrence of assignable causes of process shifts so that investigation of the process and corrective action may be undertaken before many nonconforming units are manufactured. The control chart is an on-line process-monitoring technique widely used for this purpose. Control charts may also be used to estimate the parameters of a production process, and, through this information, to determine process capability. The control chart may also provide information useful in improving the process. Finally, the eventual goal of statistical quality control is the elimination of variability in the process. It may not be possible to completely eliminate variability, but the control chart is an effective tool in reducing variability as much as possible.

3.1 STATISTICAL BASIS OF THE CONTROL CHART

A typical control chart is a graphical display of a quality characteristic that has been measured or computed from a sample versus the sample number or time. A control chart is a time plot of a statistic, such as a
sample mean, range, standard deviation, or proportion, with a centerline and upper and lower control limits. The limits give the desired range of values for the statistic. When the statistic is outside the bounds, or when its time plot reveals certain patterns, the process may be out-of-control. The chart contains a centre line that represents the average value of the quality characteristic corresponding to the in-control state. Two other horizontal lines, called the upper control limit (UCL) and the lower control (LCL), are also shown on the chart. These control limits are chosen so that if the process is in control, nearly all of the sample points will fall between them. As long as the points plot within the control limits, the process is assumed to be in control, and no action is necessary. However, a point that plots outside of the control limits is interpreted as evidence that the process is out of control, and investigation and corrective action are required to find the eliminate the assignable cause or causes responsible for this behavior. It is customary to connect the sample points on the control chart with straight-line segments, so that it is easier to visualize how the sequence of points has evolved over time.

The first modern ideas on how statistics could be used in quality control came in the mid-1920s from a colleague of Deming’s Walter Shewhart of Bell Laboratories. Shewhart invented the control chart for industrial process. A control chart is a graphical display of measurements (usually aggregated in the form of means or other statistics) of an industrial process through time. By carefully scrutinizing the chart, a quality control engineer can identify any potential problems with the production process. The idea is that when a process is in control, the variable being measured – the mean of every four (or) five observations, for example – should remain stable through time. The mean should stay somewhere around the middle line (the grand mean for the process) and not wander off “too much”. In statistics, the term “too much” means more than several standard deviations of the process. The number of required standard deviations is chosen so that there
will be a small probability of exceeding them when the process is in control. Addition and subtraction of the required number of standard deviations (generally three) give us the upper control limit (UCL) and the lower control limit (LCL) of the control chart. As long as the points plot within the control limits, the process is assumed to be in control. When these bounds are breached, the process is deemed out of control and must be corrected. A control chart is illustrated in Figure 3.1.

![Figure 3.1 A typical control chart](image)

However, a point that plots outside of the control limits is interpreted as evidence that the process is out of control, and investigation and corrective action are required to find and eliminate the assignable cause or causes responsible for this behavior. It is customary to connect the sample points on the control chart with straight-line segments, so that it is easier to visualize how the sequence of points has evolved over time.

It may not always said to be the process is in under control if all the points plot inside the control limits, if they behave in a systematic or nonrandom manner; this could be an indication that the process is out of control. For example, several points continuously plot above the center line but below the upper control limit, i.e. if 18 of 20 points plot above center line and only two plots below the center line, there is a possibility to suspect that there is something incorrect. For a process to be presumed under statistical
control, it is necessary that all the points should plot in an essentially random manner. The pattern of points plotting continuously above or below the center line but within the control limits implies the presence of assignable causes, and if it can be found and eliminated, the process can be brought back under control.

The control charts may also be used as an estimating device. It is used to estimate certain process parameters, such as the mean, standard deviation, fraction nonconforming or fallout. These estimates may then be used to determine the capability of the process to produce acceptable products. These process-capability studies have considerable impact on many management decision problems that occur over the product cycle, including make or buy decisions, plant and process improvements that reduce process variability, and contractual agreements with customers or vendors regarding product quality.

3.1.1 Categories of Control Chart

The control charts may be classified under two categories. If the quality characteristic can be measured and expressed as a number on some continuous scale measurement, it is usually called a variable. In such cases, it is convenient to describe the quality characteristic with a measure of central tendency and a measure of variability. Control charts for central tendency (usually mean) and measure of variability (usually Range and Standard deviation) are collectively called variable control charts. The X-bar chart is most widely used chart for controlling central tendency, whereas charts based on either the sample range or the sample standard deviation are used to control process variability. Many quality characteristics are not measured on a continuous scale or even a quantitative scale, and these may be judged on either conforming or nonconforming on the basis of whether or not it possesses certain attributes, or the number of nonconformities appearing on a
unit of product may be counted. Control charts for these quality characteristics are called attributes control charts.

The present study deals with only variable control charts. Some characteristics of the paper like substance, caliper and bulk density have been taken for the study and all these characteristics are measured on a metric scale, i.e. substance is measured in g/m$^2$, caliper is measured in microns (10$^{-6}$ meters) or mils (0.001 inches) and bulk density is measured m$^3$/kg. Therefore variable control charts are used to individually control these characteristics.

### 3.1.2 Selection of a Control Chart

The following Figure 3.2 illustrates the selection on a control chart on the basis of nature of quality characteristics (Douglas CMontgomery2003).

![Choose a Control Chart](image)

**Figure 3.2 Guide to selection of control chart**

If the quality characteristic under study is metric, i.e., measured in a definite unit of measurement, and subgroup sample size is eight or less we may use X-bar and R Chart; if the subgroup sample size is more than eight, we may use X-bar and S Chart.
3.1.3 Reasons for Popularity of Control Charts

Control charts have had a long history of use in U.S. industries and in many offshore industries as well. There are at least five reasons for the popularity.

1. Control charts are a proven technique for improving productivity. A successful control chart program will reduce scrap and rework, which are the primary productivity killers in any operation. If the scrap and rework are reduced, then productivity increases, cost decreases, and production capacity increases.

2. Control charts are effective in defect prevention. The control chart helps keep the process in control, which is consistent with the “do it right the first time” philosophy. It is never cheaper to sort out “good” units from “bad” units later on than it is to build it right initially. If process control is not effective, then it is presumed that the company is paying its workforce to produce nonconforming products.

3. Control charts prevent unnecessary process adjustment. A control chart can distinguish between background noise and abnormal variation; no other device including a human operator is as effective in making this distinction. If process operators adjust the process based on periodic tests unrelated to a control chart program, they will often overact to the background noise and make unneeded adjustments. These unnecessary adjustments can actually result in a deterioration of process performance.
4. Control charts provide diagnostic information. The pattern of points on the control chart will contain information of diagnostic value to an experienced operator. This information allows the implementation of a change in the process that improves its performance.

5. Control charts provide information about process capability. The control chart provides information about the value of important process parameters and their stability over time. This allows an estimate of process capability to be made. This information is of tremendous use to product and process designers.

Control charts are among the most important management control tools; they are as important as cost controls and material controls. Modern computer technology has made it easy to implement control charts in any type of process, as data collection and analysis can be performed on a microcomputer or a local area network terminal in real-time, on-line at the work center.

3.1.4 Analysis of Patterns on Control Charts

A control chart may indicate an out-of-control condition either when one or more points fall beyond the control limits or when the plotted points exhibit some nonrandom pattern of behavior, as seen in Figure 3.3. For example, it can be observed from the following figure that from 12\textsuperscript{th} sample point to 16\textsuperscript{th} sample point, five points in a row increase in magnitude. Also from 21\textsuperscript{st} sample points, five points in a row fall below the center line. This arrangement of points is called a run. A run is the sequence of the observations of some type.
If the observations are increasing, it is called a run up and if the observations are decreasing, it is called a run down. A run of 8 or more points has a very low probability and consequently, a sequence of 8 points is an indication of process in out-of-control.

Apart from the above, it may sometimes happen that all the points plot within the control limits, but however, the points may cluster around center point, may be called as cycling as illustrated in Figure 3.4. Such a pattern may indicate a problem with the process such as operator fatigue, raw material deliveries, heat or stress buildup, etc. Although the process is not really out of control, the yield may be improved by elimination or reduction of the sources of variability causing this cyclic behavior.
The Western Electric Handbook (1956) suggests a set of decision rules for detecting nonrandom patterns on control charts. Specifically, it suggests concluding that the process is out of control if either

i) One point plots outside the three-sigma control limits;

ii) Two out of three consecutive points plot beyond the two-sigma warning limits;

iii) Four out of five consecutive points plot at a distance of one-sigma or beyond from the center line;

iv) Eight consecutive points plot on one side of the center line.

These rules apply to one side of the center line at a time. Therefore, a point above the upper warning limit followed immediately by a point below the lower warning limit would not signal an out-of-control alarm. These are often used in practice for enhancing the sensitivity of control charts. That is, the use of these rules can allow smaller process shifts to be detected more quickly than would be the case if our only criterion was the usual three-sigma control limit violation.

### 3.1.5 Sensitizing Rules for Control Charts

The basic criterion is one or more points outside of the control limits. Several criteria may be applied simultaneously to a control chart to determine whether the process is out of control. The supplementary criteria are sometimes used to increase the sensitivity of the control charts to a small process shift so that we may respond quickly to the assignable cause. Some of the widely used sensitizing rules in practice are given in Table 3.1.
### Table 3.1. Sensitizing rules for shewhart control charts

<table>
<thead>
<tr>
<th>Standard action signal</th>
<th>1. One or more points outside of the control limits.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2. Two or three consecutive points outside the two-sigma warning limits but still inside the control limits.</td>
</tr>
<tr>
<td></td>
<td>3. Four of five consecutive points beyond the one-sigma limits</td>
</tr>
<tr>
<td></td>
<td>4. A run of eight consecutive points on one side of the center line.</td>
</tr>
<tr>
<td></td>
<td>5. Six points in a row steadily increasing or decreasing.</td>
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<tr>
<td></td>
<td>6. Fifteen points in a row in zone in one-sigma limit.</td>
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<tr>
<td></td>
<td>7. Fourteen points in a row alternating up and down.</td>
</tr>
<tr>
<td></td>
<td>8. Eight points in a row on both sides of the center line with none in one-sigma limit.</td>
</tr>
<tr>
<td></td>
<td>9. An unusual or nonrandom pattern in the data.</td>
</tr>
<tr>
<td></td>
<td>10. One or more points near a warning or control limits.</td>
</tr>
</tbody>
</table>

### 3.2 CONTROL CHART FOR X-BAR ($\bar{X}$)

When dealing with a quality characteristics that is a variable, it is usually necessary to monitor both the mean value of the quality characteristic and its variability. Control of the process average or mean quality level is usually done with the control chart for means, or the $\bar{X}$ (X-bar) chart.
The control chart for $\bar{X}$ uses subgroups each of size $n$ for $k$ consecutive periods of time. To compute control limits for the mean, we have to compute the mean of the subgroup means (called $\bar{X}$) and the standard deviation of the mean (which is called the standard error of the mean $\sigma_{\bar{X}}$). The estimate of the standard deviation of the mean is a function of the $d_2$ factor, which represents the relationship between the standard deviation and the range for varying sample sizes. The estimator of $\sigma_{\bar{X}}$ is the average range divided by $d_2$, a constant that depends on the sample size $n$. The estimator of $\sigma$ is $\bar{R}/d_2$. The American Society for Testing and Materials Manual on Presentation of Data and Control Chart Analysis proves values for $d_2$ and other constants involved in the computation of control limits. Equations (3.1) and (3.2) define the control limits for the $\bar{X}$ chart.

Control Limits for the $\bar{X}$ Chart (when $\sigma$ is unknown)

Central Line $\quad = \quad \bar{X}$

Upper Control Limit $\quad = \quad \bar{X} + 3 \frac{\bar{R}}{d_2 \sqrt{n}} \quad (3.1)$

Lower Control Limit $\quad = \quad \bar{X} - 3 \frac{\bar{R}}{d_2 \sqrt{n}} \quad (3.2)$

where $\bar{X} = \frac{\sum_{i=1}^{k} X_i}{k} \quad ;$

$\bar{R} = \frac{\sum_{i=1}^{k} R_i}{k} \quad ;$

$\frac{\bar{R}}{d_2}$ is estimate of $\sigma$. 

\[ 
\]
\( \bar{X}_i \) = sample mean of n observations at time \( i \)

\( R_i \) = range of n observations at time \( i \)

\( k \) = number of subgroups

\( n \) = size of the subgroups

We can simplify the calculations in Equation (3.1) by utilizing the A\(_2\) factor, equal to \( \frac{3}{d_2\sqrt{n}} \). The simplified control limits are:

Central Line \( = \bar{X} \)

Upper Control Limit \( = \bar{X} + A_2 \bar{R} \)

Lower Control Limit \( = \bar{X} - A_2 \bar{R} \)

The values of A\(_2\) can be obtained from Statistical Tables for different values of \( n \).

### 3.3 CONTROL CHART FOR R (RANGE)

The R chart enables to determine whether the variability in a process is in control or whether changes in the amount of variability are occurring over time. If the process range is in control, then the amount of variation in the process is consistent over time, and we can use the results of the R chart to develop the control limits for the mean. To develop the R chart, we need to think of the range of a sample as a random variable with its own mean and standard deviation. The average range \( \bar{R} \) provides an estimate of the mean of this random variable. The estimate of the standard deviation of the range is

\[
\hat{\sigma}_R = d_3 \frac{R}{d_2}
\]
where \( d_2 \) and \( d_3 \) are constants that depend on the sample size; values of \( d_2 \) and \( d_3 \) are also provided in the Statistical Table. Thus the control limits for R chart is as follows:

**Control Limits for the R Chart**

\[
\begin{align*}
\text{Central Line} & = \overline{R} \\
\text{Upper Control Limit} & = \overline{R} + 3\hat{\sigma}_R = \overline{R} \left(1 + 3\frac{d_2}{d_3}\right) \\
\text{Lower Control Limit} & = \overline{R} - 3\hat{\sigma}_R = \overline{R} \left(1 - 3\frac{d_2}{d_3}\right)
\end{align*}
\]

If we let \( D_4 = 1 + 3\frac{d_1}{d_2} \)

\[
D_3 = 1 + 3\frac{d_1}{d_2},
\]

the control limits for R chart can be rewritten as

\[
\begin{align*}
\text{Central Line} & = \overline{R} \\
\text{Upper Control Limit} & = D_4 \overline{R} \\
\text{Lower Control Limit} & = D_3 \overline{R}
\end{align*}
\]

where \( \overline{R} = \frac{\sum_{i=1}^{k} R_i}{k} \)

\( R_i \) = range of n observations at time \( i \)

\( k \) = number of subgroups

The values of \( D_3 \) and \( D_4 \) can be obtained from Statistical Tables for different values of \( n \).
3.4 CONTROL CHARTS FOR $\bar{X}$ AND S

Although $\bar{X}$ and R charts are widely used, it is occasionally desirable to estimate the process standard deviation directly instead of indirectly through the use of the range R. This leads to control charts for $\bar{X}$ and S, where S is the sample standard deviation. Generally, $\bar{X}$ and S charts are preferable to their more familiar counterparts, $\bar{X}$ and R charts, when either.

1. The sample size $n$ is moderately large – say, $n > 8$.
2. The sample size $n$ is variable.

Setting up and operating control charts for X-bar and S requires about the same sequence of steps as those for X-bar and R charts, except that for each sample we must calculate the sample average X-bar and the sample standard deviation S. The sample standard deviation can be calculated through the formula

$$S = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}}$$

However, the sample standard deviation is not an unbiased estimator of $\sigma$. When the underlying distribution is normal, then S actually estimates $c_4\sigma$, where $c_4$ is a constant that depends on the sample size $n$. Further, the standard deviation of S is $\sigma \sqrt{1 - c_4^2}$. This can be used to establish control charts on X-bar and S. Since $c_4\sigma$ is the expected value of S, the center line for the chart is $c_4\sigma$. The control limits for S chart is as follows (when $\sigma$ is known):
Control Limits for the S Chart

Central Line \[ = \quad c_4 \sigma \]

Upper Control Limit \[ = \quad c_4 \sigma + 3 \sigma \sqrt{1 - c_4^2} \]

Lower Control Limit \[ = \quad c_4 \sigma - 3 \sigma \sqrt{1 - c_4^2} \]

If we define two constants \( B_5 = c_4 - 3 \sqrt{1 - c_4^2} \) and 
\[ B_6 = c_4 + 3 \sqrt{1 - c_4^2}, \]

The control limits for S Chart become

Central Line \[ = \quad c_4 \sigma \]

Upper Control Limit \[ = \quad B_6 \sigma \]

Lower Control Limit \[ = \quad B_5 \sigma \]

The values of \( B_5 \) and \( B_6 \) can be obtained from Statistical Tables for different values of \( n \).

If the value of \( \sigma \) is not known, then it must be estimated by analyzing past data. Suppose that we have \( m \) preliminary samples, each of size \( n \), and \( S_i \) is the standard deviation of the \( i^{th} \) sample. The average of the \( m \) standard deviations is

\[ \bar{S} = \frac{1}{m} \sum_{i=1}^{m} S_i \]

The Statistic \( \frac{\bar{S}}{c_4} \) is an unbiased estimator of \( \sigma \). Therefore, the parameters of the S chart would be
Central Line \[= \overline{s}\]

Upper Control Limit \[= \overline{s} + 3 \frac{\overline{s}}{c_4} \sqrt{1-c_4^2} \, B_4 \overline{s}\]

Lower Control Limit \[= \overline{s} - 3 \frac{\overline{s}}{c_4} \sqrt{1-c_4^2} \, B_3 \overline{s}\]

If we define the constants \(B_3 = \frac{\overline{s}}{c_4} \sqrt{1-c_4^2}\) and \(B_4 = \frac{3 \overline{s}}{c_4 \sqrt{n}}\),

The control limits for S chart would be

Central Line \[= \overline{s}\]

Upper Control Limit \[= B_4 \overline{s}\]

Lower Control Limit \[= B_3 \overline{s}\]

When \(\overline{s}/c_4\) is used to estimate \(\sigma\), we may define the control limits on the corresponding X-bar chart as:

Central Line \[= \overline{X}\]

Upper Control Limit \[= \overline{X} + 3 \frac{\overline{s}}{c_4 \sqrt{n}}\]

Lower Control Limit \[= \overline{X} - 3 \frac{\overline{s}}{c_4 \sqrt{n}}\]

If we let \(A_3 = \frac{3}{c_4 \sqrt{n}}\),

The control limits for X-bar charts will become
Central Line = \overline{X}

Upper Control Limit = \overline{X} + A_3 \overline{S}

Lower Control Limit = \overline{X} - A_3 \overline{S}

The values of B_3, B_4, and A_3 can be obtained from Statistical Tables for different values of n.

The quality engineers have preferred the R chart to the S chart because of the simplicity of calculating R from each sample. However, the current availability of hand-held calculators with automatic calculation of S and the increased availability of microcomputers for on-line implementation of control charts directly at the workstation have eliminated any computational difficulty.

3.5 CONTROL CHART FOR INDIVIDUAL MEASUREMENT (MOVING RANGE)

In some situations where the sample size used for monitoring is \( n = 1 \), that is, the sample consists of an individual unit. In such situations, the control chart for individual units is useful. In many applications of the individual control chart, moving range of two consecutive observations is used as the basis of estimating the process variability. The moving range is defined as:

\[ \text{MR}_i = |x_i - x_{i-1}| \]

It is also possible to establish a control chart on the moving range.

On a conventional chart for variables, the R-bar value determines the distance at which the control limits are from the mean line on the x-bar
chart. In other words, the variation within sample determines the difference that can exist in the variation between samples, before the latter is considered as statistically significant. However, there are situations in which the variation inside the sample does not serve as a good basis for establishing the control limits of x-bar. Cases where this occurs are:

- In batch manufacturing of lots, where the differences between lots are pronounced due to the inherent raw material variation and there is no possibility of reducing it;
- In continuous product manufacturing (paper machine, for instance), where the machine cross direction variation is no suitable basis to establish the machine direction variation range due to their completely opposite natures.

The Moving Range (MR) control chart is, as a matter of fact, a combination of the mean and range charts (x-bar and R) with the charts for individual values and moving range (x-MR), according to Ramos (2000), so that they make it possible to simultaneously control more than two types of variation.

The R chart will monitor the variation within sample. Consequently:

\[
\text{Lower Control Limit} = D_3 \bar{R} \\
\text{Central Line} = \bar{R} \\
\text{Upper Control Limit} = D_4 \bar{R}
\]

The MR chart, for its part, will serve as a basis to establish the distance of the control limits to the mean line on the X-bar chart. Therefore:
Lower Control Limit  = $D_3\overline{MR}$

Central Line  = $\overline{MR}$

Upper Control Limit  = $D_4\overline{MR}$

Finally, the X-bar chart will be calculated by means of the formulas:

Lower Control Limit  = $\overline{X} + \frac{3}{d_2}\overline{MR}$

Central Line  = $\overline{X}$

Upper Control Limit  = $\overline{X} - \frac{3}{d_2}\overline{MR}$

3.6  CUMULATIVE SUM CONTROL CHART

Although the Shewhart Control Chart is very effective, it may sometimes fail to detect small magnitude of the shift in the process. The Shewhart Control Chart for averages is very effective if the magnitude of the shift is $1.5\sigma$ to $2\sigma$ or larger. For smaller shifts, it is not as effective. Of course, other criteria can be applied to Shewhart charts, such as tests for runs and the use of warning limits, which attempt to incorporate information from the entire set of points into the decision procedure. However, the use of these supplemental sensitizing rules reduces the simplicity and ease of interpretation of the Shewhart control chart. The cumulative sum (or cusum) control chart is a good alternative when small shifts are important. The CUSUM chart directly incorporates all the information in the sequence of sample values by plotting the cumulative sums of the deviations of the sample values from a target value. For smaller shifts, cumulative sum (or CUSUM) control chart is very effective and a good alternative when small shift is important.
The cusum chart directly incorporates all the information in the sequence of sample values by plotting the cumulative sums of the deviations of the sample values from a target value. There are two types of cusum charts available, tabular and v-mark, out of this, tabular cusum is preferable.

Let \( x_i \) be the \( i \text{th} \) observation on the process. When the process is in control, \( x_i \) has a normal distribution with mean \( \mu_0 \) and standard deviation \( \sigma \). The tabular cusum works by accumulating deviations from \( \mu_0 \) that are above target with one statistic \( C^+ \) and accumulating deviations from \( \mu_0 \) that are below target with another statistic \( C^- \). The statistics \( C^+ \) and \( C^- \) are called one-sided upper and lower cusums, respectively. They are computed as follows:

\[
C^+_i = \max [0, x_i - (\mu_0 + K) + C^+_{i-1}]
\]
\[
C^-_i = \max [0, (\mu_0 - K) - x_i + C^-_{i-1}]
\]

where the starting values are \( C^+_0 = C^-_0 = 0 \)

\( K \) is usually called the reference value (or the allowance, or the slack value), and it is often chosen about halfway between the target \( \mu_0 \) and the out-of-control value of the mean \( \mu_1 \) that we are interested in detecting quickly. Thus \( K \) is one-half the magnitude of the shift or

\[
K = \frac{|\mu_1 - \mu_0|}{2}
\]

\( C^+_i \) and \( C^-_i \) accumulate deviations from the target value \( \mu_0 \) that are greater than \( K \), with both quantities reset to zero on becoming negative. If either \( C^+_i \) or \( C^-_i \) exceed the decision interval \( H \), the process is considered to be out of control. A reasonable value for \( H \) is five times the process standard deviation \( \sigma \).
3.7 EXPONENTIALLY WEIGHTED MOVING AVERAGE CONTROL CHART

The exponentially weighted moving average (EWMA) control chart is also a good alternative to the Shewhart control chart when we are interested in detecting small shifts. The performance of the EWMA control chart is approximately equivalent to that of the cumulative sum control chart, and in some ways it is easier to set up and operate.

The exponentially weighted moving average is defined as

\[ z_i = \lambda x_i + (1-\lambda)z_{i-1} \]

where \(0 < \lambda \leq 1\) is a constant and the starting value (required with the first sample at \(I = 1\)) is the process target, so that

\[ z_0 = \mu_0 \]

Sometimes the average of preliminary data is used as the starting value of the EWMA, so that \(z_0 = \bar{x}\).

The EWMA can be viewed as a weighted average of all past and current observations, it is very insensitive to the normality assumption. It is therefore an ideal control chart to use with individual observations. (Montgomery 1990 and Box et al 1994).

The EWMA control chart would be constructed by plotting \(z_i\) versus the sample number \(i\) (or time). The center line and control limits for the EWMA control chart are as follows.
Upper Control Limit \[= \mu_0 + L\sigma \sqrt{\frac{\lambda}{2-\lambda}} \left[1 - (1-\lambda)^{1/2}\right]\]

Center Line \[= \mu_0\]

Lower Control Limit \[= \mu_0 - L\sigma \sqrt{\frac{\lambda}{2-\lambda}} \left[1 - (1-\lambda)^{1/2}\right]\]

The factor \( L \) is the width of the control limits, i.e., UCL – LCL. The term \( \left[1 - (1-\lambda)^{1/2}\right] \) approaches unity as \( i \) gets larger. This means that after the EWMA control chart has been running for several time periods, the control limits will approach steady-state values given by

Upper Control Limit \[= \mu_0 + L\sigma \sqrt{\frac{\lambda}{2-\lambda}}\]

Lower Control Limit \[= \mu_0 - L\sigma \sqrt{\frac{\lambda}{2-\lambda}}\]

The EWMA control chart is very effective against small process shifts. The design parameters of the chart are the multiple of sigma used in the control limits (\( L \)) and the value of \( \lambda \). The suggested values of \( L \) are between about 2.6 and 2.8 and \( \lambda \) is 0.1 (Montgomery 1990).

**Characteristics of the EWMA**

The EWMA methodology has some very attractive properties, in particular:

- Unlike traditional "Shewhart" control charts, which use each data point independently, EWMA charts may use all of the data collected over time to determine the control status of a process.
The EWMA is often superior to the CUSUM charting technique for detecting "larger" shifts in the process location.

The EWMA can be used to monitor both process standard deviations and averages.

Weights can be chosen for EWMA and the number of \( \lambda \) for calculating control limits to match in-control average run lengths for traditional "Shewhart" control charts.

**Comparison of EWMA, CUSUM, and moving average charts**

CUSUM, EWMA, and moving average (MA) charts are suitable for monitoring in-control processes to detect small shifts in the process average. They are not as sensitive at detecting larger shifts in process level as traditional variables control chart, such as X-bar chart.

The firm may use CUSUM, EWMA, and MA charts when not enough data are available to use traditional variables control charts; for example,

- when there are long time intervals between consecutive results
- when results are difficult or time-consuming to obtain
- when only one measurement properly represents the process at any given period in time.

**3.8 MULTIVARIATE CONTROL CHART**

The Shewhart Control Chart is very effective if there is only one process output variable or quality characteristic of interest. In practice,
however, many of the processing monitoring and control scenarios involve several related variables. Although applying univariate control charts to each individual variable is a possible solution, this may be inefficient, time consuming and can lead to erroneous conclusions. Multivariate methods that consider the variables jointly are effective in these situations.

In the present study, the bulk density of the paper is influenced by the two quality characteristics, i.e. substance \((x_1)\) and caliper \((x_2)\). We assume that these two variables have normal distributions and they could be monitored by X-bar chart to each characteristic. Also the bulk density can be plotted as an X-bar chart.

However, monitoring these two quality characteristics independently can be very misleading. The individual charts may sometimes show the process in control, but if we look at the multivariate control chart we find some variation in the process. Process-monitoring problems in which several related variables are of interest are called multivariate quality control (or process-monitoring) problems.

The multivariate quality control chart has two phases, viz. Phase-1 for establishing control; that is, testing whether the process was in control when the \(m\) preliminary subgroups were drawn and the sample statistics \(\bar{x}\) and \(S\) computed. The objective in phase 1 is to obtain an in-control set of observations so that control limits can be established for phase 2, which is monitoring of future production. This is sometimes called a retrospective analysis.

The statistic to be plotted on the process monitoring chart is

\[
T^2 = n(\bar{x} - \bar{x})'S^{-1}(\bar{x} - \bar{x})
\]
\[ S = \begin{bmatrix} S_{1k}^2 & S_{1k} S_{2k} \\ S_{1k} S_{2k} & S_{2k}^2 \end{bmatrix} \]

i.e.,
\[ T^2 = \frac{11}{S_1^2 S_2^2 - S_{12}^2} \left[ S_2^2 (x_1 - \bar{x}_1)^2 + S_1^2 (x_2 - \bar{x}_2)^2 - 2S_{12} (x_1 - \bar{x}_1)(x_2 - \bar{x}_2) \right] \]

The Upper and Lower Control Limits (UCL and LCL) are given by:

**Phase 1:**
\[ UCL = \frac{p(m-1)(n-1)}{mn-m-p+1} F_{\alpha,m,n-m-p+1} \]
LCL = 0

**Phase 2:**
\[ UCL = \frac{p(m+1)(n-1)}{mn-m-p+1} F_{\alpha,m,n-m-p+1} \]
LCL = 0

where
- \( p \) = number of quality characteristics
- \( m \) = sample size
- \( n \) = subgroup size
- \( \alpha = 0.001 \)

### 3.9 Non-Parametric Methods

If the assumptions on the population from which the samples are drawn are not known and/or the normality condition does not hold good, non-parametric is the only alternative to the parametric tests. In case where a nonparametric method as well as a parametric method can be applied, the nonparametric method is almost as good or almost as powerful as the parametric method. Because of the less restrictive data measurement requirements and the fewer assumptions needed about the population distribution, nonparametric methods are regarded as more generally applicable than parametric methods. The sign test, run test, the Wilcoxon
signed-rank test, the Mann-Whitney-Wilcoxon test, the Kruskal-Wallis test, and the Spearman rank correlation are some of the nonparametric methods. The paper industry can make use of sign test and run test for application in quality control.

3.9.1 The Run Test

The randomness of the sample drawn from a population is essential for all types of statistical testing because sample results are to be used to draw conclusions regarding the population under study. The run test helps to determine whether the order or sequence of observations (symbols, items or number) in a sample is random. The runs test examines the number of ‘runs’ of each of two possible characteristics that sample elements may have. A run is a sequence of identical occurrences of elements (symbols or numbers) preceded and succeeded by different occurrences of elements or by no element at all. For example, in our quality control problem, the occurrence of two or more sample averages above or below the grand mean would constitute a run. To quantity how many runs are acceptable before raising doubt about the randomness of the process, a probability distribution is used that leads to a statistical test for randomness. The hypothesis of variations in sample averages is randomly distributed can be tested by run test with the decision rule

Reject the hypothesis if $R \leq C_1$ or $R \geq C_2$, accept otherwise.

where $R$ is the number of runs and $C_1$, $C_2$ are critical values obtained from standard table with total tail probability $P(R \leq C_1) + P(R \geq C_2) = \alpha$. 
3.9.2 Sign Test

The *sign test* is based on the sign of a difference between related observations. A plus sign is designated to a positive difference and a minus sign for a negative difference. The sign test is not concerned with the magnitude of the difference, only the direction of the difference. The sign test is based on only two outcomes and the test statistic, the number of positive signs, follows the binomial distribution. The binomial probability distribution can be approximated to a normal distribution if \( np \geq 5 \) and \( n(1-p) \geq 5 \) are satisfied. The test statistic (with \( x \), the number of positive signs) then becomes,

\[
z = \frac{x-np}{\sqrt{npq}}
\]

The hypothesis of variations (above or below the grand mean) in sample averages is even can be tested by sign test with the decision rule

If calculated value, \( z_{cal} \) of \( z \)-test statistic is greater than or equal to its critical value, reject the hypothesis; do not reject otherwise.