Chapter 2

CHAPTER 2

Invertible fuzzy topological spaces

In this chapter we introduce the concept of invertible fuzzy topological spaces. Also we discuss and study several interesting properties of this newly introduced fuzzy topological spaces besides giving some characterizations of these spaces.

2.1 Invertible fuzzy topological spaces

In this section we introduce the concept of invertible fuzzy topological spaces and give some examples.

Motivated by the classical concept introduced in [23] we now define:

Definition 2.1.1. A fuzzy topological space (X, T) is called **invertible fuzzy topological space** if and only if for each non-zero fuzzy open set $\lambda \in T$ there exists a fuzzy homeomorphism $f : (X, T) \to (X, T)$ such that $f(1 - \lambda) \leq \lambda$ and in this case f is called an inverting fuzzy homeomorphism for λ.

Example 2.1.1. *Let λ be a fuzzy set in $I = [0,1]$ defined by*

$$\lambda(x) = \begin{cases} 1 - x, & 0 \leq x \leq \frac{1}{2}, \\ x, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Consider $T = \{0, \lambda, 1\}$. Clearly T is a fuzzy topology on I. Let $f : (I,T) \to (I,T)$ be defined by $f(x) = x$ for each $x \in I$. Clearly, f is a fuzzy homeomorphism. Now for $\lambda \in T$, we have $f(1-\lambda) = 1-\lambda \leq \lambda$. Also $f(1-1) = f(0) = 0 \leq 1$. This shows that (I,T) ***is an invertible fuzzy topological space.***

Example 2.1.2. *Let μ_1 and μ_2 be fuzzy sets in $I = [0,1]$ defined by*

$$\mu_1(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2}, \\ 2x - 1, & \frac{1}{2} \leq x \leq 1; \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{4}, \\ -4x + 2, & \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 0, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Clearly $T = \{0, \mu_1, \mu_2, \mu_1 \vee \mu_2, 1\}$ is a fuzzy topology on I. Let $f : (I,T) \to (I,T)$ be any fuzzy homeomorphism. Then for $0 \neq \mu_1 \in T$, we have $f(1-\mu_1) \nleq \mu_1$. This shows that (I,T) ***is not an invertible fuzzy topological space.***

2.2 Properties and Example

In this section we investigate some interesting properties of invertible fuzzy topological spaces.

Proposition 2.2.1. *Let (X, T) and (Y, S) be invertible fuzzy topological spaces. Then $(X \times Y, T \times S)$ is an invertible fuzzy topological space.*

Proof: Let $0 \neq \lambda \in T$ and $0 \neq \mu \in S$. Then $0 \neq \lambda \times \mu$ is a fuzzy open set in $X \times Y$. Since $0 \neq \lambda \in T$, there exists fuzzy homeomorphism $f_\lambda : X \to X$, such that $f_\lambda(1-\lambda) \leq \lambda$. Also since $0 \neq \mu \in S$, there exists a fuzzy homeomorphism $f_\mu : Y \to Y$ such that $f_\mu(1-\mu) \leq \mu$. Then $f_\lambda \times f_\mu : X \times Y \to X \times Y$ is a fuzzy homeomorphism such that $(f_\lambda \times f_\mu)\ (1-(\lambda \times \mu)) = (f_\lambda \times f_\mu)[(1-\lambda) \times (1-\mu)] = f_\lambda(1-\lambda) \times f_\mu(1-\mu) \leq \lambda \times \mu$. This proves that $X \times Y$ is an invertible fuzzy topological space.

□

Remark 2.2.1. *Let (X_α, T_α) be a family of invertible fuzzy topological space such that $X_\alpha \cap X_\beta = \phi$ whenever $\alpha \neq \beta$. Put $X = \bigcup_{\alpha \in \Delta} X_\alpha$. Let $T : \{\lambda : X \to [0,1] \mid \lambda_{/X_\alpha} \in T_\alpha\}$, then T is a fuzzy topology on X and (X, T) is an invertible fuzzy topological space.*

In [23] the following result in general topology is proved as Theorem 1.

RESULT (Theorem 1in [23]): If the invertible space S contains a non-empty open set U whose closure $\overline{U}$ is compact, then S is compact.

However in the case of fuzzy topological space a similar version of the above result is not true as the following example shows:-

Example 2.2.1. *Let X be any non-empty set. Define $\lambda_n : X \to [0,1]$ as follows: $\lambda_n = 1-\frac{1}{n}$; $n = 1, 2, 3, \cdots$. Clearly $T = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \cdots\}$ is fuzzy topology on X. We shall show that (X, T) is invertible fuzzy* *topological space. Define* $f : X \to X$ *by* $f(x) = x$. *Clearly* f *is a fuzzy homeomorphism. Now* $f(1-1) = f(0) = 0 \leq 1$ *and* $f(1-\lambda_n) = f(1-(1-\frac{1}{n})) = f(\frac{1}{n}) = \frac{1}{n} \leq 1-\lambda_n$, *for* $n = (2, 3, \cdots)$. *Therefore* (X, T) ***is invertible fuzzy topological space.*** *The fuzzy set* $\lambda_2 = 1 - \frac{1}{2} = \frac{1}{2}$ *is* ***both fuzzy open*** *and* ***fuzzy closed.*** *Since* $\lambda_1 = 0$; $\lambda_2 = \frac{1}{2}$; $\lambda_3 = \frac{2}{3}$; $\cdots$ *etc. It is easy to see that* λ_2 *is clearly* ***fuzzy compact.*** *Now we shall show that* (X, T) *is not fuzzy compact. Consider fuzzy open cover* $\{\lambda_n\}_{n=1}^{\infty}$ *of* 1_X. *This cover clearly has no finite subcover. Hence* (X, T) ***is not fuzzy compact.*** *Thus we find the invertible fuzzy topological space* (X, T) *contains a non-zero fuzzy open set* $\lambda_2 = \frac{1}{2}$ *such that* $\overline{\lambda_2} = \frac{1}{2}$ *is fuzzy compact but the space* (X, T) *is not fuzzy compact.*

2.3 Characterizations of invertible fuzzy topological spaces

In this section we study some characterizations of invertible fuzzy topological spaces.

Proposition 2.3.1. *The following assertions are equivalent*

(1) (X, T) *is an invertible fuzzy topological space.*

(2) Given a proper fuzzy closed set λ *and a non-zero fuzzy open set* μ, *there is a fuzzy homeomorphism* $f : (X, T) \to (X, T)$ *such that* $f(\lambda) \leq \mu$.

(3) Given a proper fuzzy closed set λ *and a non-zero fuzzy open set* μ, *there exists a fuzzy homeomorphism* $f : (X, T) \to (X, T)$ *such*

that $\lambda \leq f(\mu)$.

(4) *Given any proper fuzzy set* λ *such that cl* $\lambda \neq 1$ *and a non-zero fuzzy set* μ *such that int* $\mu \neq 0$, *there exists a fuzzy homeomorphism* $f : (X, T) \to (X, T)$ *such that* $f(cl\lambda) \leq int\ \mu$.

(5) *Given any proper fuzzy set* λ *such that cl* $\lambda \neq 1$ *and a non-zero fuzzy set* μ *such that int* $\mu \neq 0$ *there exists a fuzzy homeomorphism* $f : (X, T) \to (X, T)$ *such that cl* $\lambda \leq f(int\ \mu)$.

Proof: (1)$\Rightarrow$ (2) Let us assume that (X, T) is an invertible fuzzy topological space. Let λ be any proper fuzzy closed set and μ be any non-zero fuzzy open set. Since λ is a proper fuzzy closed set, $1 - \lambda$ is a non-zero fuzzy open set and since (X, T) is invertible fuzzy topological space there is a fuzzy homeomorphism $g : (X, T) \to (X, T)$ such that $g(\lambda) \leq 1 - \lambda$. If $1 - \lambda \leq \mu$, then $g(\lambda) \leq 1 - \lambda \leq \mu$, that is, $g(\lambda) \leq \mu$. If $1 - \lambda \geq \mu$, then $\lambda \leq 1 - \mu$. In this case let $f : (X, T) \to (X, T)$ be the fuzzy homeomorphism corresponding to the non-zero fuzzy open set μ. Then we have $f(1 - \mu) \leq \mu$. Also, now $\lambda \leq 1 - \mu$ implies $f(\lambda) \leq f(1 - \mu) \leq \mu$, that is; $f(\lambda) \leq \mu$. This proves (1) implies (2).

(2)$\Rightarrow$ (1) Let λ be any non-zero fuzzy open set. Then $1 - \lambda$ is a proper fuzzy closed set. Hence by assumption (1), there is a fuzzy homeomorphism $f : (X, T) \to (X, T)$ such that $f(1 - \lambda) \leq \lambda$. This proves that (X, T) is an invertible fuzzy topological space. Thus (2) implies (1) is proved.

(2)$\Rightarrow$ (3) Let λ be any proper fuzzy closed set and μ be any non-zero fuzzy open set. Then by assumption (2), there exists a fuzzy homeomorphism $g : (X, T) \to (X, T)$ such that $g(\lambda) \leq \mu$. Then $\lambda \leq$

$g^{-1}g(\lambda) \leq g^{-1}(\mu)$. Put $g^{-1} = f$. Then f is a fuzzy homeomorphism from (X, T) onto (X, T) and $\lambda \leq f(\mu)$. This proves (2) implies (3).

(3) $\Rightarrow$ (4) Let λ be any proper fuzzy set such that cl $\lambda \neq 1$ and μ be any non-zero fuzzy set such that int $\mu \neq 0$. Then cl λ and int μ are proper fuzzy closed set and non-zero fuzzy open set respectively. Hence by assumption (3), there exists a fuzzy homeomorphism $g : (X, T) \rightarrow (X, T)$ such that cl $\lambda \leq g$(int μ). Then g^{-1} (cl λ) $\leq$ $g^{-1}g$(int μ) = int μ. Therefore g^{-1} (cl λ) $\leq$ int μ . Put $g^{-1} = f$. Clearly f is a fuzzy homeomorphism and f(cl λ) $\leq$ int μ. This proves (3) implies (4).

(4) $\Rightarrow$ (5) Let λ be a proper fuzzy set such that cl $\lambda \neq 1$ and μ be any non-zero fuzzy set such that int $\mu \neq 0$. Then by assumption (4), there exists fuzzy homeomorphism $g : (X, T) \rightarrow (X, T)$ such that g(cl λ) $\leq$ int μ. Then cl $\lambda = g^{-1}g$(cl λ) $\leq g^{-1}$(int μ). Therefore cl $\lambda \leq g^{-1}$(int μ). Put $g^{-1} = f$. Clearly f is a fuzzy homeomorphism and cl$\lambda \leq f$(int μ). This proves (4) implies (5).

(5) $\Rightarrow$ (2) Let λ be a proper fuzzy closed set λ and μ be a non-zero fuzzy open set. Since λ is a proper fuzzy closed set such that cl $\lambda \neq 1$, by assumption (5), there exists a fuzzy homeomorphism $g : (X, T) \rightarrow (X, T)$ such that cl $\lambda \leq g$(int μ). That is., $\lambda \leq$ cl $\lambda \leq$ g (int μ) $\leq g(\mu)$. In other words $g^{-1}(\lambda) \leq g^{-1}g(\mu) = \mu$. Put $g^{-1} = f$. Clearly f is a fuzzy homeomorphism and $f(\lambda) \leq \mu$. This proves (5) implies (2). □

Corollary 2.3.1. *Let X be any invertible fuzzy topological space and fuzzy T_1 space. Let x_α be any fuzzy point and μ be any non-zero fuzzy open set. Then there is a fuzzy homeomorphism $f : X \rightarrow X$ such that $f(x_\alpha) \leq \mu$.*

Remark 2.3.1. *Let (X, T) be any invertible fuzzy topological space. Let λ_1 and λ_2 be two proper fuzzy closed sets. Let μ_1 and μ_2 be two fuzzy open sets such that $\mu_1 \wedge \mu_2 = 0$. Then there exists a fuzzy homeomorphism $f : X \to X$ such that $f(\lambda_1) \leq \mu_1$ and a fuzzy homeomorphism $g : X \to X$ such that $g(\lambda_2) \leq \mu_2$. Thus we find in an invertible fuzzy topological space, homeomorphic copies of any two fuzzy closed sets can be separated by two disjoint fuzzy open sets.*

****