Chapter 4

Common Fixed Points and Invariant Approximation for a Pair of Commuting Mappings

This chapter deals with the existence of invariant points from a set of best approximation for a pair of commuting mappings. We generalize, unify and extend some of the known results of Brosowski-Meinardus types on invariant approximation for a pair of commuting mappings. The chapter has been divided into two sections. In the first section we discuss the existence of common fixed points and invariant approximation for a pair of commuting mappings in the framework of convex metric spaces. The results of this section are proved in Narang and Chandok (2010 b, Communicated c). In the second section we discuss some results on the existence of common fixed points and invariant approximation for a pair of commuting mappings when the underlying spaces are metric spaces. The results of this section are proved in Chandok and Narang (2009) and Narang and Chandok (2009 b).

4.1 Common fixed points and invariant approximation for commuting mappings in convex metric spaces

Using fixed point theory, Meinardus and Brosowski established some interesting results on invariant approximation in normed linear spaces. Later various researchers obtained generalizations of their results. After Jungck’s generalization of the Banach contraction principle in 1976 to the case of two commuting self mappings, fixed point theory for a pair of commuting mappings have rapidly grown. Al-Thagafi (1996) proved common fixed point theorems, which generalized and unified well known results of Habiniak (1989), Hicks and Humphires (1982), Jungck (1976 a) and Singh (1979 b).
In this section we generalize, unify and extend some of the known results on invariant approximation for a pair of commuting mappings in the framework of convex metric spaces.

Let \((X, d)\) be a convex metric space. \(D\) and \(M\) subsets of \(X\), \(S\) and \(T\) self maps on \(X\). Let \(B_M(p) = \{x \in M : d(p, x) = d(p, M)\}\) be the set of best approximants to \(p\) in \(M\), \(C_M^S(p) = \{x \in M : Sx \in B_M(p)\}\) and \(D_M^S(p) = B_M(p) \cap C_M^S(p)\). The following results can be easily verified:

i) \(B_M(p) = C_M^S(p) = D_M^S(p)\) when \(S\) is the identity map on \(M\).

ii) \(B_M(p) \cap F(S, T) = C_M^S(p) \cap F(S, T) = D_M^S(p) \cap F(S, T)\).

iii) If \(S(B_M(p)) \subset B_M(p)\), then \(B_M(p) \subset C_M^S(p)\), and hence \(D_M^S(p) = B_M(p)\).

iv) If \(S(C_M^S(p)) \subset C_M^S(p)\), then \(S(D_M^S(p)) \subset S(C_M^S(p)) \subset D_M^S(p)\).

Using Theorems 2.2.10, and 2.2.18, we prove some results on invariant approximation for a pair of commuting mappings in convex metric spaces thereby generalizing and extending, some of the results of Al-Thagafi (1996), Beg et al. (1992), Sahab et al. (1988), Vijayaraju and Marudai (2004) and of few others.

**Theorem 4.1.1.** Let \((X, d)\) be a convex metric space with Property (I), \(M\) a non-empty complete subset of \(X\) and \(S, T\) are self mappings of \(X\) with \(p \in F(S, T)\), \(T(\partial M) \subset M\), and \(q \in F(S)\). If \(D = D_M^S(p)\) is \(q\)-starshaped, \(T\) is demicompact, \(S(D) = D\), \(S\) is continuous and affine with respect to \(q\) on \(D\), \(S\) and \(T\) are commuting on \(D\) and \(T\) is \(S\)-nonexpansive on \(D \cup \{p\}\), then \(S\) and \(T\) have a common fixed point in \(B_M(p)\).

**Proof.** Since \(T(\partial M) \subset M\), it follows from Lemma 3.1.9 that \(T(B_M(p)) \subset B_M(p)\). Let \(x \in C_M^S(p)\). Since \(S\) and \(T\) are commuting on \(D\), it follows that \(d(S(T(x)), p) = d(T(S(x)), T(p)) \leq d(S^2(x), S(p)) = d(p, M)\). Therefore \(T(x) \in C_M^S(p)\) and hence \(T(C_M^S(p)) \subset C_M^S(p)\). We have

\[
T(D) = T(B_M(p) \cap C_M^S(p)) \subset T(B_M(p)) \cap T(C_M^S(p)) \subset B_M(p) \cap T(C_M^S(p)) = D = S(D).
\]

Hence by Theorem 2.2.10, \(S\) and \(T\) have a common fixed point in \(B_M(p)\). \(\Box\)

**Corollary 4.1.2.** (Vijayaraju and Marudai (2004)-Theorem 3.3) Let \(X\) be a normed linear space, \(M\) a nonempty complete subset of \(X\) and \(I, T\) are self mappings of \(X\) with \(p \in F(I, T)\), \(T(\partial M) \subset M\), and \(q \in F(I)\). If \(D = D_M^I(p)\) is \(q\)-starshaped, \(T\) is demicompact, \(I(D) = D\), \(I\) is continuous and affine with respect to \(q\) on \(D\), \(I\) and \(T\) are
commuting on $D$ and $T$ is $I$-nonexpansive on $D \cup \{p\}$, then $I$ and $T$ have a common fixed point in $B_M(p)$.

**Theorem 4.1.3.** Let $(X,d)$ be a convex metric space with Property (I). $S,T$ are self mappings of $X$ with $p \in F(S,T)$, $M \subset X$ with $T(\partial M \cap M) \subset M$, and $q \in F(S)$. If $D = D^S_M(p)$ is closed and $q$-starshaped, $\text{cl}(T(D))$ is compact, $S(D) = D$, $S$ is continuous and affine on $D$, $S$ and $T$ are commuting on $D$ and $T$ is $S$-nonexpansive on $D \cup \{p\}$, then $S$ and $T$ have a common fixed point in $B_M(p)$.

**Proof.** Let $x \in D$. Then for all $k \in (0,1)$, $d(p, W(p,x,k)) \leq kd(p,p) + (1-k)d(p,x) < \text{dist}(p,M)$ implies $W(p,x,k) \cap M = \emptyset$ and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, $Tx \in M$. Since $x \in B_M(p)$, we have $d(Tx,p) = d(Tx,Tp) \leq d(Sx,Sp) = d(Sx,p) = \text{dist}(p,M)$ and hence $Tx \in B_M(p)$. Because $S$ and $T$ commute on $D$, $d(STx,p) = d(TSx,p) \leq d(S^2x,Sp) = \text{dist}(p,M)$ as $Sx \in D$ and $Sp = p$. Thus $STx \in B_M(p)$ and so $Tx \in C_M^S(p)$. Therefore $Tx \in D$ and hence $T(D) \subset D = S(D)$. The result now follows from Theorem 2.2.18. \hfill \Box

**Theorem 4.1.4.** Let $(X,d)$ be a convex metric space with Property (I), $M$ a complete subset of $X$ and $S,T$ are self maps on $X$ with $p \in F(S,T)$, $M \subset X$ with $T(\partial M) \subset S(M) \subset M$ and $q \in F(S)$. If $D = D^S_M(p)$ is $q$-starshaped, $T$ is demicompact, $S(M) \cap D \subset S(D) \subset D$, $S$ is continuous and affine with respect to $q$ on $D$, $S$ and $T$ are commuting on $D$ and $T$ is $S$-nonexpansive on $D \cup \{p\}$, then $S$ and $T$ have a common fixed point in $B_M(p)$.

**Proof.** Let $x \in D$, then proceeding as in Theorem 4.1.3, we can show that $Tx \in D$ and $T(D) = T(B_M(p) \cap C_M^S(p)) \subset B_M(p) \subset T(\partial M)$ (see Lemma 3.1.9) \subset $S(M)$. Since $T(D) \subset S(M)$, there exists $z \in M$ such that $T(x) = S(z)$. Therefore $d(Sz,p) = d(Tx,Tp) \leq d(Sx,p) = \text{dist}(p,M)$. Thus $z \in C_M^S(p)$ and hence $T(D) \subset S(C^S_M(p)) \subset B_M(p)$. This shows that $T(D) \subset S(M) \cap D \subset S(D) \subset D$. Hence by Theorem 2.2.10, $T$ and $S$ have a common fixed point in $B_M(p)$. \hfill \Box

**Corollary 4.1.5.** (Vijayaraju and Marudai (2004)-Theorem 3.4) Let $X$ be a normed linear space, $M$ a complete subset of $X$ and $I,T$ are self maps on $X$ with $p \in F(I,T)$, $M \subset X$ with $T(\partial M) \subset I(M) \subset M$ and $q \in F(I)$. If $D = D^I_M(p)$ is $q$-starshaped, $T$ is
demicompact, \( I(M) \cap D \subset I(D) \subset D \), \( I \) is continuous and affine with respect to \( q \) on \( D \), \( I \) and \( T \) are commuting on \( D \) and \( T \) is \( I \)-nonexpansive on \( D \cup \{p\} \), then \( I \) and \( T \) have a common fixed point in \( B_M(p) \).

**Theorem 4.1.6.** Let \( (X, d) \) be a convex metric space with Property (I), \( S \) and \( T \) are self maps on \( X \) with \( p \in F(S, T) \), \( M \subset X \) with \( T(\partial M \cap M) \subset S(M) \subset M \) and \( q \in F(S) \). If \( D = D_M^S(p) \) is closed and \( q \)-starshaped, \( \text{cl} \ (T(D)) \) is compact, \( S(M) \cap D \subset S(D) \subset D \), \( S \) is continuous and affine on \( D \), \( S \) and \( T \) are commuting on \( D \) and \( T \) is \( S \)-nonexpansive on \( D \cup \{p\} \), then \( S \) and \( T \) have a common fixed point in \( B_M(p) \).

**Proof.** Proceeding as in Theorem 4.1.4, we have \( T(D) \subset S(M) \cap D \subset S(D) \subset D \). Theorem 2.2.18 now ensures that \( S \) and \( T \) have a common fixed point in \( B_M(p) \). \( \square \)

**Remarks 4.1.1.** i) Theorems 4.1.3 and 4.1.6 were proved by Al-Thagafi (1996) in normed linear spaces when \( S \) is linear on \( D \).

ii) If \( S(B_M(p)) \subset B_M(p) \), then \( B_M(p) \subset C_M^S(p) \), and hence \( D_M^S(p) = B_M(p) \). If \( S(C_M^S(p)) \subset C_M^S(p) \), then \( S(D_M^S(p)) \subset S(C_M^S(p)) \subset D_M^S(p) \). In view of these inclusions it follows that Theorems 4.1.3 and 4.1.6 hold for \( D = B_M(p) \) and \( D = C_M^S(p) \). Therefore the following result of Sahab et al. (1988) is a special case of Theorem 4.1.3.

**Theorem 4.1.7.** Let \( T, I : X \to X \) be operators, \( C \) a subset of a normed linear space \( X \) such that \( T : \partial C \to C \) and \( p \in F(I, T) \). Further \( T \) and \( I \) satisfy \( ||Tx - Ty|| \leq ||Ix - Iy|| \) for all \( x, y \) in \( D' = B_C(p) \cup \{p\} \) and let \( I \) be linear, continuous on \( D' \), and \( ITx = TIx \) for all \( x \in B_C(p) \). If \( B_C(p) \) is nonempty, compact and starshaped with respect to a point \( q \in F(I) \) and if \( I(B_C(p)) = B_C(p) \) then \( B_C(p) \cap F(I, T) \neq \emptyset \).

**Remarks 4.1.2.** i) Theorem 4.1.7 was proved by Beg et al. (1992) for convex metric spaces satisfying Property (I).

ii) Results similar to Theorems 4.1.3 and 4.1.6 have been proved by Khan et al. (2002 a) in Hausdorff locally convex spaces.

iii) Taking \( S \) to be the identity map, we see that Corollaries 1 and 2 of Singh (1979 b) are special cases of Theorem 4.1.3.

**Theorem 4.1.8.** Let \( (X, d) \) be a convex metric space with Property (I), \( S \) and \( T \) be self mappings of \( X \) with \( p \in F(S, T) \), and \( M \in G_\alpha \) with \( T(M_p) \subset S(M) \subset M \). Suppose
that $S$ is affine and nonexpansive on $M_p$, $d(Sx, p) = d(x, p)$ for all $x \in M$ and $M$ is complete. If $S$ and $T$ are commuting on $M_p$, $T$ is $S$-nonexpansive on $M_p \cup \{p\}$, $S$ and $T$ are demicompact and $T$ is affine and $B_M(p)$ is nonempty, then

i) $B_M(p)$ is complete and convex.

ii) $T(B_M(p)) \subset S(B_M(p)) \subset B_M(p)$, and

iii) $S$ and $T$ have a common fixed point in $B_M(p)$.

Proof. As $M$ is complete and convex, $B_M(p)$ is also complete and convex. Hence condition (i) holds.

Let $y \in T(B_M(p))$. Since $T(B_M(p)) \subset T(M_p) \subset S(M)$, there exists $z \in B_M(p)$ and $w \in M$ such that $y = T(z) = S(w)$. Therefore $d(S(w), p) = d(T(z), p) \leq d(S(z), p) = d(p, M)$. As $S(w) \in M$ and $d(p, M) \leq d(S(w), p)$, it follows that $d(p, M) = d(S(w), p)$. Since $d(w, p) = d(S(w), p) = d(p, M)$, $w \in B_M(p)$, $y = S(w) \in S(B_M(p))$. Hence $T(B_M(p)) \subset S(B_M(p))$. Because $d(S(x), p) = d(x, p)$ for all $x \in M$, $S(B_M(p)) \subset B_M(p)$. Thus condition (ii) holds.

Since $I$ is nonexpansive, $S$ is demicompact and $S(B_M(p)) \subset B_M(p)$, it follows from Theorem 3.1.23 that $S$ has a fixed point in $B_M(p)$. Hence $S$ and $T$ have a common fixed point in $B_M(p)$. \hfill \Box

Corollary 4.1.9. (Vijayaraju and Marudai (2004)-Theorem 3.8) Let $X$ be a normed linear space, $I$ and $T$ be self mappings of $X$ with $p \in F(I, T)$, and $M \in G_o$ with $T(M_p) \subset I(M) \subset M$. Suppose that $I$ is affine and nonexpansive on $M_p$, $d(Ix, p) = d(x, p)$ for all $x \in M$ and $M$ is complete. If $I$ and $T$ are commuting on $M_p$, $T$ is $I$-nonexpansive on $M_p \cup \{p\}$, $I$ and $T$ are demicompact and $T$ is affine and $B_M(p)$ is nonempty, then

i) $B_M(p)$ is complete and convex.

ii) $T(B_M(p)) \subset I(B_M(p)) \subset B_M(p)$, and

iii) $I$ and $T$ have a common fixed point in $B_M(p)$.

Theorem 4.1.10. Let $(X, d)$ be a convex metric space with Property (I), $S, T$ are self mappings of $X$ with $p \in F(S, T)$, and $M \in G_o$ with $T(M_p) \subset S(M_p) \subset M_p$. Suppose that $S$ is affine and nonexpansive on $M_p$, $d(Sx, p) = d(x, p)$ for all $x \in M_p$. If $S$ and $T$ are commuting on $M_p$, $T$ is $S$-nonexpansive on $M_p \cup \{p\}$, $T$ is affine and $\text{cl}(T(M_p))$ is approximatively compact, then $S$ and $T$ have a common fixed point in $B_M(p)$. 

53
Proof. Let $D = \text{cl} \ (T(M_p))$. Then $B_D(p)$ is compact. Let us assume that $p \notin M$. Since $M_p$ is convex and $T$ is affine, $D$ is convex and hence $B_D(p)$ is compact and convex. Therefore by Theorem 2.2.10, $S$ and $T$ have a common fixed point in $B_M(p)$. \hfill \Box

Corollary 4.1.11. (Vijayaraju and Marudai (2004)-Theorem 3.9) Let $X$ be a normed linear space, $I, T$ are self mappings of $X$ with $p \in F(I, T)$, and $M \in G_0$ with $T(M_p) \subset I(M_p) \subset M_p$. Suppose that $I$ is affine and nonexpansive on $M_p$, $d(Ix, p) = d(x, p)$ for all $x \in M_p$. If $I$ and $T$ are commuting on $M_p$, $T$ is $I$-nonexpansive on $M_p \cup \{p\}$, $T$ is affine and $\text{cl} \ (T(M_p))$ is approximatively compact, then $I$ and $T$ have a common fixed point in $B_M(p)$.

The following theorem extends a result of Smoluk (1981):

Theorem 4.1.12. Let $(X, d)$ be a convex metric space with Property (I), $S, T$ are self mappings of $X$ with $p \in F(S, T)$, and $M \in G_0$ with $T(M_p) \subset S(M) \subset M$. Suppose that $S$ is affine and nonexpansive on $M_p$, $d(Sx, p) = d(x, p)$ for all $x \in M$, $S$ and $T$ are commuting on $M_p$, $T$ is $S$-nonexpansive on $M_p \cup \{p\}$, and that one of the following two conditions is satisfied:

(a) $\text{cl} \ (S(M_p))$ is compact

(b) $\text{cl} \ (T(M_p))$ is compact and $T$ is affine on $M_p$.

Then

i) $B_M(p)$ is non-empty, closed and convex,

ii) $T(B_M(p)) \subset S(B_M(p)) \subset B_M(p)$, and

iii) $S$ and $T$ have a common fixed point in $B_M(p)$.

Proof. Since $d(Tx, Tp) \leq d(Sx, Sp) = d(Sx, p) = d(x, p)$, $T$ is $S$-nonexpansive on $M_p \cup \{p\}$ and so $T$ satisfies the hypotheses of Theorem 3.1.29. Then for both (a) and (b), (i) holds, $S(B_M(P)) \subset B_M(p)$ and $T(B(M_p)) \subset B_M(p)$. Now we prove (ii).

Let $y \in T(B_M(p))$. Since $T(M_p) \subset S(M)$, there exists $z \in B_M(p)$ and $w \in M$ such that $y = Tz = Sw$. Since $T$ is $S$-nonexpansive on $M_p \cup \{p\}$ and $d(Sx, p) = d(x, p)$ for all $x \in M$, $d(Sw, p) = d(Tz, p) \leq d(Sz, p) = d(z, p) = \text{dist}(p, M)$. Thus $w \in C^M(p) = B_M(p)$ and so $T(B_M(p)) \subset S(B_M(p))$ and therefore (ii) holds. Next we obtain (iii).
If (a) is satisfied then by Theorem 3.1.29, $S$ has a fixed point $q \in B_M(p)$. Now suppose (b) is satisfied. Since $T$ is affine, $D_0 = T(B_M(p))$ is convex because $B_M(p)$ is convex as for $Ty, Tz \in T(B_M(p))$, consider

$$W(Ty, Tz, k) = TW(y, z, k) \in T(B_M(p)).$$

Since $T$ is continuous, $D_0 = T(B_M(p))$ is compact as $T(B_M(p))$ is a closed subset of the compact set $cl (T(M_p))$ in view of $B_M(p) \subset M_p$. Hence $D_1 = cl (S(D_0))$ is convex and compact. Since $S$ and $T$ commute on $B_M(p)$, $S(D_0) \subset D_0$ as $S(D_0) = S(T(B_M(p))) = T(S(B_M(p))) = D_0$. Since $S$ is continuous on $B_M(p)$, $S(D_1) \subset D_1$. It follows from Theorem 2.1.9 that $S$ has a fixed point $q \in D_1 \subset B_M(p)$. Now (iii) follows from Theorem 2.2.18 with $D = B_M(p)$. \qed

Remarks 4.1.3. i) If $S$ is identity map on $M_p$ in Theorem 4.1.12 then

a) Smoluk’s result is a special case of Theorem 4.1.12 (b), and

b) if $M_p$ is compact, Theorem 3.1.29 is Theorem 4.1.12 (a).

ii) For normed linear spaces Theorem 4.1.12 was proved by Al-Thagafi (1996)(see also Shahzad (2003)) by assuming the linearity of the mappings $S$ and $T$.

Now, we discuss the existence of invariant points from a set of best simultaneous approximation and deduce some results on invariant points from a set of best approximation for a pair of commuting mappings. For this we need the following lemma of Jungck (1976 a).

Lemma 4.1.13. Let $(X, d)$ be a compact metric space. Suppose that $T$ and $S$ are commuting mappings of $X$ into itself such that $T(X) \subseteq S(X)$, $S$ is continuous and $d(Tx, Ty) < d(Sx, Sy)$ for all $x, y \in X$ whenever $Sx \neq Sy$. Then $T$ and $S$ have a unique common fixed point in $X$.

Theorem 4.1.14. Let $K$ be a nonempty subset of a convex metric space $(X, d)$ with Property (I), $T$ and $S$ are continuous self-mappings of $K$ such that $T$ is $S$-asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that $y_1, y_2 \in X$ and the set $D$ of best simultaneous approximation to $y_1$ and $y_2$ is nonempty, compact and starshaped with respect to $z \in F(S)$, and $T$ satisfies

$$d(Tx, y_i) \leq d(x, y_i)$$

55
for all \( x \in X \) and \( i = 1, 2 \). If \( T \) and \( S \) is commuting on \( D \) and \( T \) is uniformly asymptotically regular on \( D \) and \( S \) is affine on \( D \) such that \( S(D) = D \), then \( D \) contains \( T \)- and \( S \)-invariant point.

**Proof.** Since \( D \) is the set of best simultaneous approximation to \( y_1 \) and \( y_2 \) and \( d(Tx, y_i) \leq d(x, y_i) \) for all \( x \in X \) and \( i = 1, 2 \), \( Tx \) is in \( D \). Thus \( T \) maps \( D \) into itself. Since \( T \) is \( S \)-asymptotically nonexpansive, there exists a sequence \( \{k_n\} \) of real numbers in \([1, \infty)\) with \( k_n \geq k_{n+1}, k_n \to 1 \) as \( n \to \infty \) such that \( d(T^n(x), T^n(y)) \leq k_n d(Sx, Sy) \), for all \( x, y \in K \). Suppose that \( z \) is a star-center of \( D \). Define \( T_n(x) = W(T^n x, z, a_n) \) for all \( x \in D \) where \( a_n = \frac{(1 - \frac{1}{n})}{k_n} \). Since \( z \) is a star-center of \( D \) and \( T(D) \subseteq D \), \( T_n \) is a self map of \( D \) for each \( n \). Consider

\[
T_n(Sx) = W(T^n(Sx), Sx, a_n) = W(S(T^n x), Sx, a_n) = SW(T^n x, z, a_n) = S(T_n x).
\]

Therefore \( T_n \) and \( S \) commute for each \( n \). Since \( T_n(D) \subseteq D \) and \( S(D) = D \), so \( T_n(D) \subseteq S(D) \). Suppose \( x, y \in D \) and \( Sx \neq Sy \). Then we have

\[
d(T_n x, T_n y) = d(W(T^n x, z, a_n), W(T^n y, z, a_n)) \\
\leq a_n d(T^n x, T^n y) \\
\leq a_n k_n d(Sx, Sy) \\
= \frac{(1 - \frac{1}{n})}{k_n} k_n d(Sx, Sy) \\
= \frac{1}{n} d(Sx, Sy).
\]

Also, \( D \) is compact and \( S \) is continuous on \( D \) and so by Lemma 4.1.13, there is a point \( x_n \) in \( D \) such that \( x_n = T_n x_n = S x_n \). Therefore

\[
d(x_n, T^n x_n) = d(T_n x_n, T^n x_n) \\
= d(W(T^n x_n, z, a_n), T^n x_n) \\
\leq a_n d(T^n x_n, T^n x_n) + (1 - a_n) d(z, T^n x_n) \\
\to 0.
\]

Since \( T \) is uniformly asymptotically regular and \( S \)-asymptotically nonexpansive on \( D \),
$S$ commutes with $T^n$ and $x_n = Sx_n$, it follows that

\[ d(x_n, Tx_n) \leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \]
\[ \leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(S(T^n x_n), S(x_n)) \]
\[ \leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n x_n, x_n) \]
\[ \rightarrow 0. \]

Since $D$ is compact, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to x \in D$. Since $T$ is continuous, $T(x_{n_i}) \to T(x)$, it follows that

\[ d(x, Tx) \leq d(x, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, Tx) \to 0, \]

which gives $Tx = x$. Since $S$ is continuous and $x_{n_i} = S(x_{n_i})$, it follows that $Sx = x$. Hence $x \in F(T, S)$.

\[ \square \]

**Corollary 4.1.15.** (Vijayaraju(1993)-Corollary 2.4) Let $K$ be a nonempty subset of a normed linear space $X$, $T$ and $S$ are continuous self-mappings of $K$ such that $T$ is $S$-asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that $y_1, y_2 \in X$ and the set $D$ of best simultaneous approximation to $y_1$ and $y_2$ is nonempty, compact and starshaped with respect to $z \in F(S)$, and $T$ satisfies

\[ d(Tx, y_i) \leq d(x, y_i) \]

for all $x \in X$ and $i = 1, 2$. If $T$ and $S$ is commuting on $D$, $T$ is uniformly asymptotically regular on $D$ and $S$ is affine on $D$ such that $S(D) = D$, then $D$ contains $T$- and $S$-invariant point.

If $y_1 = y_2 = x$, we have

**Corollary 4.1.16.** Let $K$ be a nonempty subset of a convex metric space $(X, d)$ with Property (I), $T$ and $S$ are continuous self-mappings of $K$ such that $T$ is $S$-asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that the set $D$ of best $K$-approximants is nonempty, compact and starshaped with respect to $z \in F(S)$, and $D$ is invariant under $T$. If $T$ and $S$ is commuting on $D$, $T$ is uniformly asymptotically regular on $D$ and $S$ is affine on $D$ such that $S(D) = D$, then $D$ contains $T$- and $S$-invariant point.
4.2 Common fixed points and invariant approximation for a pair of commuting mappings in metric spaces

In this section we discuss some results on the existence of common fixed points of best approximation for a pair of commuting mappings on a certain class of nonconvex sets introduced by Dotson (1973) when the underlying spaces are metric spaces thereby extending many known results.

We start with the following example of a pair of commuting mappings on a subset (not necessarily starshaped).

**Example 4.2.1.** Let $X = \mathbb{R}$, be endowed with the usual metric and $M = \{x : 0 \leq x \leq 1, x \in \mathbb{Q}\}$. Clearly, $M$ is not starshaped. Let $T, g$ be self-map on $M$ defined as $T(x) = x^2$ and $g(x) = x$, for all $x \in M$. Here $C(T, g) = \{0, 1\}$ and $Tgx = gTx$, so $(T, g)$ is a commuting pair.

The following common fixed point theorem for two maps in metric spaces was proved by Khan et al. (2000) for $p$-normed spaces and it is easy to see that proceeding as in Theorem 2.2.25, the proof given by Khan et al. (2000) can easily be extended to metric spaces.

**Theorem 4.2.2.** Let $I$ and $T$ be self maps on a metric space $(X, d)$, $u \in F(T) \cap F(I)$ and $M$ is $T$-invariant subset of $X$. Suppose $I$ and $T$ are commuting on $D = P_M(u)$, $I$ is continuous on $D$, $T$ is $I$-nonexpansive on $D \cup \{u\}$ and $I(D) = D$. Suppose that $D$ has a contractive jointly continuous family $F$ such that $I(f_x(\alpha)) = f_{Ix}(\alpha)$ for all $x \in D$ and $\alpha \in [0, 1]$. Then $I$ and $T$ have a common fixed point on $D$.

**Remark 4.2.1.** Theorem 4.1.7 is special case of Theorem 4.2.2 which was proved by Beg et al. (1992) for convex metric spaces with Property (I).

Jungck (1976 b) obtained the following common fixed point result for $G$-contractive mappings:

**Lemma 4.2.3.** Let $F$ and $G$ be commuting mappings of a compact metric space $(X, d)$ into itself such that $F(X) \subset G(X)$, and $G$ is continuous. If $F$ is $G$-contractive map on the metric space $X$, then there is a unique common fixed point of $F$ and $G$.

Using Lemma 4.2.3, we have the following theorem on common fixed points of commuting mappings in metric spaces:
**Theorem 4.2.4.** Let \((X,d)\) be a metric space, \(F\) and \(G : X \to X\) be commuting mappings such that \(F\) is \(G\)-nonexpansive where \(G\) satisfies \(G^2 = G\). Let \(C\) be a subset of \(X\) and \(x\) a point of \(X\) such that both are invariant under \(F\) and \(G\). Let \(D = P_C(x)\) be the set of best approximant of \(x\) in \(C\). If \(G\) is continuous on \(D\), \(F(D) \subseteq G(D)\) and \(D\) is nonempty compact and has jointly continuous contractive family \(\mathcal{F}\) such that \(G(f_y(\alpha)) = f_{Gy}(\alpha)\), for all \(y \in D\) and \(\alpha \in [0,1]\), then \(D\) contains a point invariant under both \(F\) and \(G\).

**Proof.** We first observe that both \(F\) and \(G\) are self maps on \(D\). Let \(y \in D\). Consider

\[
d(x, GFy) = d(x, FGy) = d(Fx, FGy) \leq d(x, G^2y) = d(x, Gy) = d(x, C). \tag{4.2.1}
\]

Also,

\[
d(x, GGy) = d(x, Gy) = d(x, C). \tag{4.2.2}
\]

From relations (4.2.1) and (4.2.2) we have \(F(y) \in D\) and \(G(y) \in D\). Thus \(F\) and \(G\) are self mappings on \(D\).

Define \(F_n : D \to D\) as \(F_n(x) = f_{F(x)}(t_n), x \in D\) and \(\{t_n\}\) is a sequence of real numbers in \((0,1)\) such that \(t_n \to 1\).

Since \(F\) and \(G\) commute on \(D\), it follows from the property of the family \(\mathcal{F}\) that \(F_n(G(x)) = f_{G(F(x))}(t_n) = Gf_{F(x)}(t_n) = G(F_n(x))\) for each \(x \in D\). Thus for each \(n\), \(F_n\) commutes with \(G\) and \(F_n(D) \subseteq G(D)\). Since \(\mathcal{F}\) is contractive and \(F\) is \(G\)-nonexpansive, we get

\[
d(F_ny, F_nz) = d(f_{F(y)}(t_n), f_{F(z)}(t_n)) \leq \Phi(t_n)d(F(y), F(z)) \leq \Phi(t_n)d(G(y), G(z))
\]

for every \(y, z \in D\) and so \(F_n\) is \(G\)-contraction for each \(n\).

It follows from Lemma 4.2.3 that there is a unique common fixed point, say \(x_n \in D\), of \(F_n\) and \(G\) for each \(n\) i.e. \(Gx_n = x_n = F_nx_n\) for each \(n\). Since \(D\) is compact, \(\{x_n\}\) has a subsequence \(\{x_{n_i}\} \to x_o \in D\) and hence \(F(x_{n_i}) \to F(x_o)\). The joint continuity of \(\mathcal{F}\) gives

\[
x_{n_i} = F_nx_{n_i} = f_{Fx_{n_i}}(t_{n_i}) \to f_{Fx_o}(1) = F(x_o)
\]

and so \(x_o = F(x_o)\). Also continuity of \(G\) gives \(G(x_o) = G(\lim x_{n_i}) = \lim G(x_{n_i}) = \lim x_{n_i} = x_o\). Thus \(x_o \in D\) is invariant under both \(F\) and \(G\).\qed
**Corollary 4.2.5.** (Sahab and Khan (1987)-Theorem 3.1) Let $F$ and $G$ be commuting operators on a normed linear space $X$ such that $F$ is $G$-nonexpansive, where $G$ is linear, continuous and satisfies $G^2 = G$. Let $C$ be a subset of $X$, $x$ a point of $X$ such that both of them are invariant under both $F$ and $G$. Let $D = \{ y \in C : Gy \text{ is a best } C\text{-approximant to } x \}$. If $F(D) \subseteq G(D)$, and also $D$ is nonempty, compact and $G$-starshaped with respect to $G$, then $D$ contains a point invariant under both $F$ and $G$.

Assuming $G = I$ (the identity mapping), we obtain the following:

**Corollary 4.2.6.** (Sahab and Khan (1987)-Theorem 3.4) Let $F$ be a nonexpansive operator on a normed linear space $X$. Let $C$ be an $F$-invariant subset of $X$ and $x$ an $F$-invariant point. If the set of best $C$-approximants to $x$ is nonempty, compact and for which there exists a contractive jointly continuous family $\mathcal{F}$ of functions, then it contains an $F$-invariant point.

We shall be using the following result of Jungck (1988) to prove our next theorem:

**Lemma 4.2.7.** Let $f$ be a continuous self map of a compact metric space $(X, d)$. If $f(x) \neq f(y)$ implies $d(fx, fy) < d(gx, hy)$ for some $g, h \in C_f$, where $C_f$ is the set of all maps $g : X \to X$ such that $gf = fg$, then there is a unique point $a \in X$ such that $a = f(a)$. In fact $a = h(a)$ for all $h \in C_f$.

Using Lemma 4.2.7, we have:

**Theorem 4.2.8.** Let $(X, d)$ be a metric space, $I$ and $T : C \to C$ be commuting maps where $C$ is a compact subset of $X$ and has jointly continuous contractive family $\mathcal{F}$ satisfying $I(f_x(\alpha)) = f_{Ix}(\alpha)$ for all $x \in C$ and $\alpha \in [0, 1]$. If for each $x, y \in C$, there exists $I = I(x, y), J = J(x, y) \in C_T$ such that

$$d(Tx, Ty) \leq d(Ix, Jy) \quad (4.2.3)$$

then there exists $a \in C$ such that $a = Ta$ and $a = Ia$ for all continuous $I \in C_T$.

**Proof.** Let $\langle t_n \rangle$ be a sequence of real numbers in $(0, 1)$ such that $t_n \to 1$. Let $T_n : C \to C$ be defined as $T_n(x) = f_{T_n}(t_n)$. Since $I$ and $T$ commute, it follows from the property of
the family $\mathcal{F}$ that
\[
T_n(I(x)) = f_{T(I(x))}(t_n) = f_{I(T(x))}(t_n) \text{ as } I \text{ and } T \text{ commute}
\]
\[
= If_{T(x)}(t_n)
\]
\[
= IT_n(x)
\]
for each $x \in C$. Thus for each $n$, $T_nI = IT_n$.

Now fix $n$. By hypothesis, for each $x, y \in C$ there exists $I, J \in C_T$ such that
\[
d(Tx, Ty) \leq d(Ix, Jy).
\]
So,
\[
d(T_nx, T_ny) = d(f_{T_n}(t_n), f_{Ty}(t_n))
\]
\[
\leq \Phi(t_n)d(Tx, Ty)
\]
\[
\leq \Phi(t_n)d(Ix, Jy).
\]
Therefore, for all $x, y \in C$, $T_n(x) \neq T_n(y)$ implies $d(T_nx, T_ny) \leq d(Ix, Jy)$ for some $I, J \in C_T$. Since $T_n$ is continuous, by Lemma 4.2.7 there is unique $x_n \in C$ such that for all $I \in C_T$, $x_n = T_nx = Ix_n$. Since $C$ is compact, \langle x_n \rangle has a subsequence $\langle x_{n_i} \rangle \to a \in C$. Now
\[
a = \lim x_{n_i} = \lim T_nx_{n_i} = \lim f_{T_n}(t_{n_i}) \to f_Ta(1) = Ta
\]
Also, $a = \lim x_{n_i} = \lim Ix_{n_i} = I \lim x_{n_i} = Ia$ for all continuous $I \in C_T$. Thus $a = Ta$ and $a = Ia$ for all continuous $I \in C_T$. \hfill \Box

**Corollary 4.2.9.** Let $C$ be a compact subset of a metric space $(X, d)$, $T, I : C \to C$ be continuous commuting maps, $C$ has jointly continuous contractive family $\mathcal{F}$ satisfying $I(f_x(\alpha)) = f_{Ix}(\alpha)$ for all $x \in C$ and $\alpha \in [0, 1]$. If for $x, y \in C$, there exists $n = n(x, y)$, $m = m(x, y)$ in $\mathbb{N} \cup \{0\}$ such that
\[
d(Tx, Ty) \leq d(I^nx, I^ny) \tag{4.2.4}
\]
then there exists $a \in C$ such that $a = Ta$ and $a = Ia$.

**Proof.** For each $n$,
\[
T_n(I^n x) = f_{T(I^n x)}(t_n) = f_{I(T(I^{n-1} x))}(t_n)
\]
\[
= f_{I(T(I^{n-1} x))}(t_n) = I f_{T(I^{n-1} x)}(t_n) = ... = I^n(T_n x)
\]
i.e. $T_n I^n = I^n T_n$ for each $n$ and $I^n : C \to C$. Therefore (4.2.4) implies that members of $\mathfrak{I}$ satisfy (4.2.3) and so the result follows from Theorem 4.2.8.

As a direct application of Theorem 3.2.1, we now prove the following result for a pair of commuting mapping.

**Theorem 4.2.10.** Let $F$ and $G$ be commuting mappings on a metric space $(X, d)$ such that $F$ is $G$-nonexpansive and $G^2 = I$. Let $C$ be a $FG$-invariant subset of $X$ and $x$ an $FG$-invariant point of $X$. If the set $P_C(x)$ is a singleton set $\{x_o\}$, then $x_o$ is a simultaneous invariant point of $F$ and $G$.

**Proof.** For any $x, y \in X$, consider $d(FGx, GFy) \leq d(G(Gx), G(Gy)) = d(x, y)$ as $G^2 = I$ i.e. $FG$ is nonexpansive on $X$. By Theorem 3.2.1, $P_C(x)$ contains an $FG$-invariant point $x_o$ as $P_C(x) = \{x_o\}$. Consider $FG(Fx_o) = F(GFx_o) = F(FGx_o) = Fx_o$. By the unicity of $x_o$, we get $Fx_o = x_o$. Also $G(x_o) = G(Fx_o) = GFx_o = FGx_o = x_o$. Thus $x_o$ is invariant under both $F$ and $G$.

**Remark 4.2.2.** For normed linear spaces, Theorem 4.2.10 was proved by (Sahab and Khan (1987)-Theorem 4.1).

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