# Chapter 7

# CHAPTER 7

# Pre $\alpha$-irresolute and somewhat pre $\alpha$-irresolute functions

In this chapter we introduce and study the concepts of pre $\alpha$-irresolute functions and somewhat pre $\alpha$-irresolute functions and investigate some characterizations and basic properties of these classes of functions. Relations between this class and some other existing classes are also obtained.

## 7.1 Pre $\alpha$-irresolute functions

In this section we introduce the concept of pre $\alpha$-irresolute function and discuss its relations with other known functions given below. We also discuss characterizations and some of its interesting properties.

**Definition 7.1.1.** A function $f$ : $(X, T) \to (Y, S)$ is said to be $\alpha$-irresolute [41] (semi $\alpha$-irresolute [102], almost $\alpha$-irresolute [101], resp.) if $f^{-1}(V)$ is $\alpha$-open (semi-open $\beta$-open, resp.) in $X$ for every $\alpha$-open set $V$ in $Y$.

**Definition 7.1.2.** A function $f$ : $(X, T) \to (Y, S)$ is said to be strongly $\alpha$-continuous [100] (irresolute [21], resp.) if $f^{-1}(V)$ is $\alpha$-open (semi-open, resp.) in $X$ for every semi-open $V$ in $Y$.

**Definition 7.1.3.** A function $f$ : $(X, T) \to (Y, S)$ is said to be $\beta$-continuous [1] (pre-continuous[42], resp.) if $f^{-1}(V)$ is $\beta$-open (pre-open, resp.) in $X$ for every open set $V$ in $Y$.

**Definition 7.1.4.** A function $f$ : $(X, T) \to (Y, S)$ is said to be strongly $\alpha$-irresolute [37] if $f^{-1}(V)$ is open in $X$ for every $\alpha$-open set $V$ in $Y$.

Motivated by the concepts of almost $\alpha$-irresolute functions and semi $\alpha$-irresolute functions introduced and studied in [102] and [101], respectively. We now define

**Definition 7.1.5.** A function $f$ : $(X, T) \to (Y, S)$ is said to be **pre $\alpha$-irresolute** if $f^{-1}(V)$ is pre-open in $(X, T)$ for every $\alpha$-open set $V$ in $(Y, S)$.

From the above definitions we obtain the following diagram :

| | | |
|---|---|---|
| irresolute | $\nRightarrow$ <br> $\Leftarrow$ | strongly $\alpha$-continuous |
| $\Downarrow\not\Uparrow$ | | $\Downarrow\not\Uparrow$ |
| semi $\alpha$-irresolute | $\nRightarrow$ <br> $\Leftarrow$ | $\alpha$-irresolute |
| $\Downarrow\not\Uparrow$ | | $\Downarrow\not\Uparrow$ |
| almost $\alpha$-irresolute | $\nRightarrow$ <br> $\Leftarrow$ | pre $\alpha$-irresolute |
| $\Downarrow\not\Uparrow$ | | $\Downarrow\not\Uparrow$ |
| $\beta$-continuous | $\nRightarrow$ <br> $\Leftarrow$ | pre-continuous |

The examples given below show that the converses of the above implications are not true in general.

**Example 7.1.1.** *Let* $X = \{a, b, c\}$ *with* $T_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. *Let* $f : (X, T_1) \to (X, T_1)$ *be a function defined by:* $f(a) = f(c) = a$ *and* $f(b) = b$. *Now* $f^{-1}(X) = X$; $f^{-1}(\phi) = \phi$; $f^{-1}(\{a\}) = \{a, c\}$; $f^{-1}(\{b\}) = \{b\}$; $f^{-1}(\{a, b\}) = X$; $f^{-1}(\{a, c\}) = \{a, c\}$; $f^{-1}(\{b, c\}) = \{b\}$. *It can be easily seen that* $int\ \{c\} = \phi$; $int(\{a, c\}) = \{a\}$; $cl\{a\} = \{a, c\}$; $int\{b, c\} = \{b\}$; $cl\{b\} = \{b, c\}$; $cl\{a, c\} = \{a, c\}$; $cl\{b, c\} = \{b, c\}$; $cl\{c\} = \{c\}$ *in* $(X, T_1)$. *Then* $\{c\}$, $\{a, c\}$, $\{b, c\}$ *are not* $\alpha$*-open sets and not pre-open sets in* $(X, T_1)$. $\{a, c\}$, $\{b, c\}$ *are semi-open sets and* $\beta$*-open sets in* $(X, T_1)$. *In* $(X, T_1)$ *the only semi-open sets are* $X$, $\phi$, $\{a\}$, $\{b\}$, $\{a, b\}$ $\{b, c\}$ *and* $\{a, c\}$. *Now* $f^{-1}(X)$, $f^{-1}(\phi)$, $f^{-1}(\{a\})$, $f^{-1}(\{b\})$, $f^{-1}(\{a, b\})$, $f^{-1}(\{a, c\})$ *and* $f^{-1}(\{b, c\})$ *are semi-open sets in* $(X, T_1)$. *Then* $f$ ***is irresolute*** *from*

$(X, T_1)$ *to* $(X, T_1)$. *We have* $f^{-1}(\{a\}) = \{a, c\}$, *which is not* $\alpha$*-open set in* $(X, T_1)$. *Therefore* $f$ ***is not strongly*** $\alpha$***-continuous.*** *In* $(X, T_1)$ *the only* $\alpha$*-open sets are* $X$, $\phi$, $\{a\}$, $\{b\}$ *and* $\{a, b\}$. *Then* $f^{-1}(X)$, $f^{-1}(\phi)$, $f^{-1}(\{a\}), f^{-1}(\{b\})$ *and* $f^{-1}(\{a, b\})$ *are semi-open sets and also* $\beta$*-open sets in* $(X, T_1)$. *Hence* $f$ ***is semi*** $\alpha$***-irresolute*** *and also* $f$ ***is both almost*** $\alpha$***-irresolute*** *and* $\beta$***-continuous*** *from* $(X, T_1)$ *to* $(X, T_1)$. *Now* $f^{-1}(\{a\}) = \{a, c\}$, *which is not* $\alpha$*-open set in* $(X, T_1)$. *Thus* $f$ ***is not*** $\alpha$***-irresolute***. *Since* $f^{-1}(\{a\}) = \{a, c\}$ *and* $\{a, c\}$ *is not a pre-open set in* $(X, T_1)$, $f$ ***is not pre*** $\alpha$***-irresolute*** *and also* $f$ ***is not pre continuous***. *Thus from this example we find (1)* $f$ *is irresolute* ***does not imply*** $f$ *is strongly* $\alpha$*-continuous*, $f$ *is* $\alpha$*-irresolute*, $f$ *is pre* $\alpha$*-irresolute and* $f$ *is pre-continuous. (2)* $f$ *is semi* $\alpha$*-irresolute* ***does not imply*** $f$ *is strongly* $\alpha$*-continuous*, $f$ *is* $\alpha$*-irresolute*, $f$ *is pre* $\alpha$*-irresolute and* $f$ *is pre-continuous. (3)* $f$ *is almost* $\alpha$*-irresolute* ***does not imply*** $f$ *is strongly* $\alpha$*-continuous*, $f$ *is* $\alpha$*-irresolute*, $f$ *is pre* $\alpha$*-irresolute and* $f$ *is pre-continuous. (4)* $f$ *is* $\beta$*-continuous* ***does not imply*** $f$ *is strongly* $\alpha$*-continuous*, $f$ *is* $\alpha$*-irresolute*, $f$ *is pre* $\alpha$*-irresolute and* $f$ *is pre-continuous.*

**Example 7.1.2.** *Let* $X = \{a, b, c\}$ *with* $T_2 = \{X, \phi, \{a, b\}\}$ *and* $T_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. *Let* $f : (X, T_2) \to (X, T_1)$ *be a function defined by :* $f(a) = a$; $f(b) = c$ *and* $f(c) = c$ . *Now* $f^{-1}(\{a\}) = \{a\}$; $f^{-1}(X) = X$; $f^{-1}(\phi) = \phi$; $f^{-1}(\{b\}) = \phi$; $f^{-1}(\{a, b\}) = \{a\}$. *It can be easily seen that* $int\{a\} = \phi$; $int\{b\} = \phi$; $int\{c\} = \phi$; $cl\{a\} = X = cl\{b\}$; $cl\{c\} = \{c\}$; $int\ \{a, c\} = \phi$ *and* $int\{b, c\} = \phi$ *in* $(X, T_2)$. $int\{a, c\} = \{a\}$; $cl\{a\} = \{a, c\}$; $int\{b, c\} = \{b\}$; $cl\{b\} = \{b, c\}$; $int\ \{c\} = \phi$; $cl\{c\} = \{c\}$; $cl\{a, c\} = \{a, c\}$; $cl\{b, c\} = \{b, c\}$ *in* $(X, T_1)$. *In* $(X, T_1)$

*only $\alpha$-open sets are $X, \phi, \{a\}, \{b\}$ and $\{a, b\}$. Then $f^{-1}(\{a\}), f^{-1}(X)$, $f^{-1}(\phi)$, $f^{-1}(\{b\})$ and $f^{-1}(\{a, b\})$ are pre-open sets and also $\beta$-open sets in $(X, T_2)$. Therefore $f$* ***is pre $\alpha$-irresolute*** *and also $f$* ***is almost $\alpha$-irresolute****. Now $f^{-1}(\{a\}), f^{-1}(X), f^{-1}(\phi), f^{-1}(\{b\})$ and $f^{-1}(\{a, b\})$ are pre-open set in $(X, T_2)$. Hence $f$* ***is pre-continuous****. In $(X, T_1)$ the only $\alpha$-open sets are $X, \phi, \{a\}, \{b\}$ and $\{a, b\}$. Now $f^{-1}\{a, b\} = \{a\}$, which is not semi-open set and also not $\alpha$-open set in $(X, T_2)$. Therefore $f$* ***is not semi $\alpha$-irresolute*** *and also $f$* ***is not $\alpha$-irresolute*** *from $(X, T_2)$ to $(X, T_1)$. Thus from this example we find (1) $f$ is pre $\alpha$-irresolute* ***does not imply*** *$f$ is semi $\alpha$-irresolute and $f$ is $\alpha$-irresolute. (2) $f$ is almost $\alpha$-irresolute* ***does not imply*** *$f$ is semi $\alpha$-irresolute and $f$ is $\alpha$-irresolute. (3) $f$ is pre-continuous* ***does not imply*** *$f$ is semi $\alpha$-irresolute and $f$ is $\alpha$-irresolute.*

**Example 7.1.3.** *Let $X = \{a, b, c\}$, $T_3 = \{X, \phi, \{b\}\}$. Let $f : (X, T_3) \rightarrow (X, T_3)$ be a function defined by $f(a) = f(b) = a$ and $f(c) = c$. Now $f^{-1}(X) = X$; $f^{-1}(\phi) = \phi$; $f^{-1}(b) = \phi$. Then $f^{-1}(X)$, $f^{-1}(\phi)$ and $f^{-1}(\{b\})$ are pre-open sets and also $\beta$-open sets in $(X, T_3)$. This shows that $f$* ***is pre-continuous*** *and also $f$* ***is $\beta$-continuous*** *from $(X, T_3)$ to $(X, T_3)$. It can be easily seen that int $\{a\}$ = int $\{c\}$ = int $\{a, c\}$ = $\phi$; int $\{a, b\}$= $\{b\}$; cl$\{b\}$ = $X$; cl$\{a, b\}$ = $X$; int $\{b, c\}$ = $\{b\}$; cl$\{a\}$ = cl$\{c\}$ = $\{a, c\}$; cl$\{b, c\}$ = $X$ in $(X, T_3)$. In $(X, T_3)$ the only $\alpha$-open sets are $X$, $\phi$, $\{a, b\}$, $\{b\}$ and $\{b, c\}$. Now $f^{-1}(\{a, b\}) = \{a\}$ and $f^{-1}\{b, c\} = \{c\}$, it is not a pre-open set in $(X, T_3)$ and also not a $\beta$-open set in $(X, T_3)$. Hence $f$* ***is not pre $\alpha$-irresolute*** *and also $f$* ***is not almost $\alpha$-irresolute*** *from $(X, T_3)$ to $(X, T_3)$. In $(X, T_3)$ the only semi-open sets are $X$, $\phi$, $\{a, b\}$, $\{b\}$ and $\{b, c\}$. Now $f^{-1}(\{a, b\}) =$*

*$\{a\}$, which is not a $\alpha$-open set in $(X, T_3)$. This shows $f$* **is not strongly $\alpha$-continuous** *from $(X, T_3)$ to $(X, T_3)$. Thus from this example we find (1) $f$ is pre-continuous* **does not imply** *$f$ is pre $\alpha$-irresolute, $f$ is almost $\alpha$-irresolute and $f$ is strongly $\alpha$-continuous. (2) $f$ is $\beta$-continuous* **does not imply** *$f$ is pre $\alpha$-irresolute, $f$ is almost $\alpha$-irresolute and $f$ is strongly $\alpha$-continuous.*

**Example 7.1.4.** *Let $X = \{a, b, c\}$ with $T_4 = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. Let $f : (X, T_4) \to (X, T_4)$ be an identity function. Now $f^{-1}(X) = X$; $f^{-1}(\phi) = \phi; f^{-1}(\{b\}) = \{b\}$; $f^{-1}(\{c\}) = \{c\}$; $f^{-1}(\{b, c\}) = \{b, c\}$. It can be easily seen that $int\{a\} = \phi$; $int\{a, b\} = \{b\}$; $cl\{b\} = \{a, b\}$; $int\{a, c\} = \{c\}$; $cl\{c\} = \{a, c\}$; $cl\{a, b\} = \{a, b\}$, $cl\{a\} = \{a\}$ and $cl\{a, c\} = \{a, c\}$ in $(X, T_4)$. In $(X, T_4)$ the only $\alpha$-open sets are $X$, $\phi$, $\{b\}$, $\{c\}$ and $\{b, c\}$. Then $f^{-1}(X) = X$, $f^{-1}(\phi) = \phi$, $f^{-1}(\{b\}) = \{b\}$, $f^{-1}(\{c\}) = \{c\}$ and $f^{-1}(\{b, c\}) = \{b, c\}$ are $\alpha$-open sets, semi-open sets and pre-open sets in $(X, T_4)$. Therefore $f$* **is $\alpha$-irresolute** *and hence $f$* **is semi $\alpha$-irresolute** *and $f$* **is pre $\alpha$-irresolute**. *We have $\{a, b\}$ and $\{a, c\}$ are semi-open sets in $(X, T_4)$, but $\{a, b\}$ and $\{a, c\}$ are not $\alpha$-open sets in $(X, T_4)$. Now $f^{-1}(\{a, b\}) = \{a, b\}$ and $f^{-1}(\{a, c\}) = \{a, c\}$, which are not $\alpha$-open sets in $(X, T_4)$. Therefore $f$* **is not strongly $\alpha$-continuous.** *Thus from this example we find (1) $f$ is $\alpha$-irresolute* **does not imply** *$f$ is strongly $\alpha$-continuous. (2) $f$ is semi $\alpha$-irresolute* **does not imply** *$f$ is strongly $\alpha$-continuous. (3) $f$ is pre $\alpha$-irresolute* **does not imply** *$f$ is strongly $\alpha$-continuous.*

**Example 7.1.5.** *Let $X = \{a, b, c\}$ with $T_4 = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. Let $f : (X, T_4) \to (X, T_4)$ be function defined by $f(a) = a$; $f(b) =$*

*$f(c) = c$. Now $f^{-1}(X) = X$; $f^{-1}(\phi) = \phi$; $f^{-1}(\{b\}) = \phi$; $f^{-1}(\{c\})$ $= \{b, c\}$; $f^{-1}(\{b, c\}) = \{b, c\}$. It can be easily seen that $int\{a\} = \phi$; $int\{a, b\} = \{b\}$; $cl\{b\} = \{a, b\}$; $int\{a, c\} = \{c\}$; $cl\{c\} = \{a, c\}$ in $(X, T_4)$. In $(X, T_4)$ the only $\alpha$-open sets are $X$, $\phi$, $\{b\}$, $\{c\}$ and $\{b, c\}$. Then $f^{-1}(\{X\}), f^{-1}(\phi), f^{-1}(\{b\})$, $f^{-1}(\{c\})$ and $f^{-1}(\{b, c\})$ are the semi-open sets and $\beta$-open sets in $(X, T_4)$. Thus $f$* ***is semi $\alpha$-irresolute*** *and hence $f$* ***is almost $\alpha$-irresolute***. *It is easily verified that $\{a, b\}$ and $\{a, c\}$ are semi-open sets in $(X, T_4)$, but $\{a, b\}$ and $\{a, c\}$ are not $\alpha$-open sets in $(X, T_4)$. Now $f^{-1}\{a, b\} = \{a\}$, it is not semi-open set in $(X, T_4)$. Therefore $f$* ***is not irresolute***. *Thus from this example we find (1) $f$ is semi $\alpha$-irresolute* ***does not imply*** *$f$ is irresolute. (2) $f$ is almost $\alpha$-irresolute* ***does not imply*** *$f$ is irresolute.*

**Example 7.1.6.** *Let $X = \{a, b, c\}$ with $T_5 = \{X, \phi, \{b, c\}\}$ and $T_4 = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. Let $f : (X, T_5) \to (X, T_4)$ be a function defined by $f(a) = a$; $f(b) = f(c) = b$. Now $f^{-1}(X) = X$; $f^{-1}(\phi) = \phi$; $f^{-1}(\{c\}) = \phi$; $f^{-1}(\{b\}) = \{b, c\}$; $f^{-1}(\{b, c\}) = \{b, c\}$. It can be easily seen that $int\{a\} = \phi$; $int\{a, b\} = \{b\}$; $cl\{b\} = \{a, b\}$; $int\{a, c\} = \{c\}$; $cl\{c\} = \{a, c\}$; $cl\{a, b\} = \{a, b\}$, $cl\{a\} = \{a\}$ and $cl\{a, c\} = \{a, c\}$ in $(X, T_4)$. $int\{a\} = int\{b\} = int\{c\} = \phi$; $int\{a, b\} = int\{a, c\} = \phi$; $cl\{a\} = \{a\}$; $cl\{b\} = cl\{c\} = cl\{a, b\} = cl\{a, c\} = X$ in $(X, T_5)$. In $(X, T_4)$ the only $\alpha$-open sets are $X$, $\phi$, $\{b\}, \{c\}$ and $\{b, c\}$. Then $f^{-1}(X)$, $f^{-1}(\phi), f^{-1}(\{c\}), f^{-1}(\{b\})$ and $f^{-1}(\{b, c\})$ are $\alpha$-open sets and also pre-open sets in $(X, T_5)$. Therefore $f$* ***is $\alpha$-irresolute*** *and also $f$* ***is pre $\alpha$-irresolute***. *In $(X, T_4)$ the only open sets are $X$, $\phi$, $\{b\}, \{c\}$ and $\{b, c\}$. Now $f^{-1}(X)$, $f^{-1}(\phi)$, $f^{-1}(\{c\}), f^{-1}(\{b\})$ and $f^{-1}(\{b, c\})$ are pre-open sets in $(X, T_5)$. This shows that $f$* ***is pre-continuous***. *We* *obtain that* $\{a, b\}$ *and* $\{a, c\}$ *are semi-open sets in* $(X, T_4)$, *but* $\{a, b\}$ $\{a, c\}$, $\{a\}$ *are not semi-open sets in* $(X, T_5)$. *Then* $f^{-1}\{a, c\} = \{a\}$, *which is not semi-open set in* $(X, T_5)$. *Therefore* $f$ ***is not irresolute.*** *Thus from this example we find (1)* $f$ *is* $\alpha$*-irresolute* ***does not imply*** $f$ *is irresolute. (2)* $f$ *is a pre* $\alpha$*-irresolute* ***does not imply*** $f$ *is irresolute. (3)* $f$ *is pre-continuous* ***does not imply*** $f$ *is irresolute.*

**Example 7.1.7.** *Let* $X = \{a, b, c\}$ *with* $T_2 = \{X, \phi, \{a, b\}\}$ *and* $T_6 = \{X, \phi, \{a\}\}$. *Let* $f : (X, T_2) \to (X, T_6)$ *be a function defined by* $f(a) = a = f(c)$; $f(b) = b$. *It can be easily seen that int* $\{a\}$ = *int* $\{b\}$ = *int* $\{c\}$ = *int* $\{a, c\}$ = *int* $\{b, c\} = \phi$; *cl*$\{a\}$ = *cl* $\{b\}$ = *cl*$\{a, c\}$ = *cl*$\{b, c\} = X$; *cl*$\{c\} = \{c\}$ *in* $(X, T_2)$. *int* $\{a, c\}$ = *int* $\{a, b\} = \{a\}$; *int* $\{b\}$ = *int* $\{c\}$ = *int* $\{b, c\} = \phi$ ; *cl*$\{a\}$ = *cl*$\{a, c\}$ = *cl*$\{a, b\} = X$; *cl*$\{b, c\} = \{b, c\}$; *cl*$\{b\} = \{b, c\}$ = *cl*$\{c\}$ *in* $(X, T_6)$. *Now* $f^{-1}(X) = X$, $f^{-1}(\phi) = \phi$ *and* $f^{-1}(\{a\}) = \{a, c\}$. *Then* $f^{-1}(X)$, $f^{-1}(\phi)$ *and* $f^{-1}(\{a\})$ *are* $\beta$*-open sets in* $(X, T_2)$. *Hence* $f$ ***is*** $\beta$***-continuous.*** *In* $(X, T_6)$ *the only* $\alpha$*-open sets are* $X$, $\phi$, $\{a\}$, $\{a, c\}$ *and* $\{a, b\}$ *and also the only semi-open sets are* $X$, $\phi$, $\{a\}$, $\{a, c\}$ *and* $\{a, b\}$. *Now* $f^{-1}(\{a\}) = \{a, c\}$, $f^{-1}(X) = X$; $f^{-1}(\phi) = \phi$; $f^{-1}(\{a, c\}) = \{a, c\}$ *and* $f^{-1}(\{a, b\}) = X$. *Then* $f^{-1}(\{a\}) = f^{-1}(\{a, c\}) = \{a, c\}$, *which is not a semi-open set in* $(X, T_2)$. *This proves that* $f$ ***is not semi*** $\alpha$***-irresolute*** *and also* $f$ ***is not irresolute.*** *Thus from this example we find* $f$ *is* $\beta$*-continuous* ***does not imply*** $f$ *is semi* $\alpha$*-irresolute and* $f$ *is irresolute.*

**Remark 7.1.1.** *From Examples 7.1.1 to 7.1.7 and the diagram given after the Definition 7.1.5, we have the following table of implications.*

Table

| $\Rightarrow$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $b$ | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $c$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| $d$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $e$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $f$ | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| $g$ | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| $h$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

1 represent "**implies**" and 0 represent "**does not imply**".

In the above table,

$a-$irresolute function

$b-$semi $\alpha$-irresolute function

$c-$almost $\alpha$-irresolute function

$d-\beta$-continuous function

$e-$strongly $\alpha$-continuous function

$f-\alpha$-irresolute function

$g-$pre $\alpha$-irresolute function

$h-$pre-continuous function

**Notation 7.1.1.** *The family of all $\alpha$-open sets in a topological space $(Y, S)$ is denote by $S^\alpha$ and it is shown in [56] that $S^\alpha$ is a topology for $Y$.*

**Theorem 7.1.1.** *The following assertions are equivalent for a function $f : (X, T) \rightarrow (Y, S)$ :*

*(1) $f$ is pre $\alpha$-irresolute;*

*(2) $f : (X, T) \rightarrow (Y, S^\alpha)$ is pre-continuous;*

*(3) For each $x \in X$ and each $\alpha$-open set $V$ in $Y$ containing $f(x)$, there exists a pre-open set $U$ containing $x$ such that $f(U) \subset V$;*

*(4) $f^{-1}(V) \subset int(cl(f^{-1}(V)))$ for every $\alpha$-open set $V$ in $Y$;*

*(5) $f^{-1}(C)$ is pre-closed in $X$ for every $\alpha$-closed set $C$ in $Y$;*

*(6) $cl(int(f^{-1}(B))) \subset f^{-1}(\alpha\text{-}cl(B))$ for every subset $B$ in $Y$;*

*(7) $f(cl(int(A))) \subset \alpha\text{-}cl(f(A))$ for every subset $A$ in $X$.*

***Proof***: (1) $\Rightarrow$ (2) Let $x \in X$ and let $V$ be any $\alpha$-open set in $Y$ containing $f(x)$. By definition 7.1.5, $f^{-1}(V)$ is pre-open set in $X$ containing $x$. Hence $f : (X, T) \to (Y, S^{\alpha})$ is pre-continuous.

(2) $\Rightarrow$ (3) Let $x \in X$ and let $V$ be any $\alpha$-open set in $Y$ containing $f(x)$. Put $U = f^{-1}(V)$, then by (2), $U$ is a pre-open subset of $X$ containing $x$ and $f(U) \subset V$.

(3) $\Rightarrow$ (4) Let $V$ be any $\alpha$-open set in $Y$ and $x \in f^{-1}(V)$. By (3), there exists a pre-open set $U$ in $X$ containing $x$ such that $f(U) \subset V$. Thus, we have $x \in U \subset int(cl(U)) \subset int(cl(f^{-1}(V)))$ and hence $f^{-1}(V) \subset int(cl(f^{-1}(V)))$.

(4) $\Rightarrow$ (5) Let $C$ be any $\alpha$-closed subset in $Y$. Put $V = Y - C$ , then $V$ is $\alpha$-open in $Y$. By (4), we obtain $f^{-1}(V) \subset int(cl(f^{-1}(V)))$ and hence $f^{-1}(C) = X - f^{-1}(Y - C) = X - f^{-1}(V)$ is pre-closed in $X$.

(5) $\Rightarrow$ (6) Let $B$ be any subset in $Y$. Since the set $\alpha\text{-}cl(B)$ is $\alpha$-closed subset in $Y$, by(5) $f^{-1}(\alpha\text{-}cl(B))$ is pre-closed in $X$ and hence $cl(\text{int}(f^{-1}(\alpha\text{-}cl(B))) \subset f^{-1}(\alpha\text{-}cl(B))$. Thus, we obtain $cl(\text{int}(f^{-1}(B)) \subset f^{-1}(\alpha\text{-}cl(B))$.

(6) $\Rightarrow$ (7) Let $A$ be any subset in $X$. By (6), we have $cl(\text{int}(A)) \subset cl(\text{int}(f^{-1}(f(A))) \subset f^{-1}(\alpha\text{-}cl\ f(A))$ and hence $f(cl(\text{int}(A))) \subset \alpha\text{-}cl(f(A))$.

(7) $\Rightarrow$ (1) Let $V$ be any $\alpha$-open set in $Y$. Since $f^{-1}(Y-V) = X - f^{-1}(V)$ is a subset of $X$, by (7), we have

$$f(cl(int(f^{-1}(Y-V))) \subset \alpha\text{-}cl(f(f^{-1}(Y-V))) \subset \alpha\text{-}cl(Y-V) = Y - \alpha\text{-}int(V) = Y - V \tag{7.1.1}$$

and hence $X-\ int\ (cl\ (f^{-1}(V))) = cl\ (\text{int}\ (X - f^{-1}(V))) = cl(\text{int}(f^{-1}(Y-V))) \subset f^{-1}(f(\ cl\ (int\ (f^{-1}(Y-V)))))$ (using (7.1.1)) $\subset f^{-1}(Y-V) = X - f^{-1}(V)$. Therefore, we obtain $f^{-1}(V) \subset int(cl(f^{-1}(V)))$ and hence $f^{-1}(V)$ is pre-open in $X$. Thus $f$ is pre $\alpha$ -irresolute. □

The following four lemmas taken from [18, 59, 43, 44] are given here for convenience of the reader.

**Lemma 7.1.1.** *([18], [59]): Let $\{X_\alpha : \alpha \in A\}$ be any family of topological spaces and $A = \prod_{j=1}^{n} A_{\alpha_j} \times \prod_{\beta \neq \alpha_j} X_\beta$ a non-empty subset of the product space $\Pi\ X_\alpha$, where $n$ is a positive integer. Then $A$ is a $\alpha$-open ( pre-open, resp.) in $\Pi\ X_\alpha$ if and only if $A_{\alpha_j}$ is $\alpha$-open ( pre-open, resp.) in $X_{\alpha_j}$ for each $j (j = 1, 2 \cdots n)$.*

**Lemma 7.1.2.** *[44]: If $f : X \to Y$ is $\alpha$-continuous and pre-open, then the inverse image of each $\alpha$-open set in $Y$ is an $\alpha$-open set.*

**Notation 7.1.2.** *If $X$ is any topological space, then $PO(X)$ stand for the* *collection of all pre-open sets in $X$ and $SO(X)$ stand for the collection of all semi-open sets in $X$.*

**Lemma 7.1.3.** *[43]: If $V \in PO(X)$ and $U \in SO(X)$, then $U \cap V \in PO(U)$.*

**Lemma 7.1.4.** *[43]: If $U \in PO(X)$ and $V \in PO(U)$, then $V \in PO(X)$.*

**Theorem 7.1.2.** *Let $f : (X, T) \to (Y, S)$ be a mapping from a topological space $X$ to another topological space $Y$. If the graph $g : X \to X \times Y$ of $f$ is pre $\alpha$-irresolute, then $f$ is also pre $\alpha$-irresolute.*

***Proof:*** Let $x \in X$ and $V$ be any $\alpha$-open set in $Y$ containing $f(x)$. Then, by Lemma 7.1.1, the set $X \times V$ is $\alpha$-open in $X \times Y$ containing $g(x)$. Since $g$ is pre $\alpha$-irresolute, there exists a pre-open set $U$ of $X$ containing $x$ such that $g(U) \subset X \times V$ and hence $f(U) \subset V$. Thus, $f$ is pre $\alpha$-irresolute. □

**Theorem 7.1.3.** *If a function $f : X \to \Pi\, Y_\lambda$, is pre $\alpha$-irresolute, then $P_\lambda \circ f : X \to Y_\lambda$ is pre $\alpha$-irresolute for each $\lambda \in \Delta$, where $P_\lambda$ is the projection map of $\Pi\, Y_\lambda$ onto $Y_\lambda$.*

***Proof:*** Let $V_\lambda$ be any $\alpha$-open set in $Y_\lambda$. Since $P_\lambda$ is continuous and open, it is $\alpha$-irresolute by Lemma 7.1.2 and hence $P_\lambda^{-1}(V_\lambda)$ is $\alpha$-open in $\Pi\, Y_\lambda$. Since the function $f$ is pre $\alpha$-irresolute, we obtain $f^{-1}(P_\lambda^{-1}(V_\lambda)) = (P_\lambda \circ f)^{-1}(V_\lambda)$ is pre-open in $X$. Hence $P_\lambda \circ f$ is pre $\alpha$-irresolute for each $\lambda \in \Delta$. □

**Theorem 7.1.4.** *If the product function $f = \Pi f_\lambda : \Pi\, X_\lambda \to \Pi\, Y_\lambda$ is pre $\alpha$-irresolute, then $f_\lambda : X_\lambda \to Y_\lambda$ is pre $\alpha$-irresolute for each $\lambda \in \Delta$.*

***Proof:*** Let $\lambda_0 \in \Delta$ be an arbitrary fixed index and $V_{\lambda_0}$ be any $\alpha$-open set in $Y_{\lambda_0}$. Then $\Pi\, Y_\gamma \times V_{\lambda_0}$ is $\alpha$-open in $\Pi\, Y_\lambda$ by Lemma 7.1.1, where $\lambda_0 \neq \gamma \in \Delta$. Since $f$ is pre $\alpha$-irresolute, then $f^{-1}$ $(\Pi\, Y_\gamma \times V_{\lambda_0})$ $= \Pi\, X_\gamma \times f_{\lambda_0}^{-1}(V_{\lambda_0})$ is pre-open in $\Pi\, X_\lambda$ and hence, by Lemma 7.1.1, $f_{\lambda_0}^{-1}(V_{\lambda_0})$ is pre-open in $X_{\lambda_0}$. This proves that $f_{\lambda_0}$ is pre $\alpha$-irresolute. □

**Theorem 7.1.5.** *If $f : (X, T) \to (Y, S)$ is pre $\alpha$-irresolute and $A$ is a semi-open subset in $X$, then the restriction $f\,|A : A \to Y$ is pre $\alpha$-irresolute.*

***Proof:*** Let $V$ be any $\alpha$-open set in $Y$. Since $f$ is pre $\alpha$-irresolute, $f^{-1}(V)$ is pre-open in $X$. Then, by Lemma 7.1.3, we obtain $(f|A)^{-1}(V)$ $= A \cap f^{-1}(V) \in PO(A)$ because $A$ is semi-open subset in $X$. Hence, $f|A$ is pre $\alpha$-irresolute. □

**Theorem 7.1.6.** *Let $f : (X, T) \to (Y, S)$ be a function and $\{A_\lambda : \lambda \in \Delta\}$ be a cover of $X$ by pre-open sets of $(X, T)$. Then $f$ is pre $\alpha$-irresolute if $f \mid A_\lambda : A_\lambda \to Y$ is pre $\alpha$-irresolute for each $\lambda \in \Delta$.*

***Proof:*** Let $V$ be any $\alpha$-open set in $Y$. Since $f\,|A_\lambda$ is pre $\alpha$-irresolute, then $(f\,|A_\lambda)^{-1}$ $(V)$ is pre-open in $A_\lambda$. By Lemma 7.1.4 and since $A_\lambda \in PO(X)$, then $(f\,|A_\lambda)^{-1}$ $(V) \in PO(X)$ for each $\lambda \in \Delta$. Therefore, $f^{-1}$ $(V) = X \cap f^{-1}(V) = \cup\{A_\lambda \cap f^{-1}(V) : \lambda \in \Delta\} =$ $\cup$ $\{(f\,|A_\lambda)^{-1}$ $(V) : \lambda \in \Delta\}$ is pre-open in $X$ because the union of pre-open sets is pre-open set. Hence $f$ is pre $\alpha$-irresolute. □

**Theorem 7.1.7.** *If a function $f : (X, T) \to (Y, S)$ is pre $\alpha$-irresolute, then $f^{-1}(B)$ is pre-closed in $X$ for any nowhere dense set $B$ in $Y$.*

***Proof:*** Let $B$ be any nowhere dense subset in $Y$. Then $Y - B$ is a $\alpha$-open set in $Y$. Since $f$ is pre $\alpha$-irresolute, $f^{-1}(Y - B) = X - f^{-1}(B)$ is pre-open in $X$ and hence $f^{-1}(B)$ is pre-closed in $X$. □

**Theorem 7.1.8.** *A function $f : (X, T) \to (Y, S)$ is pre $\alpha$-irresolute if and only if, for each $y \in Y$ and each open set $V$ in $Y$ such that $y \in int(cl(V))$, the inverse image of $V \cup \{y\}$ is pre-open in $X$.*

***Proof:*** Necessity : Since $V \subset V \cup \{y\} \subset \text{int}(cl(V))$, then $V \cup\{y\}$ is an $\alpha$-open set in $Y$. Then $f^{-1}(V \cup \{y\})$ is pre-open in $X$ because $f$ is pre $\alpha$-irresolute.
Sufficiency: Let $V$ be an $\alpha$-open set in $Y$. Then, there exists an open set $B$ of $Y$ such that $B \subset V \subset \text{int}(cl(B))$. By hypothesis, $f^{-1}(B \cup \{y\})$ is pre-open in $X$ for each $y \in V$. This shows that $f^{-1}(V) = \cup\{f^{-1}(B \cup \{y\}) : y \in V\}$ is pre-open in $X$ and hence $f$ is pre $\alpha$-irresolute. □

**Theorem 7.1.9.** *The following hold for functions $f : X \to Y$ and $g : Y \to Z$:*

*(1) If $f$ is pre-continuous and $g$ is strongly $\alpha$-irresolute, then $g \circ f$ is pre $\alpha$-irresolute.*

*(2) If $f$ is pre $\alpha$-irresolute and $g$ is $\alpha$-irresolute, then $g \circ f$ is pre $\alpha$-irresolute.*

***Proof:*** $(a)$ Let $W$ be any $\alpha$-open subset of Z. Since $g$ is strongly $\alpha$-irresolute, $g^{-1}$ (W) is open in $Y$. Now $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)))$. Since $f$ is pre-continuous and $g^{-1}$ (W) is open in $Y$, we conclude that $f^{-1}(g^{-1}(W)))$ is pre-open set in $X$. Hence $(g \circ f)$ is pre $\alpha$-irresolute.

$(b)$ Let $W$ be any $\alpha$-open set of $Z$. Since $g$ is $\alpha$-irresolute, $g^{-1}(W)$ is $\alpha$-open in $Y$. Now $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Since $f$ is pre $\alpha$-irresolute and $g^{-1}(W)$ is $\alpha$-open in $Y$, then $f^{-1}(g^{-1}(W))$ is pre-open in $X$ and hence $(g \circ f)$ is pre $\alpha$-irresolute. □

## 7.2 Somewhat pre $\alpha$-irresolute functions

In this section we introduce the concept of somewhat pre $\alpha$-irresolute function and discuss some of its properties.

Motivated by the concept of somewhat continuous functions introduced and studied in [34] we now define

**Definition 7.2.1.** A function $f : (X, T) \to (Y, S)$ is said to be somewhat pre $\alpha$-irresolute if for any non-empty $\alpha$-open set $V$ of $Y$ and $f^{-1}(V) \neq \phi$, then there exists a pre-open set $U \neq \phi$ in $X$ such that $U \subset f^{-1}(V)$.

Clearly every pre $\alpha$-irresolute function is somewhat pre $\alpha$-irresolute; but however the converse is not true as the following example shows:-

**Example 7.2.1.** *Let $X = \{a, b, c\}$, $T_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, T_1) \to (X, T_1)$ be the function defined by $f(a) = f(c) = a$; $f(b) = b$. It can be easily seen that cl $\{a, c\} = \{a, c\}$; int $\{a, c\} = \{a\}$ in $(X, T_1)$. Now $f^{-1}(\phi) = \phi$; $f^{-1}(X) = X$; $f^{-1}\{a\} = \{a, c\}$, $f^{-1}\{b\} = \{b\}$; $f^{-1}\{a, b\} = X$. In $(X, T_1)$ the only non-empty $\alpha$-open set are $X, \{a\}, \{b\}$, and $\{a, b\}$. Also $X, \{a\}, \{b\}, \{a, b\}$ are non-empty pre-open set in $(X, T_1)$. Then $X \subseteq f^{-1}(X) = X$, $\{a\} \subseteq f^{-1}\{a\} = \{a, c\}$, $\{b\} \subseteq f^{-1}\{b\} = \{b\}$ and $\{a, b\} \subseteq f^{-1}\{a, b\} = X$. This proves that* ***$f$ is somewhat pre $\alpha$-irresolute.*** *Now $f^{-1}\{a\} = \{a, c\}$, which is not a pre-open set in $(X, T_1)$. This shows that $f$* ***is not pre $\alpha$-irresolute.***

If $f : (X, T) \to (Y, S)$ is pre $\alpha$-irresolute and $g : (Y, S) \to (Z, Q)$ is somewhat pre $\alpha$-irresolute, then $g \circ f : (X, T) \to$ (Z, Q) is not necessarily somewhat pre $\alpha$-irresolute. The following example serves this purpose.

**Example 7.2.2.** *Let $X = \{a, b, c\}$ with $T = \{\phi, X, \{c\}\}$; $S = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$; $Q = \{X, \phi, \{b\}\}$. Define $f : (X, T) \to (X, S)$ by $f(a) = a$; $f(b) = c = f(c)$. It can be easily seen that int $\{a\}$ = int $\{b\}$ = int $\{a,b\} = \phi$; int $\{b,c\} = \{c\}$ = int$\{a, c\}$; cl$\{c\}$ = cl$\{b,c\}$ = cl$\{a,c\} = X$; cl$\{a\}$ = cl$\{b\}$ = cl$\{a, b\} = \{a, b\}$ in $(X,T)$. $\{a,c\}$ and $\{b, c\}$ are pre-open sets in $(X,T)$. In $(X, S)$ the only $\alpha$-open sets are $X, \phi, \{b\}, \{c\}$ and $\{b, c\}$. Now $f^{-1}\{\phi\} = \phi$; $f^{-1}(X) = X$; $f^{-1}\{b\} = \phi$; $f^{-1}\{b, c\} = \{b, c\} = f^{-1}\{c\}$. Then $f^{-1}\{\phi\}, f^{-1}(X), f^{-1}\{b\}, f^{-1}\{b, c\}$ and $f^{-1}\{c\}$ are pre-open sets in $(X, T)$. Hence f* ***is pre $\alpha$-irresolute function*** *from $(X, T)$ to $(X, S)$. Let g be the identity function from $(X, S)$ onto $(X, Q)$. In $(X, Q)$ the only $\alpha$-open sets are $X, \phi, \{b\}, \{a, b\}$ and $\{b, c\}$ and also the only pre-open sets are $X, \phi, \{b\}, \{a, b\}$ and $\{b, c\}$. Now $g^{-1}\{X\} = X$; $g^{-1}\{\phi\} = \phi$; $g^{-1}\{b\} = \{b\}$; $g^{-1}\{a, b\} = \{a, b\}$; $g^{-1}\{b, c\} = \{b, c\}$. Then $g^{-1}\{a, b\} = \{a, b\}$, which is not a pre-open set in $(X, S)$. Hence g* ***is not pre $\alpha$-irresolute function.*** *The pre-open set $\{b\}$ in $(X, S)$ is contained in $g^{-1}\{X\}$; $g^{-1}\{b\}$; $g^{-1}\{a, b\}$; $g^{-1}\{b, c\}$. This proves that g* ***is somewhat pre $\alpha$-irresolute function*** *from $(X, S)$ to $(X, Q)$.*

*Now consider the function $(g \circ f) : (X, T) \to (X, Q)$. Then $(g \circ f)^{-1}(X) = X$; $(g \circ f)^{-1}\{b\} = \phi$; $(g \circ f)^{-1}\{a, b\} = \{a\}$ and $(g \circ f)^{-1}\{b, c\} = \{b, c\}$. But $(g \circ f)^{-1}\{a, b\} = \{a\}$ and there is no nonempty pre-open set in $(X, T)$ such that it is contained in*

$(g \circ f)^{-1}\{a,b\} = \{a\}$. *Therefore* $(g \circ f)$ ***is not somewhat pre $\alpha$-irresolute function.***

**Definition 7.2.2.** A subset $V$ in a topological space $(X, T)$ is called **pre-dense** if there exists no pre-closed set $U$ such that $V \subset U \subset X$.

**Definition 7.2.3.** A subset $V$ in a topological space $(X, T)$ is called **$\alpha$-dense** if there exists no $\alpha$-closed set $U$ such that $V \subset U \subset X$.

**Theorem 7.2.1.** *If $f : (X, T) \to (Y, S)$ is a function, then the following assertions are equivalent.*

*(1) $f$ is somewhat pre $\alpha$-irresolute*

*(2) If $C$ is $\alpha$-closed subset in $Y$ such that $f^{-1}(C) \neq X$, then there is a proper pre-closed subset $D$ in $X$ such that $D \supset f^{-1}(C)$*

*(3) If $M$ is a pre-dense subset of $X$, then $f(M)$ is a $\alpha$-dense subset in $Y$.*

***Proof:*** (1) $\Rightarrow$ (2) Suppose $f$ is somewhat pre $\alpha$-irresolute function and $C$ is a $\alpha$-closed set in $Y$ such that $f^{-1}(C) \neq X$. Therefore clearly $X - C$ is $\alpha$-open in $Y$ and $f^{-1}(X - C) = X - f^{-1}(C) \neq \phi$ (Since $f^{-1}(C) \neq X$). By (1), there exists a non-empty pre-open set $K$ in $X$ such that $K \subseteq f^{-1}(X - C)$. That is $K \subseteq X - f^{-1}(C)$ which implies that $f^{-1}(C) \subseteq X - K$. Clearly $X - K$ is a pre-closed set and taking $D = X - K$, we have therefore $f^{-1}(C) \subseteq D$. Thus we find that (1) $\Rightarrow$ (2) is proved.

(2) $\Rightarrow$ (3) Let $M$ be a pre-dense set in $X$ and suppose $f(M)$ is not $\alpha$-dense in $Y$. Then there exists a $\alpha$-closed set A (say) in $Y$ such that $f(M) \subset A \subset Y$ - - -(1). Since $A \subset Y$, $f^{-1}(A) \neq X$ and so by (2) there

exists a proper pre-closed set $B$ such that $B \supset f^{-1}(A)$. But $B \supset f^{-1}(A) \supset f^{-1}(f(M)) \supset M$ (from (1)). That is, there exists a proper pre-closed set $B$ such that $B \supset M$, which is a contradiction to the assumption on $M$. Therefore (2) $\Rightarrow$ (3) is proved.

(3) $\Rightarrow$ (1) Suppose $C$ is $\alpha$-open set in $Y$ and $f^{-1}(C) \neq \phi$ and therefore $C \neq \phi$. Suppose there exists no pre-open set $D$ in $X$ such that $D \subseteq f^{-1}(C)$. Then $X - f^{-1}(C)$ is subset in $X$ such that there is no pre closed set $B$ in $X$ with $X - f^{-1}(C) \subset B \subset X$ ( otherwise $X - f^{-1}(C) \subset B \Rightarrow X - B \subset f^{-1}(C)$ and $X - B$ is pre-open, which a contradiction). This means that $X - f^{-1}(C)$ is pre-dense in $X$. Then by (3), $f(X - f^{-1}(C))$ is $\alpha$-dense in $Y$. But $f\ (X - f^{-1}(C)) = f(f^{-1}(X - C)) \subset X - C \subset X$ (since $C \neq \phi$). This is a contradiction to the fact that $(C \neq \phi) f(X- f^{-1}(C))$ is pre-dense in $X$. Therefore, there exists a pre-open set $D$ in $X$ such that $D \subseteq f^{-1}(C)$. Hence $f$ is somewhat pre $\alpha$-irresolute function. □

**********