Chapter 3

Portfolio Optimization

The investors face broad classes of assets in the universe of investment. The decision process seems overwhelming by considering the number of possible assets and the various possible proportions in which each can be held. The questions of how to optimize the returns by investing in appropriate asset classes and constructing optimal portfolios and how to manage the optimal portfolios by selecting a proper portfolio management strategy have always attracted the attention of the investors. The appropriate solutions for investment decisions pave the way for perfect achievements. To construct the optimal portfolios, the first step is defining an opportunity set and delineating the efficient frontier and the next step is selection of the optimal portfolio according to investor’s attitude toward risk. In the management of optimal portfolios, it is necessary to analyse alternative portfolio management strategies and find the most profitable investment strategy. Rational choice among the alternatives helps to achieve optimal results.

3.1. Financial Securities

Securities\(^1\) are legal contracts “representing the right to receive future benefits under a stated set of conditions.”\(^2\) There are a large number of financial securities which can be selected for the purpose of investment. The investment can be done directly or indirectly. In direct investment, the investors can choose to purchase directly any one of a number of different securities, many of which represent a type of claim on a private or government entity. As an alternative, the investors can invest in intermediaries (e.g. mutual funds) which bundle a set of direct investments and then sell shares in the portfolios of financial instruments they hold (Elton et al., 2010, p. 28). The financial securities which can be traded directly can be classified by the time horizon of investment into money market instruments, capital market instruments and derivatives instruments. In addition, capital market instruments can be categorized into debt and equity instruments. The classification of financial securities according to the above scheme is shown in the following diagram:

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1 Throughout the study the terms financial assets, financial instruments and securities shall be used interchangeably.
2 Source: Elton et al. (2010), “Modern Portfolio Theory and Investment Analysis”, p. 28
3.1.1. Direct Investing

In direct investing, the investors buy securities directly as individuals and decide which securities to hold in their portfolios. They can select any one of the number of different securities according to their investment viewpoints. In direct investment, the final decision of selection of securities is taken by the investors, although the investors can consult the financial advisors over the investment process, ultimately, they can substitute the advisors’ offers.

3.1.1.1. Money Market Securities

Money market securities are short term debt instruments with maturity ranging from overnight to one year issued by the governments, financial institutions and corporations. They generally have a relatively high degree of liquidity. Trading in the money markets is done over the counter and is wholesale. Individual investors cannot invest in money market as the value of the investments is large. Many individuals interested to obtain these instruments do so by holding a mutual fund (money market fund). The Indian money market offers different instruments such as treasury bills, money at call and short notice, commercial bills, repurchase agreements (repos), commercial papers, certificate of deposits, money market mutual funds (MMMF), banker’s acceptances.

3.1.1.2. Capital Market Securities

Capital market securities include financial instruments with maturities greater than one year and those with no designated maturities at all. These instruments are basically divided into two types according to whether they contain a promised set of cash flows over time, or offer participation in the future profitability of a company. These are referred to as fixed income and equity instruments, respectively (Elton et al., 2010, p. 31).
Fixed income securities are financial instruments offering a specified payment schedule. They create obligation to pay specific amount at specific times. The term fixed refers to both the schedule of obligatory payments and the amount. The fixed income securities include government bonds and corporate bonds offering different promised returns which are due to differences in the maturity, the creditworthiness of the issuer and the taxable status. On the other hand, equity securities are financial instruments creating no obligation to pay dividends or any other form of income. Equity securities, often referred to as stocks and shares, represent ownership claims on the earnings and assets of corporations. There are two main types of stocks; common and preferred stocks. Common stocks usually entitle the owners to certain voting rights regarding company matters. On the other hand, preferred stocks do not have voting rights, but have higher claims on assets and earnings than the common stocks. The owners of preferred stocks always receive dividends before common stockholders and have priority in the event of the company becoming bankrupt.

3.1.1.3. Derivative Securities

Derivative securities are basically financial instruments whose values are derived from the values of underlying assets. There are four main types of derivative instruments, namely, forwards, futures, options and swaps. Forwards are customized contracts between two parties, where payments take place at specific times in the future at pre-determined prices. Futures are standardized contracts to buy or sell assets on or before pre-defined future dates. There are two types of options, namely, call options and put options which are contracts that give the holders the rights, but not the obligations, to buy (call options) or sell (put options) the assets. Swaps are contracts between two parties to exchange cash flows on or before pre-defined future dates based on the underlying value of currencies exchange rates, bonds/interest rates or other market variables.

3.1.2. Indirect Investing

The investors can choose to invest indirectly by purchasing the shares/units of investment companies (mutual funds). The mutual funds pool money from many investors to purchase securities in order to produce capital gain and income for the investors of fund. Mutual funds hold portfolios of securities which are selected and managed by professional investment team on behalf of many investors. Mutual funds issue units to the investors in accordance with amount of money invested by them. The investors of mutual funds are known as unitholders. Mutual funds are an affordable and convenient way for investors to gain access to investments that would otherwise only be available to large institutions or wealthy individuals. In other words, investors have access to diversified portfolios of securities which
would otherwise be quite difficult to create with a small amount of capital. Mutual funds offer liquidity, diversification and professional money management. Mutual funds can be categorized into open-ended and close-ended schemes. Open-ended schemes offer units for sale without specifying any duration for redemption. It means, these schemes do not have a fixed maturity period. The investors can buy and sell units through the mutual funds. Close-ended schemes have specified maturity periods. The funds are open for subscription only during specified periods that is at the time of launch of the schemes. Investors can invest in the schemes at the time of the initial public issue and thereafter they can buy or sell the units of the schemes on the stock exchanges where the units are listed.

The mutual funds normally come out with a number of schemes with different investment objectives. Mutual funds can be classified as growth schemes/equity oriented schemes, income schemes/debt oriented schemes or balanced schemes according to their investment objectives. Such schemes may be open-ended or close-ended schemes. The performance of a particular scheme of a mutual fund is denoted by net asset value (NAV) which is the market value of the securities held by the scheme and declared on the daily basis. The NAV per unit is the market value of securities of a scheme divided by the total number of units of the scheme on any particular date. Some costs may be incurred in connection with particular investor transactions, such as investor purchases, exchanges and redemptions. There are also regular fund operating costs that are not necessarily associated with any particular investor transaction, such as investment advisory fees, marketing and distribution expenses, brokerage fees. These expenses must be taken into consideration for investment as they affect the yields/returns of the investors. However, the performance track records and service standards of the mutual funds also must be considered which are more important. Efficient funds may give higher returns in spite of expenses.  

3.2. The Risk and Return Characteristics of Financial Securities

Financial securities are commonly judged based on their risks and returns. Return is measured by the sum of the change in the market price of a security plus any income received over a holding period divided by the price of a security at the beginning of the holding period. Total return can be formulated as follows:

\[
R_i = \left( \frac{P_{H} - P_{H-1}}{P_{H-1}} \right) + 1
\]

(3.1)

Where, \((P_t - P_{t-1})\) is price change over the holding period

\(I\) is income received during the period

\(P_{t-1}\) is purchase price of a security

The variability in returns of a security represents the risk of that security. The existence of risk means that the investor can no longer associate a single number or payoff with investment in any asset. According to Markowitz (1952), dispersion of returns of security can be considered as a measure of security risk. He quantified the risk and defined the variance of the returns as a measure of risk which is the average squared deviation of actual returns from its expected value. Total risk of security can be formulated as follows:

\[
\sigma_i^2 = \left(\frac{1}{n}\right) \times \sum_{j=1}^{n} (R_{ij} - E(R_i))^2
\]  

(3.2)

Where, \(R_{ij}\) is actual return on security \(i\)

\(n\) is number of observations

\(E(R_i)\) is expected return on security \(i\) which is calculated as follows:

\[
E(R_i) = \left(\frac{1}{n}\right) \times \sum_{j=1}^{n} R_{ij}
\]  

(3.3)

“Several factors that could affect risk of financial securities are as follows:

- The maturity of an instrument (in general, the longer the maturity the more risky it is).
- The risk characteristic and creditworthiness of the issuer or guarantor of the investment.
- The nature and priority of the claims the investment has on income and assets.
- The liquidity of the instrument and the type of market in which it is traded.

If risk is related to these elements, then measures of risk such as the variability of returns should be related to these same factors.”

The process of investment either direct or indirect initiates with the construction of optimal portfolios. The construction of optimal portfolios plays a crucial role to achieve the pre-defined investment objectives of the investors. The selection of the proper model for constructing the optimal portfolios paves the way for perfect achievement.

**3.3. Modern Portfolio Theory**

Portfolio theory or modern portfolio theory, originally proposed by Markowitz in the 1950s is grounded on diversification concept which aims to reduce the total risk of portfolio without sacrificing portfolio return. Modern portfolio theory looks at portfolios as a whole rather than focusing on the performance of individual securities. It quantifies the risk of a portfolio and

\[\text{Source: Elton et al. (2010), "Modern Portfolio Theory and Investment Analysis", p. 36}\]
assists in the selection of the most efficient portfolio by analyzing different possible portfolios of the given securities and choosing securities that do not move exactly together. Modern portfolio theory is the first formal attempt to develop a methodology for determining the optimal portfolio. Markowitz portfolio theory is also called the mean-variance optimization model due to the fact that it is based on expected return (mean) and variance of the portfolio. Modern portfolio theory is based on the following assumptions:

- All investors aim to maximize profit and minimize risk that is they choose the optimal portfolios on the basis of the expected returns and standard deviations of returns. In other words, the investors wish to maximize expected utility of total wealth.
- All investors are rational and risk averse that is they accept a higher risk if they are compensated with a higher expected return.
- All investors receive the same information at the same time.
- There are no taxes or transaction costs involved in buying or selling assets.
- Investors can buy any security of any size implying that securities are indefinitely divisible.
- Investors can lend or borrow any amount of securities at a risk free rate.
- Investors are price takers and their actions do not influence prices.
- The correlations between assets are always fixed and constant.
- Returns on assets are normally distributed.
- Investors have an accurate conception of possible returns that is the probability beliefs of investors match the true distribution of returns. In other words, investors have exact idea of potential returns.
- All investors have the same single period investment horizon.
- The investor’s utility is a quadratic function of portfolio return with decreasing marginal utility.

### 3.3.1. Portfolio Return and Risk

According to Markowitz (1952), the expected return on a portfolio is a weighted average of the expected returns on the individual securities constituting the portfolio. The weight applied to each return is the fraction of the portfolio invested in that security. Portfolio expected return is formulated as follows:

$$E(R_p) = \sum_{i=1}^{n} w_i E(R_i)$$  \hspace{1cm} (3.4)

Where, \(w_i\) is weight of security \(i\) in the portfolio.
E(R)_i\) is expected return on security \(i\)

\(n\) is total number of securities in the portfolio

Markowitz (1952) argued that although portfolio expected return is a weighted average of the returns on the individual securities in the portfolio, portfolio risk is not the weighted average of the risks of the individual securities in the portfolio. He formulated portfolio risk by the statistical notion of covariances between individual security returns. According to him, portfolio variance is formulated as follows:

\[
\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \quad (3.5)
\]

Where, \(w_i\) and \(w_j\) are weights of securities \(i\) and \(j\) in the portfolio

\(n\) is total number of securities in the portfolio

\(\sigma_{ij}\) is covariance of security \(i\) with security \(j\) which is calculated as follows:

\[
\sigma_{ij} = \left( \frac{1}{n} \right) \sum_{k=1}^{n} (R_{ik} - E(R_i))(R_{jk} - E(R_j)) \quad (3.6)
\]

Where, \(E(R_i)\) and \(E(R_j)\) are expected returns on securities \(i\) and \(j\), respectively.

\(\sigma_{ij}\) may be expressed in terms of the correlation coefficient (\(\rho_{ij}\)) that is:

\[
\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \quad (3.7)
\]

Therefore, portfolio variance can be written as follows:

\[
\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j \quad (3.8)
\]

The variance on a portfolio can be written in the other form as follows:

\[
\sigma_p^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \quad (3.9)
\]

### 3.3.2. Diversification

The total risk of an individual security consists of two risks which are systematic risk and unsystematic risk. Systematic risk comprises factors that are external to companies (macro in nature), uncontrollable and affect a large number of securities. Systematic risk can be further subdivided into market risk, interest rate risk and purchasing power risk. Unsystematic risk includes factors which are internal to companies (micro in nature) and affect only those particular companies. These are controllable to a large degree. Unsystematic risk is subdivided into business risk and financial risk. Unsystematic risk can be reduced through diversification. As the number of securities which are not perfectly positively correlated increases, the portfolio risk decreases.
Figure 3.1 The effect of number of securities on risk of the portfolio

According to Markowitz (1952), “the adequacy of diversification is not thought by investors to depend solely on the number of different securities held.” In order to reduce the portfolio variance it is necessary to consider the correlation coefficient among securities in the portfolio. The correlation coefficients may range from -1 to +1. A value of -1 indicates perfect negative correlation between returns of two securities implying perfect comovement in the opposite directions, while a value of +1 indicates perfect positive correlation between returns of two securities implying perfect comovement in the same direction. A value of zero indicates no correlation implying the movements of returns are independent of each other.
Figure 3.2 Relationship between expected return and standard deviation for various correlation coefficients

The various cases where the correlation between two securities ranges from -1 to +1 are shown in Figure 3.2. Points R and S represent pure holdings (100%) of securities R and S. The intermediate points along the line segment RS represent portfolios containing various combinations of the two securities. The line segment identified as $\rho = +1$ is a straight line which shows the inability of a portfolio of perfectly positively correlated securities to serve as a means to reduce risk. The segments labelled $\rho = 0$ and -0.5 are hyperbolas. The lower the correlation, the greater the potential benefit from diversification. In the extreme case of perfect negative correlation, there is a perfect hedging opportunity and the segment labelled $\rho = -1$ shows that portfolio risk can be reduced to zero (Fischer and Jordan, 2009, p. 579). A graphical representation of the risk and return on these portfolios shows that except in portfolio opportunity set for $\rho = +1$, there are some portfolios whose risks are less than the risk of each security in the portfolio taken individually.
3.3.3. The Efficient Frontier with No Risk-free Asset

Investors face with an infinite number of possible combinations of risky assets. On the other hand, the investors are rational as they would like to have higher return and they are risk averse as they would prefer to have lower risk. Therefore, by finding a set of portfolios that offers a higher return for the same risk or a lower risk for the same return, all portfolios that investors could consider holding would be identified. This set of portfolios is called the efficient frontier.

![Efficient Frontier Diagram](image)

Figure 3.3 The efficient frontier with no risk-free asset

Figure 3.3 represents the portfolio possibilities which are plotted in risk-return space. To find efficient portfolios offering less risk for the same return or more return for the same risk, it is useful to move as far as possible in the direction of increasing return and as far as possible in the direction of decreasing risk. Over this process, all portfolios which are dominated by efficient portfolios can be eliminated. According to definition of efficient portfolio, point R cannot be eliminated since there is no portfolio that has less risk for the same return or more return for the same risk. Point R is the global minimum variance portfolio. The global minimum variance portfolio is portfolio that has the lowest risk of any feasible portfolio.
(Elton et al., 2010, p. 80). Portfolio S cannot be eliminated since there is no portfolio that has the same return and less risk or the same risk and more return than portfolio S. Portfolio S (usually holding an individual security) offers the highest expected return of all portfolios. As Figure 3.3 illustrates the efficient frontier is a concave function in expected return standard deviation space that shows all efficient portfolios and extends from the minimum variance portfolio to the maximum return portfolio (Elton et al., 2010, p. 82).

The set of efficient portfolios can be determined using quadratic programming approach that technically manipulates the portfolio weights. The process is as follows: A desired expected return is specified. Then all portfolios that produce this expected return are considered and the portfolio that has the smallest variance of return is chosen as the efficient portfolio. This is continued for other levels of portfolio return until all the possible expected returns are considered (Chandra, 2005, p. 256). Mathematically, by minimizing the portfolio risk for a specified portfolio expected return one point on the efficient frontier is determined. The problem can be formulated as follows:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \\
\text{Subject to} & \quad \sum_{i=1}^{n} w_i E(R_i) = \mu_p \\
& \quad \sum_{i=1}^{n} w_i = 1 \\
& \quad w_i \geq 0 \quad i = 1, 2, \ldots, n
\end{align*}
\]

(3.10)

Where, \( w_i \) and \( w_j \) are weights of securities \( i \) and \( j \) in the portfolio.

\( \sigma_{ij} \) is covariance of security \( i \) with security \( j \).

\( \mu_p \) is specified portfolio expected return.

\( n \) is total number of securities in the portfolio.

By varying \( \mu_p \) between the return on the minimum variance portfolio and the return on the maximum return portfolio, the efficient frontier can be traced out (Elton et al., 2010, p. 106).

3.3.4. The Efficient Frontier with Risk-free Lending and Borrowing

The introduction of risk-free asset into risky portfolio possibility set changes the efficient frontier considerably. In the absence of risk-free asset the efficient frontier is concave curve, whereas, the introduction of risk-free borrowing and lending options transforms the efficient frontier into a straight line. Lending at risk-free rate can be considered as investing in risk-free security (e.g., treasury bills). Likewise borrowing can be considered as selling such a security short, thus borrowing can take place at the risk-free rate. The combinations of risky portfolios with risk-free asset result in a series of straight line with the same origin but

\(^5\) Short sales not allowed.
potentially different slopes. Each of these lines called capital allocation line (CAL). Among all CALs, one which is tangent to the efficient frontier of risky portfolios provides the investor the best possible opportunity set. The formula for capital asset line can be shown to be:

\[
E(R_c) = R_f + \left[ \frac{E(R_p) - R_f}{\sigma_p} \right] \sigma_c
\]

where, \(E(R_c)\) is expected return on the combination of risk free asset and risky portfolio, \(R_f\) is risk-free rate, \(E(R_p)\) is expected return of risky portfolio, \(\sigma_p\) is standard deviation of risky portfolio and \(\sigma_c\) is standard deviation of a combination of risk free asset and risky portfolio.

The intercept of the capital allocation line is \(R_f\) and the slope is \(\frac{E(R_p) - R_f}{\sigma_p}\).

Figure 3.4 Combinations of the risk-free asset and various risky portfolios

The tangent line has the steepest slope. The slope can be referred to Sharpe ratio, therefore, the tangency portfolio has highest Sharpe ratio.
Figure 3.5 The efficient frontier with risk-free asset

All investors who believe they face the efficient frontier and risk-free lending and borrowing rate would hold the same portfolio of risky assets. This portfolio maximizes the slope of the line connecting the risk-free asset and a risky portfolio. Thus, identification of this portfolio constitutes a solution to find the efficient frontier. The problem can be formulated as follows:

Maximize \( \frac{E(R_P) - R_f}{\sigma_P} \)  

Subject to \( \sum_{i=1}^{n} w_i = 1 \quad i = 1, 2, \ldots, n \)

Where, \( E(R_P) \) is expected return of risky portfolio.
\( R_f \) is risk free rate.
\( \sigma_p \) is standard deviation of risky portfolio.
\( w_i \) is weight of security i.
\( n \) is total number of securities in the portfolio.

This maximization problem is quadratic programming problem. As Figure 3.5 shows, a portfolio with risk-free asset enables an investor to achieve a higher level of satisfaction. For every point on the curve, there is at least one point on the straight line which is superior to the point on the curve. Since straight line offers higher expected return for the same standard deviation or lower standard deviation for the same return, every investor would do well to choose some combination of risk free asset and tangency portfolio. A conservative investor
may choose a point on the segment to the left of tangency portfolio, whereas an aggressive investor may choose a point on the segment to the right of tangency portfolio. All portfolio combinations to the left of tangency portfolio show combinations of risky and risk-free assets, and all those to the right of tangency portfolio represent purchases of risky assets made with funds borrowed at the risk-free rate that is in these portfolio combinations, the weights assigned to risk-free asset are negative and those assigned to risky portfolios are more than one.

3.3.5. Optimal Portfolio

After delineating the efficient frontier, the next step is finding the optimal portfolio. To determine the optimal portfolio on the efficient frontier, the investor’s risk-return preference must be known. Utility functions or indifference curves are normally used to represent the investor’s preferences. By assuming that the investor is risk averse and the utility function is quadratic, indifference curves can be derived in the form of convex curves in expected return standard deviation space. Each investor has a map of indifference curves. Figure 3.6 shows an indifference map for a hypothetical investor.

Figure 3.6 Risk-return indifference curves

All the points lying on a given indifference curve offer the same level of satisfaction, however, they show different combinations of risk and return. For example, point C and D, which lie on the indifference curve I₁, offer the same level of satisfaction; likewise points R
and S, which lie on the indifference curve $I_2$, offer the same level of satisfaction. The level of satisfaction increases as one moves leftward (Chandra, 2005, p. 258).

By finding the risk-return indifference curves and delineating the efficient frontier, the optimal portfolio can be determined. The optimal portfolio is the tangent point of the indifference curve and efficient frontier. This point indicates the highest utility that an investor can achieve. With this point, the investor will attain highest utility as well as the best risk-return combination. The investors with different indifference map might have different optimal portfolios.

Figure 3.7 The optimal portfolio with no risk-free asset

By introducing a risk-free asset into risky portfolio possibility set the process of selection of optimal portfolio can be divided into two steps: the first step is identification of tangency portfolio irrespective of the investor’s risk preference and the next step is selection of the optimal portfolio according to investor’s risk preference.
Mathematically, by maximizing the expected utility function, the optimal portfolio can be determined. The portfolio problem is expressed as a choice between expected return and standard deviation. Therefore any utility function can alternately be expressed the same way. The utility function can be maximized by maximizing the portfolio expected return and minimizing the portfolio risk at a specified risk aversion coefficient. The problem can be formulated as follows:

Maximize $E(R_p) - \frac{1}{2} A \sigma_p^2$  
Subject to $\sum_{i=1}^{n} w_i = 1$  
$w_i \geq 0 \quad i = 1, 2, \ldots, n$  

Where, the objective function is utility function, $E(R_p)$ and $\sigma_p^2$ are expected return and variance of the portfolio, respectively. $w_i$ is weight of security $i$ and $n$ is total number of securities in the portfolio. “$A$” is risk aversion coefficient that expresses the investor’s trade-off between expected return and variance of return. The higher “$A$” represents more risk.

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6. This constraint will be removed, if there is risk-free borrowing.
aversion or less risk tolerance. In other word, the higher “A”, the more averse the investor is of risk. This maximization problem is quadratic programming problem.

Many research studies on portfolio optimization have concentrated on methods for implementing the portfolio theory. The Markowitz model has been criticised because its application requires computation of covariance of each security with the other securities. In addition, the construction of efficient frontier needs lengthy calculations which may not add value for investors. In the Markowitz model the inputs required for portfolio analysis for N securities involve N expected returns, N variances of returns and (N^2 – N)/2 covariance terms or correlation coefficients as measure of interrelationship between the returns on assets considered; therefore, in total, N(N+3)/2 pieces of information are required. Theoretically, the Markowitz model is considered as a superior approach in constructing the optimal portfolio. However, it has hardly become an operational tool for portfolio managers and investors, since this model requires a large number of inputs and involves the computational complexity. Several algorithms have been developed to produce solutions based on the mean–variance model. In addition, various research studies have been carried out to simplify Markowitz’s assumptions in an attempt to make the model operational. The Sharpe single index model is one of the prominent results of these simplifications requiring fewer inputs and computational simplicity.

3.4. The Single Index Model

The single index model proposed by Sharpe (1963) estimates the correlation structure between securities by assuming that the only reason securities move together is a common response to market movements. The single index model aims to reduce the number of inputs by abandoning the calculation of covariance of each security with the other securities and substituting information on the relationship of each security to the market index. It is based on calculation of expected covariance for each security relative to the market index instead of calculation of expected covariance for each pair of securities.

According to the Sharpe single index model, the return of each security is a linear function of return of the market index as follows:

\[
R_i = \alpha_i + \beta_i R_m + \epsilon_i
\]

\[(3.14)\]

\(^7\) Bodie et al. (2008, p. 173) opine “the factor of \( \frac{1}{2} \) is a scaling convention for simplifying calculations. It has no economic significance, and it could be eliminated simply by defining a “new” \( A \) with half the value of the A used here.”
Where, \( R_i \) is return on security \( i \), \( R_m \) is return on the market index, \( \alpha_i \) is constant return, \( \beta_i \) is measure of the sensitivity of the security \( i \)’s return to the market index return, \( e_i \) is error term. The estimates of \( \alpha_i \), \( \beta_i \) and \( \sigma_{ei}^2 \) can be obtained by regressing the security \( i \)’s return on the corresponding market index’s return.

The single index model is based on the following assumptions:

- The error term \( (e_i) \) has an expected value of zero and a constant variance.
- The error term is not correlated with the return on the market index. This implies that market index is unrelated to unique return:
  \[
  \text{Cov} (e_i, R_m) = 0
  \]
- The error term for security \( i \) is not correlated with the error term for any other security. This implies that securities are related only through common response to the return on the market index:
  \[
  \text{Cov} (e_i, e_j) = 0
  \]

By using the single index model, the inputs required for portfolio problem, namely, the expected return on each security, the variance of return on each security, and the covariance of returns between each pair of securities can be formulated as follows:

- The mean return: \( E(R_i) = \alpha_i + \beta_i E(R_m) \)  \hspace{1cm} (3.15)
- The variance of a security’s return: \( \sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2 \)  \hspace{1cm} (3.16)
- The covariance of returns between securities \( i \) and \( j \): \( \sigma_{ij} = \beta_i \beta_j \sigma_m^2 \)  \hspace{1cm} (3.17)

The expected return has two components: a unique part \( \alpha_i \) and a market-related part \( \beta_i E(R_m) \). Likewise, a security’s variance has the same two parts; unique risk \( \sigma_{ei}^2 \) and market-related risk \( \beta_i^2 \sigma_m^2 \). In contrast, the covariance depends only on market risk representing the joint movement of securities (Elton et al., 2010, p. 134).

If the single index model holds, by substituting the above inputs in equations (3.4) and (3.9), the portfolio expected return and variance can be formulated as follows:

\[
E(R_p) = \sum_{i=1}^{n} w_i \alpha_i + \sum_{i=1}^{n} w_i \beta_i E(R_m)
\]
\[
\sigma_p^2 = \sum_{i=1}^{n} w_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{n} w_i^2 \sigma_{ei}^2
\]  \hspace{1cm} (3.19)

The parameters of the single index model can be estimated using alternative ways. From equations (3.15) and (3.16) it is clear that expected return and risk can be calculated for any portfolio by estimating \( \alpha_i \), \( \beta_i \) and \( \sigma_{ei}^2 \) for each security and estimates of the expected return \( E(R_m) \) and the variance \( \sigma_m^2 \) for the market (Elton et al., 2010, p. 136). The Sharpe single index model is a simplification over the Markowitz model. In portfolio analysis with N
securities, the Sharpe single index model requires $3N+2$ estimates. This model requires $N$ alphas ($\alpha$), $N$ betas ($\beta$), and $N$ variances of the error terms ($\sigma^2$), the expected return on market and the variance of the market return. The single index model can also be employed if analysts supply estimates of expected return for each security, the variance of the return on each security, the Beta ($\beta_i$) for each security and the variance of the market return (Elton et al., 2010, p. 137).

3.5. Portfolio Management Strategies

Selection of an appropriate strategy to manage the optimal portfolio is essential to achieve investment objectives. Basically there are two portfolio management strategies that are applied to manage portfolio effectively in order to yield the highest possible returns at lowest possible risks, namely, active and passive portfolio management strategies. The portfolio manager who applies active strategy attempts to beat the performance of an investment benchmark index or market by exploiting the market inefficiencies. On the other hand, the portfolio manager who adopts passive strategy believes that market is efficient and it is not possible to beat the market returns regularly over time and tries to match that benchmark performance and replicate the return pattern of the benchmark.

An active portfolio strategy is applied by most investment professionals who strive to earn superior risk-adjusted return over the market index. Active strategy involves purchasing securities and continuously monitoring the price movements in order to exploit profitable conditions. Active managers use analytical researches, forecasts and their own judgment and experience in making investment decisions on what securities to buy, hold, and sell. Active managers can be categorized into three groups: market timers, sector selectors and security selectors. Market timers change the beta on the portfolio according to forecasts of how the market will do (Elton et al., 2010, p. 699). They increase the beta of the portfolio if they forecast the market will be bullish and decrease it if they forecast the market will be bearish. Sector selectors increase the weights of sectors when they believe that these sectors will outperform other sectors in the future and decrease the weights of the sectors when they believe that these sectors will underperform other sectors in the future. “Stocks can be divided into different sectors according to different classification as follow:

- Broad industrial classification (e.g., industrial, financial, utilities).
- Major product classification (e.g., consumer goods, industrial goods, services).
- Perceived characteristics (e.g., growth, cyclical, stable, size, yield, quality).
• According to sensitivity to basic economic phenomena (e.g., stocks sensitive to changes in exchange rates, interest rates)."\(^8\)

Security selectors try to identify the undervalued and overvalued securities. They increase the weight for undervalued securities and decrease it for overvalued securities. In other words, they buy undervalued securities and sell overvalued securities.

Passive portfolio strategy is followed by fund managers who believe that in the long-term the investment will be profitable. Passive strategy involves creating a well-diversified portfolio and holding the portfolio with the aim of long-term appreciation. Passive strategy requires good initial research, patience and a well-diversified portfolio. In the case of managing stock portfolios, the simplest case of passive strategy is the index fund that is designed to replicate exactly a well-defined index of common stock. The fund managers attempt to hold all stocks of index, in exactly the same proportions as used by the index. The standard capital asset pricing model proposed by Sharpe, Lintner and Mossin could be considered the theoretical justification for such a fund (Elton et al., 2010, p. 696).

Among different strategies used to manage the optimal portfolios, the buy-and-hold approach may be linked with a passive investment strategy and tactical asset allocation is considered as a comparatively active strategy.

Buy-and-hold is a long term investment strategy and the managers applying this approach believe that in the long run, financial markets offer a good rate of return despite periods of volatility or decline. Then they actively select assets, but once in a position, are not concerned with short-term price movements and technical indicators. On the other hand, tactical asset allocation is a dynamic investment strategy that actively adjusts asset allocations to improve the risk-adjusted returns of passive management investing by exploiting information that affects the markets such as changes in monetary policy, changes to company management and expected inflation. Tactical asset allocation approach attempts to find investment opportunities through monitoring momentum indicators, monetary indicators, sentiment indicators, seasonal indicators and other market indicators in order to take advantage of perceived differences in relative values of the various asset classes. Adherents of tactical asset allocation believe that this approach allows investors to participate in economic conditions more favourable for one asset class than for others.

\(^8\) Source: Elton et al. (2010), "Modern Portfolio Theory and Investment Analysis", p. 700