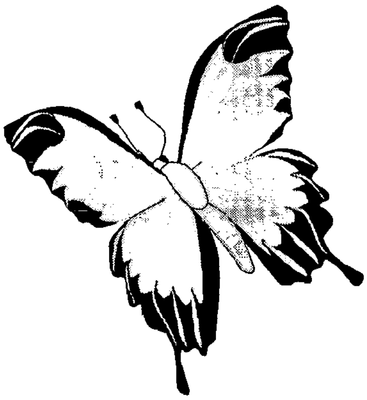

Chapter 6

CHAPTER 6

The complementary fuzzy topology

In this chapter we introduce the concept of complementary fuzzy topology and this is an extension of Norman Levine's work in [65] to fuzzy setting. We discuss and study several interesting properties of complementary fuzzy topology.

6.1 Definitions

Motivated by the classical concept introduced in [65] we now define:

Definition 6.1.1. Let (X, T) be any fuzzy topological space and let $\mathfrak{B}^* = \{\text{int}\,\lambda \mid (1 - \lambda) \in T\}$. It is clear that $\mathfrak{B}^*$ is a base for a fuzzy topology T^* on X and $T^* \subseteq T$. We call T^* **the complementary fuzzy topology**.

cl and cl* will denote the closure operators and int and int* will denote the interior operators relative to T and T^* respectively.

6.2 Basic Properties

In this section we investigate and study several interesting properties by making use of the interior and closure operators.

Proposition 6.2.1. *Let (X, T) be a fuzzy topological space. Then $(T^*)^* = T^*$.*

Proof: It is simple to prove. □

Proposition 6.2.2. *Let (X, T) be any fuzzy topological space and let T^* be the complementary fuzzy topology for X. If $\lambda \in I^X$, then int cl $\lambda \leq$ int* cl* λ.*

Proof: It is easy to see that

$$\text{int cl } \lambda \leq \text{cl } \lambda \leq \text{cl}^* \lambda. \tag{6.2.1}$$

But

$$\text{int cl}\lambda \in \mathfrak{B}^* \subset T^*. \tag{6.2.2}$$

Therefore int* (int cl λ) $\leq$ int* cl*λ (by (6.2.1)) and hence int cl $\lambda \leq$ int* cl*λ (by (6.2.2)). Hence the proposition. □

Proposition 6.2.3. *Let (X, T) be any fuzzy topological space. Then T^* is indiscrete if and only if every fuzzy set λ in X is either T-dense or nowhere T-dense in X.*

Proof: Now T^* is indiscrete $\Leftrightarrow T^* = \{0, 1\} \Leftrightarrow \mathfrak{B}^* = \{0, 1\} \Leftrightarrow$ For every T-closed set λ, int $\lambda = 0$ or int $\lambda = 1 \Leftrightarrow$ for any fuzzy set λ in X, we have int cl $\lambda = 0$ or int cl $\lambda = 1 \Leftrightarrow$ every fuzzy set λ in X is such that λ is nowhere T-dense or T-dense. □

Remark 6.2.1. *It is of interest to note that if T_1 and T_2 are fuzzy topologies for X such that $T_1 \subset T_2$, then in general $T_1^* \not\subset T_2^*$. The following example serves the purpose.*

Example 6.2.1. *Let $X = \{a, b, c\}$. Define $T_1 = \{0, \lambda, \mu, \rho, 1\}$ and $T_2 = \{0, \lambda, \mu, \rho, \sigma, 1\}$ where $\lambda : X \longrightarrow [0,1]$ is such that $\lambda(a) = 1, \lambda(b) = 0, \lambda(c) = 0; \mu : X \longrightarrow [0,1]$ is such that $\mu(a) = \mu(b) = 0, \mu(c) = 1; \rho : X \longrightarrow [0,1]$ is such that $\rho(a) = \rho(c) = 1, \rho(b) = 0$; and $\sigma : X \longrightarrow [0,1]$ is such that $\sigma(a) = \sigma(b) = 1, \sigma(c) = 0$. Clearly T_1 and T_2 are fuzzy topologies on X. It can be easily verified that $T_1 \subset T_2$ but $T_1^* \not\subset T_2^*$.*

Corollary 6.2.1. *Let (X, T) be a fuzzy topological space and suppose λ be any fuzzy set in X. If λ is T^*-nowhere dense, then λ is T-nowhere dense.*

Proof. Follows from Proposition 6.2.2.

Proposition 6.2.4. *(X, T) is fuzzy connected if and only if (X, T^*) is fuzzy connected.*

Proof: Suppose (X, T) is fuzzy connected. Since $T^* \subset T$, it follows that (X, T^*) is fuzzy connected. Conversely, suppose that (X, T^*) is fuzzy connected. If (X, T) is not fuzzy connected, then there exists a fuzzy set λ such that it is T-fuzzy open and T-fuzzy closed. It is easy to see that λ is both T^*-fuzzy open and T^*-fuzzy closed which is a contradiction to the assumption. Therefore (X, T) is fuzzy connected. □

Proposition 6.2.5. *Let* (X,T) *and* (Y,S) *be fuzzy topological spaces and suppose that* $f : X \to Y$ *is fuzzy continuous and fuzzy open. Then for each fuzzy set* λ *in* Y*, we have*

(1) f^{-1} *(int* λ*)* $=$ *int* $f^{-1}(\lambda)$

(2) f^{-1}*(cl* λ*)* $=$ *cl* $f^{-1}(\lambda)$

(3) f^{-1}*(int cl* λ*)* $=$ *int cl* $f^{-1}(\lambda)$

Proof: (1) Now int $\lambda \leq \lambda$ and also $f^{-1}(\text{int } \lambda) \leq f^{-1}(\lambda)$ implies that

$$f^{-1}(\text{int}\lambda) \leq \text{int} f^{-1}(\lambda).$$

Similarly we can show that

$$\text{int} f^{-1}(\lambda) \leq f^{-1}(\text{int}\lambda).$$

Since int $f^{-1}(\lambda) = f^{-1}(\text{int } \lambda)$. Therefore (1) is verified.

(2) $f^{-1}(\text{cl } \lambda) = f^{-1}(1 - \text{int}(1 - \lambda)) = 1 - f^{-1}(\text{int}(1 - \lambda)) = 1 - \text{int}(1 - f^{-1}(\lambda)) = \text{cl } f^{-1}(\lambda)$. Hence (2) is verified.

(3) Follows from (1) and (2). □

Proposition 6.2.6. *Let* $f : (X,T) \to (Y,S)$ *be fuzzy continuous and fuzzy open map. Then* $f : (X,T^*) \to (Y,S^*)$ *is fuzzy continuous.*

Proof: Let λ be any S-closed fuzzy set in Y. Then $f^{-1}(\text{int } \lambda) = \text{int } f^{-1}(\lambda)$ by Proposition 6.2.5 and so $f^{-1}(\text{int } \lambda) \in T^*$. □

Remark 6.2.2. *If* $f : (X,T) \to (Y,S)$ *is fuzzy continuous and fuzzy open relative to* T *and* S*, then in general* f *need not be fuzzy open relative to* T^* *and* S^**. The following example serves the purpose.*

Example 6.2.2. *Let* $X = \{a, b\}$. *Define* $T = \{0, 1, \lambda, \mu\}$ *where* $\lambda(a) = 1, \lambda(b) = 0; \mu(a) = 0, \mu(b) = 1$. *Then* T *is fuzzy topology on* X. *Clearly* $T = T^*$. *Let* $Y = \{a, b\}$ *and* $S = \{0, \lambda, 1\}$ *is fuzzy topology on* Y, *then* $S^* = \{0, 1\}$. *If* $f(x) = a$ *for all* $x \in X$, *then* f ***is fuzzy continuous and fuzzy open relative to*** T ***and*** S. *But* f ***is not fuzzy open relative to*** T^* ***and*** S^*.

Proposition 6.2.7. *Let* $(X_\alpha, T_\alpha)_{\alpha\in\Delta}$ *be a non-empty family of fuzzy topological spaces and let* (X, T) *denote the fuzzy product space. If* $(X, T^\#)$ *denote the fuzzy product of the family* $(X_\alpha, T^*_\alpha)_{\alpha\in\Delta}$, *then* $T^\# = T^*$.

Proof: We shall first show that $T^\# \subseteq T^*$. Now for each $\alpha \in \Delta$, the projection map $p_\alpha : X \rightarrow X_\alpha$ is fuzzy continuous and fuzzy open relative to T and T_α and so by Proposition 6.2.6, p_α is fuzzy continuous relative to T^* and T^*_α. Since $T^\#$ is the smallest fuzzy topology for X such that $p_\alpha : X \rightarrow X_\alpha$ is fuzzy continuous relative to T^*_α for all $\alpha \in \Delta$, it follows that

$$T^\# \subseteq T^*.$$

Conversely, let $\lambda^* \in T^*$. Then $\lambda^* = \text{int}\ \mu, \mu$ is T-closed. Therefore we have,

$$\begin{aligned} \lambda^* &= \text{int}\mu = \vee\{\wedge p^{-1}_{\alpha_j}[\text{int cl}\lambda_{\alpha_j}]\} \\ &= \vee\ \text{intcl}\{\wedge p^{-1}_{\alpha_j}[\lambda_{\alpha_j}]\}. \end{aligned}$$

But int cl $\lambda_{\alpha_j} \in T^*_{\alpha_j}$ and so $p^{-1}_{\alpha_j}$ [int cl λ_{α_j}] $\in T^\#$. This implies that $\lambda^* \in T^\#$. Therefore $T^* \subseteq T^\#$. Hence $T^* = T^\#$. Hence the proposition.

□

Proposition 6.2.8. *Let (X, T) be any fuzzy topological space and T^* be the complementary fuzzy topology. Then (X, T) is fuzzy Hausdorff if and only if (X, T^*) is fuzzy Hausdorff.*

Proof: If (X, T^*) is fuzzy Hausdorff then it follows easily that (X, T) is also fuzzy Hausdorff since $T^* \subseteq T$. To prove the converse, suppose (X, T) is fuzzy Hausdorff. Let p and q be any two distinct fuzzy points in X. Then there exists fuzzy open sets $\lambda, \mu \in T$ such that

$$p \in \lambda, q \in \mu \text{ and } \lambda \leq 1 - \mu. \tag{6.2.3}$$

Hence it follows that $p \notin \text{cl } \mu$ and $q \notin \text{cl } \lambda$. Now put $\lambda^* = \text{int } (1 - \mu) \wedge$ int (cl λ) and $\mu^* = \text{int } (1 - \lambda) \wedge$ int (cl μ). Clearly $\lambda^*, \mu^* \in T^*$ and $1 - \mu^* = 1 - [\text{int } (1 - \lambda) \wedge \text{ int cl } \mu] = [1 - \text{int } (1 - \lambda)] \vee [1 - \text{ int cl } \mu] =$ cl $\lambda \vee$ cl $(1 - \mu) =$ cl $(1 - \mu) \geq$ int $(1 - \mu) \geq \lambda^*$[since $\lambda \leq 1 - \mu$] by (6.2.3). That is, $1 - \mu^* \geq \lambda^*$. It is easy to see that $p \in \lambda^*, q \in \mu^*$. This shows that (X, T^*) is fuzzy Hausdorff. Hence the proposition. □

Corollary 6.2.2. *A fuzzy topological space (X, T) is fuzzy Hausdorff if and only if for two distinct fuzzy points p and q of X, there exists fuzzy sets $\lambda, \mu \in T$ such that*

(1) $1 - p \notin cl\, \lambda, 1 - q \notin cl\, \mu$

(2) $cl\, \lambda + cl\, \mu \leq 1.$

Proof: Let p and q be any two distinct fuzzy points and $\lambda, \mu \in T$ be such that $1 - p \notin$ cl $\lambda, 1 - q \notin$ cl μ and cl $\lambda +$ cl $\mu \leq 1$. Now $1 - p \notin$ cl λ implies $1 - p >$ clλ. That is $p < 1-$cl λ implies that $p \in 1-$ cl λ. Similarly one can see that $q \in 1-$ cl μ. Now clearly $1-$ cl λ and $1-$ cl μ

are in T^* and from the assumption through cl $\lambda+$ cl $\mu \leq 1$, it follows that $1-$ cl $\lambda \leq$ cl μ or $1-$cl $\mu \leq$ cl λ. This proves that (X, T^*) is fuzzy Hausdorff. Therefore by proposition 6.2.8, (X, T) is fuzzy Hausdorff. Conversely, suppose that (X, T) is fuzzy Hausdorff. Let p and q be two distinct fuzzy points of X. Then by proposition 6.2.8, (X, T^*) is fuzzy Hausdorff. Therefore there exists fuzzy set λ and μ which are T-fuzzy closed and for which $p \in$int $\lambda, q \in$ int μ and $1-$ int $\lambda \leq$ int μ. Then $1-p \notin$ cl $(1-\lambda)$ and $1-q \notin$ cl $(1-\mu)$. For $p \in$ int λ implies $p \leq$ int λ. That is, $-p \geq -$int λ. Now $1-p \geq 1-$int $\lambda =$ cl $(1-\lambda)$ implies that $1-p \notin$ cl $(1-\lambda)$. Similarly $1-q \notin$ cl $(1-\mu)$. Now cl $(1-\lambda)+$ cl $(1-\mu) = 1-$ int $\lambda + 1-$int $\mu \leq$ int $\mu + 1-$int $\mu = 1$. This proves the converse. Hence the corollary. □

Proposition 6.2.9. *Let T^* be the complementary fuzzy topology in (X, T). Then $T = T^*$ if and only if for each fuzzy point p such that $1-p \notin \lambda$, where λ is T-fuzzy closed, there exists fuzzy sets μ_1 and μ_2 in T such that $p \in \mu_1, \lambda \leq 1-\mu_2$ and $\mu_1 \leq 1-\mu_2$.*

Proof: Assume $T = T^*$. Suppose p is a fuzzy point such that $1-p \notin \lambda$ where λ is T-fuzzy closed. Then $p < 1-\lambda \in T = T^*$. Hence there exists a T-fuzzy closed set σ such that $p \in$ int $\sigma \leq 1-\lambda$. Put $\mu_1 =$ int σ and $\mu_2 = 1-\sigma$. Then $p \in \mu_1, \lambda \leq 1-$ int σ and $1-\mu_2 = 1-(1-\sigma) = \sigma \geq$ int $\sigma = \mu_1$. Therefore $1-\mu_2 \geq \mu_1$. Sufficiency, we will show that $T \subseteq T^*$. Let $p \in \sigma \in T$ and let $\lambda \in$cl(μ). Then there exists fuzzy sets μ_1 and μ_2 in T such that $p \in \mu_1, \lambda \leq 1-\mu_2$ and $\mu_1 \leq 1-\mu_2$. But $p \in 1-$ cl$(\mu_2) =$ int $(1-\mu_2) \leq$ int $\lambda \leq$ int cl(μ). Since $1-$cl(μ_2)=int $(1-\mu_2) \in \mathfrak{B}^*$. Therefore $p \in$ int cl $(\mu) \in \mathfrak{B}^* \subset T^*$, it follows that $p \in T^*$. Therefore $T \subseteq T^*$. Then $T = T^*$. □