Chapter 2

Reliability performance measures of systems with Marshall - Olkin Trivariate Exponential failuretime distribution

2.0 Summary

This chapter deals with systems having Marshall-Olkin trivariate exponential component failure time distributions. By considering four three unit systems with Marshall-Olkin trivariate exponential failure time distribution, we derive the reliability performance measures of the system. The mgf of the sum of the failure times is expressed as the weighted average of three exponential mgf’s, which aids in finding the reliability performance measures of standby system. Three component standby, parallel, series and relay systems with trivariate exponential failure times are discussed and their reliability performance measures are obtained. In addition, the location-scale MOTVE is considered as the failuretime distribution and the corresponding reliability measures are obtained.

In Section 2.1, we express the mgf of MOTVE distribution, as the weighted average of three exponential mgf’s and obtain the MTBF and reliability function of a three unit standby system. In Section 2.2, we consider a three unit parallel system and obtain the MTBF and Reliability function of the system, we also consider the case when the component failuretimes are identical and obtain the MTBF and Reliability function. Section 2.3 gives the MTBF and reliability function of a three unit series system, and Section 2.4 gives the MTBF and reliability function of a three unit relay system. In addition, for all the four systems,
the location-scale MOTVE is considered as the failure time distribution and the corresponding reliability measures are obtained. The results of this chapter are based on Chandrasekar and Amala Revathy (2013).

The random vector \((Y_1, Y_2, Y_3)\) is said to have a location-scale MOTVE distribution with the location-scale parameter \((\xi, \tau)\) if the survival function of \(\left(\frac{Y_1 - \xi}{\tau}, \frac{Y_2 - \xi}{\tau}, \frac{Y_3 - \xi}{\tau}\right)\) at \((x_1, x_2, x_3)\) is \(\overline{F}(x_1, x_2, x_3)\) given in (1.1.1).

### 2.1 Standby system

Consider a three unit standby system with component failure times \(X_1, X_2, X_3\) respectively. Then the system failure time is \(T = \sum_{i=1}^{3} X_i\). Assume that the component failure times are identically distributed.

As in Samanta (1983), take

\[
\lambda_1 = \lambda_2 = \lambda_3 = \beta_1,
\]

\[
\lambda_{12} = \lambda_{13} = \lambda_{23} = \beta_2,
\]

\[
\lambda_{123} = \beta_3.
\]

Let us first find the mgf of \(T\).

Define

\[
\alpha_1 = \beta_1 + 2\beta_2 + \beta_3,
\]

\[
\alpha_2 = 2\beta_1 + 3\beta_2 + \beta_3 \text{ and}
\]

\[
\alpha_3 = 3\beta_1 + 3\beta_2 + \beta_3.
\]
From Theorem 1.4.1,

\[
M(t_1, t_2, t_3) = \beta_1 (\beta_1 + \beta_2) \alpha_1 \\
= \frac{1}{(\alpha_1 - t_1)(\alpha_2 - t_1 - t_2)(\alpha_3 - t_1 - t_2 - t_3)} + \frac{1}{(\alpha_1 - t_2)(\alpha_2 - t_1 - t_2)(\alpha_3 - t_1 - t_2 - t_3)} \\
+ \frac{1}{(\alpha_1 - t_2)(\alpha_2 - t_1 - t_2)(\alpha_3 - t_1 - t_2 - t_3)} + \frac{1}{(\alpha_1 - t_3)(\alpha_2 - t_1 - t_2)(\alpha_3 - t_1 - t_2 - t_3)} \\
+ \beta_1 (\beta_2 + \beta_3) \left[ \frac{1}{(\alpha_2 - t_1 - t_2)(\alpha_3 - t_1 - t_2 - t_3)} + \frac{1}{(\alpha_2 - t_1 - t_2)(\alpha_3 - t_1 - t_2 - t_3)} \right] \\
+ \frac{1}{(\alpha_2 - t_1 - t_2)(\alpha_3 - t_1 - t_2 - t_3)} + \frac{1}{(\alpha_3 - t_1 - t_2 - t_3)}
\]

Thus the mgf of \(X_1 + X_2 + X_3\) at \(t\) is,

\[
M^*(t) = \frac{6\beta_1 (\beta_1 + \beta_2) \alpha_1}{(\alpha_1 - t)(\alpha_2 - 2t)(\alpha_3 - 3t)} + \frac{3\beta_1 (\beta_2 + \beta_3)}{(\alpha_2 - 2t)(\alpha_3 - 3t)} + \frac{3\beta_2 \alpha_1}{(\alpha_1 - t)(\alpha_3 - 3t)} + \frac{\beta_3}{(\alpha_3 - 3t)}
\]

Note that the mgf exists for \(t < \min \left\{ \alpha_1, \frac{\alpha_2}{2}, \frac{\alpha_3}{3} \right\}\).

Resolving the first three terms into partial fractions and simplifying we get

\[
\frac{6\beta_1 (\beta_1 + \beta_2) \alpha_1}{(\alpha_1 - t)(\alpha_2 - 2t)(\alpha_3 - 3t)} = \frac{6\beta_1 (\beta_1 + \beta_2) \alpha_1}{(\beta_2 + \beta_3)(3\beta_2 + 2\beta_3)(\alpha_1 - t)} \\
+ \frac{24\beta_1 (\beta_1 + \beta_2) \alpha_1}{(\beta_2 + \beta_3)(3\beta_2 + 2\beta_3)(\alpha_2 - 2t)} \\
+ \frac{54\beta_1 (\beta_1 + \beta_2) \alpha_1}{(3\beta_2 + 2\beta_3)(3\beta_2 + \beta_3)(\alpha_3 - 3t)}
\]

\[
\frac{3\beta_1 (\beta_2 + \beta_3)}{(\alpha_2 - 2t)(\alpha_3 - 3t)} = -\frac{6\beta_1 (\beta_2 + \beta_3)}{(3\beta_2 + \beta_3)(\alpha_2 - 2t)} + \frac{9\beta_1 (\beta_2 + \beta_3)}{(3\beta_2 + \beta_3)(\alpha_3 - 3t)}
\]

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Let us express the mgf as the weighted average of three exponential mgfs.

Define \( M_1(t) = \left(1 - \frac{t}{\alpha_1}\right)^{-1} \), 
\( M_2(t) = \left(1 - \frac{2t}{\alpha_2}\right)^{-1} \) and 
\( M_3(t) = \left(1 - \frac{3t}{\alpha_3}\right)^{-1} \)

Then \( M^*(t) = w_1 M_1(t) + w_2 M_2(t) + w_3 M_3(t) \), where

\[
\begin{align*}
w_1 &= \frac{6 \beta_1 (\beta_1 + \beta_2)}{(\beta_2 + \beta_3)(3 \beta_2 + 2 \beta_3)} - \frac{3 \beta_2}{(3 \beta_2 + 2 \beta_3)} \\
w_2 &= -\frac{24 \beta_1 (\beta_1 + \beta_2) \alpha_1}{(\beta_2 + \beta_3)(3 \beta_2 + 2 \beta_3) \alpha_2} - \frac{6 \beta_1 (\beta_2 + \beta_3)}{(3 \beta_2 + \beta_3) \alpha_2} \quad \text{and} \\
w_3 &= \frac{54 \beta_1 (\beta_1 + \beta_2) \alpha_1}{(3 \beta_2 + 2 \beta_3)(3 \beta_2 + \beta_3) \alpha_3} + \frac{9 \beta_1 (\beta_2 + \beta_3)}{(3 \beta_2 + \beta_3) \alpha_3} + \frac{9 \beta_2 \alpha_1}{(3 \beta_2 + 2 \beta_3) \alpha_3} + \frac{\beta_3}{\alpha_3}.
\end{align*}
\]

It can be verified that \( w_1 + w_2 + w_3 = 1 \).
Therefore, the reliability function

\[
R(t) = \sum_{i=1}^{3} w_i \exp \left\{ -\left( \frac{\alpha_i}{i} \right) t \right\}, \quad t > 0,
\]

\[
= \left[ \frac{6 \beta_i(\beta_1 + \beta_2)\alpha_i}{(\beta_2 + \beta_3)(3\beta_2 + 2\beta_3)} - \frac{24 \beta_i(\beta_1 + \beta_2)\alpha_i}{(\beta_2 + \beta_3)(3\beta_2 + 3\beta_3)} \right] \exp \left\{ -\left( \frac{\beta_1 + 2\beta_2 + \beta_3}{1} \right) t \right\} \\
+ \left[ \frac{54 \beta_i(\beta_1 + \beta_2)\alpha_i}{(3\beta_2 + 2\beta_3)(3\beta_2 + \beta_3)} - \frac{6 \beta_i(\beta_2 + \beta_3)}{(3\beta_2 + \beta_3)} \right] \exp \left\{ -\left( \frac{2\beta_1 + 3\beta_2 + \beta_3}{2} \right) t \right\} \\
+ \left[ \frac{9 \beta_i(\beta_2 + \beta_3)}{(3\beta_2 + \beta_3)} - \frac{3\beta_2\alpha_1}{(3\beta_2 + 2\beta_3)} + \frac{9 \beta_2\alpha_1}{(3\beta_2 + 3\beta_3)} + \beta_3 \right] \exp \left\{ -\left( \frac{3\beta_1 + 3\beta_2 + \beta_3}{3} \right) t \right\}
\]

and

\[
MTBF = \sum_{i=1}^{3} \frac{i w_i}{\alpha_i}
\]

\[
= \left[ \frac{6 \beta_i(\beta_1 + \beta_2)\alpha_i}{(\beta_2 + \beta_3)(3\beta_2 + 2\beta_3)} - \frac{24 \beta_i(\beta_1 + \beta_2)\alpha_i}{(\beta_2 + \beta_3)(3\beta_2 + 3\beta_3)} \right] \frac{1}{\beta_1 + 2\beta_2 + \beta_3} \\
+ \left[ \frac{54 \beta_i(\beta_1 + \beta_2)\alpha_i}{(3\beta_2 + 2\beta_3)(3\beta_2 + \beta_3)} - \frac{6 \beta_i(\beta_2 + \beta_3)}{(3\beta_2 + \beta_3)} \right] \frac{1}{2\beta_1 + 3\beta_2 + \beta_3} \\
+ \left[ \frac{9 \beta_i(\beta_2 + \beta_3)}{(3\beta_2 + \beta_3)} - \frac{3\beta_2\alpha_1}{(3\beta_2 + 2\beta_3)} + \frac{9 \beta_2\alpha_1}{(3\beta_2 + 3\beta_3)} + \beta_3 \right] \frac{1}{3\beta_1 + 3\beta_2 + \beta_3}
\]

If the component failure times have a location-scale MOTVE distribution, then the
reliability function
\[ R(t) = \sum_{i=1}^{3} w_i \exp \left\{ -\left( \frac{\alpha_i}{\tau} \right) (t - \xi) \right\}, \quad t > \xi, \]

\[
= \left[ \frac{6 \beta_1 (\beta_1 + \beta_2) \alpha_i}{(\beta_2 + \beta_3)(3 \beta_2 + 2 \beta_3)} - \frac{24 \beta_1 (\beta_1 + \beta_2) \alpha_i}{(\beta_2 + \beta_3)(3 \beta_2 + \beta_3)} \right] \exp \left\{ -\left( \frac{2 \beta_1 + 3 \beta_2 + \beta_3}{2 \tau} \right) (t - \xi) \right\}
\]

\[
+ \left[ \frac{54 \beta_1 (\beta_1 + \beta_2) \alpha_i}{(3 \beta_2 + 2 \beta_3)(3 \beta_2 + \beta_3)} - \frac{6 \beta_1 (\beta_2 + \beta_3)}{(3 \beta_2 + \beta_3)} \right] \exp \left\{ -\left( \frac{2 \beta_1 + 3 \beta_2 + \beta_3}{3 \tau} \right) (t - \xi) \right\}
\]

\[
+ \left[ \frac{9 \beta_1 (\beta_2 + \beta_3)}{(3 \beta_2 + \beta_3)} - \frac{3 \beta_2 \alpha_i}{(3 \beta_2 + 2 \beta_3)} + \frac{9 \beta_2 \alpha_i}{(3 \beta_2 + 2 \beta_3)} + \beta_3 \right] \exp \left\{ -\left( \frac{3 \beta_1 + 3 \beta_2 + \beta_3}{3 \tau} \right) (t - \xi) \right\}
\]

and

\[
MTBF = \sum_{i=1}^{3} \frac{i \tau w_i}{\alpha_i} + 3 \xi
\]

\[
= \left[ \frac{6 \beta_1 (\beta_1 + \beta_2) \alpha_i}{(\beta_2 + \beta_3)(3 \beta_2 + 2 \beta_3)} - \frac{24 \beta_1 (\beta_1 + \beta_2) \alpha_i}{(\beta_2 + \beta_3)(3 \beta_2 + \beta_3)} \right] \frac{\tau}{\beta_1 + 2 \beta_2 + \beta_3}
\]

\[
+ \left[ \frac{54 \beta_1 (\beta_1 + \beta_2) \alpha_i}{(3 \beta_2 + 2 \beta_3)(3 \beta_2 + \beta_3)} - \frac{6 \beta_1 (\beta_2 + \beta_3)}{(3 \beta_2 + \beta_3)} \right] \frac{\tau}{2 \beta_1 + 3 \beta_2 + \beta_3}
\]

\[
+ \left[ \frac{9 \beta_1 (\beta_2 + \beta_3)}{(3 \beta_2 + \beta_3)} - \frac{3 \beta_2 \alpha_i}{(3 \beta_2 + 2 \beta_3)} + \frac{9 \beta_2 \alpha_i}{(3 \beta_2 + 2 \beta_3)} + \beta_3 \right] \frac{\tau}{3 \beta_1 + 3 \beta_2 + \beta_3} + 3 \xi
\]

### 2.2 Parallel system

Consider a three unit parallel system with component failure times \( X_1, X_2, X_3 \) respectively. Then the system failure time is \( T = \max_{1 \leq i \leq 3} X_i \).

The distribution function of \( T \) at \( x \) is

\[
G(x) = 1 - F(0,0) - F(0,x,0) - F(0,0,x) + F(x,0,0) + F(x,0,x) + F(0,x,x)
\]

Therefore the reliability function of the system is,
\[ R(t) = 1 - G(x) \]
\[ = F(x,0,0) + F(0,x,0) + F(0,0,x) - F(x,x,0) - F(x,0,x) \]
\[ - F(0,x,x) + F(x,x,x) \]
\[ = \exp(-\lambda_1't) + \exp(-\lambda_2't) + \exp(-\lambda_3't) - \exp\{-\lambda_1't\} - \exp\{-\lambda_2't\} \]
\[ - \exp\{-\lambda_3't\} + \exp(-\lambda t), \ t > 0 \]

Thus,

\[ R(t) = \sum_{i=1}^{3} \exp(-\lambda_i't) - \sum_{i=1}^{3} \exp\{-\lambda_i't\} + \exp(-\lambda t), \ t > 0. \]

The MTBF is given by

\[ MTBF = \frac{3}{\lambda_1'} - \frac{3}{\lambda_2'} + \frac{1}{\lambda}. \]

When the component failure times are identical, then

\[ R(t) = 3\exp(-\alpha_1't) - 3\exp\{-\alpha_2't\} + \exp(-\alpha_3't), \ t > 0 \]

and

\[ MTBF = \frac{3}{\alpha_1} - \frac{3}{\alpha_2} + \frac{1}{\alpha_3}. \]

If the component failure times have a location-scale MOTVE distribution, then the reliability function,

\[ R(t) = \sum_{i=1}^{3} \exp\left(-\frac{\lambda_i'}{\tau}(t-\xi)\right) - \sum_{i=1}^{3} \exp\left(-\frac{\lambda_i}{\tau}(t-\xi)\right) + \exp\left(-\frac{\tau}{\xi}(t-\xi)\right), \ t > \xi. \]

and

\[ MTBF = \left\{ \frac{3}{\lambda_1'} - \frac{3}{\lambda_2'} + \frac{1}{\lambda} \right\}^{\tau + \xi}. \]
2.3 Series system

Consider a three unit series system with component failure times $X_1, X_2, X_3$ respectively. Then the system failure time is $T = \min_{1 \leq i \leq 3} X_i$.

The reliability function is $R(t) = \overline{F}(t,t,t)$

$$= \exp\{-\lambda t\}, \ t > 0.$$ 

$$MTBF = \frac{1}{\lambda}.$$ 

If the components failure times are identically distributed, then

$$MTBF = \frac{1}{\alpha_3}.$$ 

If the component failure times have a location-scale MOTVE distribution, then the reliability function,

$$R(t) = \exp\left\{-\frac{\lambda}{\tau} (t-\xi)\right\}, \ t > \xi.$$ 

$$MTBF = \frac{\tau}{\lambda} + \xi.$$ 

2.4 Relay system

A three component relay system operates if and only if component 1 and at least one of the remaining two components operate. The system failure time is $T = X_1 \land (X_2 \lor X_3)$, (Barlow and Proschan, 1975). It seems natural to assume that $X_2$ and $X_3$ are identically distributed. Thus we assume that
\[ F(x_1, x_2, x_3) = \exp \left\{ -\lambda_1 x_1 - \lambda_2 (x_2 + x_3) - \lambda_{12} (x_1 \lor x_2 + x_1 \lor x_3) - \lambda_{23} (x_2 \lor x_3) \right\} \]

Note that \((X_1, X_2)\) and \((X_1, X_3)\) are identically distributed.

The reliability function is

\[ R(t) = F(t, t, 0) + F(t, 0, t) - F(t, t, t). \]

Here

\[ F(t, t, 0) = \exp \left\{ -\lambda_1 t - \lambda_2 (t) - \lambda_{12} (t + t) - \lambda_{23} (t) - \lambda_{123} (t) \right\} \]

\[ F(t, 0, t) = \exp \left\{ -\lambda_1 t - \lambda_2 (t) - \lambda_{12} (t + t) - \lambda_{23} (t) - \lambda_{123} (t) \right\} \]

\[ F(t, t, t) = \exp \left\{ -\lambda_1 t - \lambda_2 (t + t) - \lambda_{12} (t + t) - \lambda_{23} (t) - \lambda_{123} (t) \right\} \]

Therefore,

\[ R(t) = 2 \exp \left\{ -\left( \lambda_1 + 2\lambda_2 + \lambda_{12} + \lambda_{23} + \lambda_{123} \right) t \right\} - \exp \left\{ -\left( \lambda_1 + 2\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123} \right) t \right\}, \quad t > 0. \]

The MTBF is

\[ MTBF = \frac{2}{\left( \lambda_1 + \lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123} \right)} - \frac{1}{\left( \lambda_1 + 2\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123} \right)} = \frac{\left( \lambda_1 + 3\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123} \right)}{\left( \lambda_1 + \lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123} \right)} \left( \lambda_1 + 2\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123} \right) \]

If the component failure times have a location-scale MOTVE distribution, then the reliability function,
\begin{align*}
R(t) &= 2 \exp\left\{- \frac{(\lambda_1 + \lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})}{\tau} (t - \xi) \right\} \\
&\quad - \exp\left\{- \frac{(\lambda_1 + 2\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})}{\tau} (t - \xi) \right\}, \quad t > \xi
\end{align*}

and,

\begin{align*}
MTBF &= \frac{2\tau}{(\lambda_1 + \lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})} - \frac{\tau}{(\lambda_1 + 2\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})} + \xi \\
&= \frac{(\lambda_1 + 3\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})\tau}{(\lambda_1 + \lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})(\lambda_1 + 2\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})} + \xi.
\end{align*}

If the component failure times are identically distributed, then

\begin{align*}
R(t) &= 2 \exp\left\{- \frac{(\alpha_2)}{\tau} (t - \xi) \right\} \\
&\quad - \exp\left\{- \frac{(\alpha_3)}{\tau} (t - \xi) \right\}, \quad t > \xi
\end{align*}

and,

\begin{align*}
MTBF &= \frac{2\tau}{\alpha_2} - \frac{\tau}{\alpha_3} + \xi \\
&= \frac{(2\alpha_3 - \alpha_2)\tau}{\alpha_2 \alpha_3} + \xi.
\end{align*}