Chapter 3

Gram Matrices and Stirling Numbers of a Class of Diagram Algebras
In this chapter, we define Gram matrices of signed partition algebras, algebra of \( \mathbb{Z}_2 \)-relations and partition algebras. We show that the Gram matrices are similar to the matrices which is a direct sum of block sub matrices. In this connection, \((s_1, s_2, r_1, r_2, p_1, p_2)\)-Stirling numbers of the second kind are introduced. We also establish that the algebra of \( \mathbb{Z}_2 \)-relations and signed partition algebras are “generically semisimple” over a field of characteristic zero.

3.1 Gram Matrices and \((s_1, s_2, r_1, r_2, p_1, p_2)\)-Stirling Numbers

In this section, we deal with the Gram matrices \( G_{2s_1+s_2} \), \( \tilde{G}_{2s_1+s_2} \) and \( G_s \) of the algebra of \( \mathbb{Z}_2 \)-relations, signed partition algebras and partition algebras respectively. We prove that the Gram matrices \( G_{2s_1+s_2} \) and \( \tilde{G}_{2s_1+s_2} \) are similar to matrices \( G'_{2s_1+s_2} \) which is a direct sum of block sub matrices \( A'_{2r_1+r_2,2r_1+r_2} \) and \( \tilde{A}'_{2r_1+r_2,2r_1+r_2} \) of sizes \( f_{2s_1+s_2}^{2r_1+r_2} \) and \( \tilde{f}_{2s_1+s_2}^{2r_1+r_2} \) respectively. The diagonal entries of the matrices \( A'_{2r_1+r_2,2r_1+r_2} \) and \( \tilde{A}'_{2r_1+r_2,2r_1+r_2} \) are the same and the diagonal element is a product of \( r_1 \) quadratic polynomials and \( r_2 \) linear polynomials which could help in determining the roots of the determinant of the Gram matrix. In this connection, \((s_1, s_2, r_1, r_2, p_1, p_2)\)-Stirling numbers of the second kind for the algebra of \( \mathbb{Z}_2 \)-relations and signed partition algebras are introduced and their identities were established. Similarly, we have also established that the Gram matrix \( G_s \) of a partition algebra is similar to a matrix \( G'_s \) which is a direct sum of block matrices \( A'_{r,r} \) of size \( f'_s \). The diagonal entries of the matrices \( A'_{r,r} \) are the same and the diagonal element is a product of \( r \) linear polynomials which could help in determining the roots of the determinant of the Gram matrix. Stirling numbers of second kind corresponding to the partition algebras are also introduced and their identities were established.

**Definition 3.1.1.** Define,

(a) \( \Omega_{s_1,s_2}^{r_1,r_2} = \left\{ \left[ \lambda_1^2 \right] [2\lambda_2] [3\lambda_3^3] [4\lambda_4] \left| \lambda_1 \vdash k_1, \lambda_2 \vdash k_2, \lambda_3 \vdash k_3, \lambda_4 \vdash k_4 \right. \right. \) with \( \lambda_1 \in \)
\( \mathbb{P}(k_1, s_1), \lambda_2 \in \mathbb{P}(k_2, s_2), \lambda_3 \in \mathbb{P}(k_3, r_1), \lambda_4 \in \mathbb{P}(k_4, r_2) \) such that \( k_1 + k_2 + k_3 + k_4 = 4 \)

where \( \lambda_2^1 = (\lambda_{1,1}^1, \lambda_{1,2}^1, \ldots, \lambda_{1,s_1}^1) \), \( 2\lambda_2 = (2\lambda_{21}, 2\lambda_{22}, \ldots, 2\lambda_{2s_2}) \),

\( \lambda_2^3 = (\lambda_{3,1}^3, \lambda_{3,2}^3, \ldots, \lambda_{3,m}^3) \) and \( 2\lambda_4 = (2\lambda_{41}, 2\lambda_{42}, \ldots, 2\lambda_{4r_2}) \).

(b) \( \Omega_s^5 = \{ [\lambda_1]^1 | [\lambda_2]^2 \mid \lambda_1 \in \mathbb{P}(k_1, s), \lambda_2 \in \mathbb{P}(k_2, r) \text{ such that } k_1 + k_2 = 4 \} \).

**Definition 3.1.2.** Let \( (\tilde{d}, \tilde{P}) \) to be the standard diagram, where \( (\tilde{d}, \tilde{P}) \) is the diagram corresponding to the partition \( \tilde{\lambda} \in \tilde{\Omega}_{s_1, s_2} \) defined as follows:

(i) Let \( \tilde{P} = \tilde{P}_{1,1}^e \cup \tilde{P}_{1,1}^g \cup \tilde{P}_{1,1}^{\cdot} \cup \tilde{P}_{1,2}^e \cup \tilde{P}_{1,2}^g \cup \tilde{P}_{1,2}^{\cdot} \cup \tilde{P}_{2,2}^e \cup \tilde{P}_{2,2}^g \cup \tilde{P}_{2,2}^{\cdot} \) and

\( H = \bigcup_{1 \leq i \leq r_1} \tilde{P}_{i,i}^{g'} \cup \bigcup_{1 \leq i \leq r_2} \tilde{P}_{i,i}^{z_{2}} \) where \( \tilde{P}_{j,j}^{g'} = \left( \sum_{r=1}^{j-1} k_r + \sum_{l=1}^{i-1} \lambda_{jl} + 1, g' \right) \), \( 1 \leq j \leq 4, k_0 = 0 \) and \( H \) is the complement of \( \tilde{P} \) in \( \tilde{d} \).

(ii) The partition \( \lambda_{2,1}^1 \) corresponds to the connected component \( \tilde{P}_{1,1}^g \) \( \forall g' \in \mathbb{Z}_2 \) and \( 1 \leq i \leq s_1 \), the partition \( 2\lambda_{2j} \) corresponds to the connected component \( \tilde{P}_{2,2}^z \) \( \forall 1 \leq i \leq s_2 \), the partition \( \lambda_{3,m}^2 \) corresponds to the connected component \( \tilde{P}_{3,m}^g \) \( \forall g' \in \mathbb{Z}_2 \) and \( 1 \leq m \leq r_1 \) and the partition \( 2\lambda_{4l} \) corresponds to the connected component \( \tilde{P}_{4,l}^z \) \( \forall 1 \leq l \leq r_2 \) where \( \lambda_{11}, 2\lambda_{2j}, \lambda_{3,m} \) and \( 2\lambda_{4l} \) are the number of vertices in the connected component \( \tilde{P}_{1,1}^g, \tilde{P}_{2,2}^z, \tilde{P}_{3,m}^g \) and \( \tilde{P}_{4,l}^z \) respectively.

(iii) \( H_2(\tilde{d}^+ - \tilde{d}^-) = r_1 = H_2(\tilde{d}^+ - \tilde{d}^-) \) and \( H_{2z}(\tilde{d}^+ - \tilde{d}^-) = r_2 = H_{2z}(\tilde{d}^+ - \tilde{d}^-) \) where \( H_2(\tilde{d}^+ - \tilde{d}^-) \) (or \( H_{2z}(\tilde{d}^+ - \tilde{d}^-) \)) are the number of \( \{ e \} \)-horizontal edges \( C \) in the top and bottom row of \( R^{\tilde{d}} \) such that \( H_2^C = \{ e \} (\mathbb{Z}_2) \), \( r_1 + r_2 + s_1 + s_2 \leq k - 1 \), if \( r_1 + r_2 + s_1 + s_2 = k \) then \( r_1 \neq 0 \) or \( s_1 = k \).

(iv) Denote the standard diagram as \( U(\tilde{d}, \tilde{P}) \) \( \tilde{U}(\tilde{d}, \tilde{P}) \).

**Example 3.1.3.** The following are some examples of standard diagrams of \( U(\tilde{d}, \tilde{P}) \) type in signed partition algebras \( \tilde{A}_s^{\mathbb{Z}_2} \) with their corresponding partitions.
where $\phi$ is as in definition 1.3.8.

**Definition 3.1.4.** Let $[\lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^4] \in \Omega_{s_1,s_2}^{r_1,r_2}$. Define,

$$St^c \left( U_{(d,\bar{P})}^{(d,\bar{P})} \right) = \left\{ \sigma \in \mathbb{Z}_2 \mid \mathfrak{S}_k \mid \sigma U_{(d,\bar{P})}^{(d,\bar{P})} \sigma^{-1} = U_{(d,\bar{P})}^{(d,\bar{P})} \right\}$$

where $[\lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^4]$ is the partition corresponding to the diagram $U_{(d,\bar{P})}$ and $\bar{P}$ are as in Definition 3.1.2. Also, $\widetilde{P}_{2k} \simeq \mathbb{Z}_2 \mathfrak{S}_k$.

**Note 3.1.5.**

(i) Let $U_{(d,\bar{P})}^{(d,\bar{P})}$ denote the standard diagram in algebra of $\mathbb{Z}_2$-relations corresponding to the partition $\lambda \in \Omega_{s_1,s_2}^{r_1,r_2}$ and $R_{(d,\bar{P})}^{(d,\bar{P})}$ denote the standard diagram in partition algebra corresponding to the partition $\lambda \in \Omega_{s_1,s_2}^{r_1,r_2}$ which can be defined as in Definition 3.1.2, $St^c \left( U_{(d,\bar{P})}^{(d,\bar{P})} \right)$ and $St^c \left( R_{(d,\bar{P})}^{(d,\bar{P})} \right)$ can also be defined as in Definition 3.1.4 for algebra of $\mathbb{Z}_2$-relations $A_{\mathbb{Z}_2}^2$ and the partition algebras $\mathbb{P}_k(x)$.

(ii) All other diagrams of $U_{(d,\bar{P})}^{(d,\bar{P})}$, $U_{(d,\bar{P})}^{(d,\bar{P})}$, and $R_{(d,\bar{P})}^{(d,\bar{P})}$ whose underlying partition is same as the underlying partition of $U_{(d,\bar{P})}^{(d,\bar{P})}$, $U_{(d,\bar{P})}^{(d,\bar{P})}$ and $R_{(d,\bar{P})}^{(d,\bar{P})}$ respectively can be obtained as follows:

$$U_{(d',\bar{P}')}(d,\bar{P}) = \tau U_{(d,\bar{P})}^{(d',\bar{P}')}, \ U_{(d,\bar{P})}^{(d',\bar{P})} = \tau U_{(d,\bar{P})}^{(d,\bar{P})}, \ U_{(d',\bar{P}')}(d,\bar{P}) = \rho R_{(d,\bar{P})}^{(d',\bar{P}')}, \ \rho^{-1}$$

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\hline
d_1 & \phi(d_1) = (\lambda_1) & (4.1) & \Phi & \Phi & \Phi \\
d_2 & \phi(d_2) = (\lambda_2, \lambda_3) & \Phi & (4.1) & (1) & \Phi \\
d_3 & \phi(d_3) = (\lambda_1, \lambda_2, \lambda_3) & (2) & (1) & (2) & \Phi \\
d_4 & \phi(d_4) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) & (1) & (1) & (2) & (1) \\
\hline
\end{array}
\]
where \( \tau, \tilde{\tau} \in \mathbb{Z}_2 \wr \mathcal{S}_k \) and \( \rho \in \mathcal{S}_k \) are the coset representatives of \( \text{St}^c \left( U_{(d,P)}^{(d,P)} \right) \), \( \text{St}^c \left( U_{(d,P)}^{(d,P)} \right) \) and \( \text{St}^c \left( R_{(d,P)}^{(d,P)} \right) \) respectively. Also, \( U_{(d,P)}^{(d,P)} \), \( U_{(d,P)}^{(d,P)} \) and \( R_{(d,P)}^{(d,P)} \) are the standard diagrams as in Definition 3.1.2.

**Notation 3.1.6.**

(a) For \( 0 \leq r_1 \leq k - s_1 - s_2, 0 \leq r_2 < k - s_1 - s_2 \), \( 0 \leq s_1 \leq k \) and \( 0 \leq s_2 \leq k \),

\[
J_{2s_1+s_2}^{2k} = \bigcup_{0 \leq r_1 \leq k - s_1 - s_2, 0 \leq r_2 < k - s_1 - s_2} \mathbb{J}_{2s_1+s_2}^{2r_1+r_2}
\]

where \( \mathbb{J}_{2s_1+s_2}^{2r_1+r_2} = \left\{ d \in \mathcal{J}_{2s_1+s_2}^{2k} \mid d = U_{(d,P)}^{(d,P)}, \eta_e \left( U_{(d,P)}^{(d,P)} \right) = s_1, \eta_{2z} \left( U_{(d,P)}^{(d,P)} \right) = s_2 \right\} \) has \( r_1 \) number \( \{ e \} \)—of horizontal edges, \( r_2 \) number of \( \mathbb{Z}_2 \)—horizontal edges and \( \| P \| = 2s_1 + s_2 \).

Also,

\[
\left| \mathcal{J}_{2s_1+s_2}^{2r_1+r_2} \right| = \sum_{0 \leq r_1 \leq k - s_1 - s_2, 0 \leq r_2 \leq k - s_1 - s_2} \text{index of } \text{St}^c \left( U_{(d,P)}^{(d,P)} \right) = J_{2s_1+s_2}^{2r_1+r_2}.
\]

(b) For \( 0 \leq r_1 \leq k - s_1 - s_2, 0 \leq r_2 < k - s_1 - s_2 - 1 \), \( 0 \leq s_1 \leq k \) and \( 0 \leq s_2 \leq k - 1, s_1 + s_2 + r_1 + r_2 \leq k - 1 \) and if \( s_1 + s_2 + r_1 + r_2 = k \) then \( s_1 = k \) or \( r_1 \neq 0 \)

\[
\tilde{J}_{2s_1+s_2}^{2k} = \bigcup_{0 \leq r_1 \leq k - s_1 - s_2, 0 \leq r_2 < k - s_1 - s_2} \tilde{\mathbb{J}}_{2s_1+s_2}^{2r_1+r_2}
\]

where \( \tilde{\mathbb{J}}_{2s_1+s_2}^{2r_1+r_2} = \left\{ d \in \mathcal{J}_{2s_1+s_2}^{2k} \mid d = U_{(d,P)}^{(d,P)}, \eta_e \left( U_{(d,P)}^{(d,P)} \right) = s_1 \right\} \) and \( \eta_{2z} \left( U_{(d,P)}^{(d,P)} \right) = s_2 \), \( U_{(d,P)}^{(d,P)} \) has \( r_1 \) number \( \{ e \} \)—horizontal edges, \( r_2 \) number of \( \mathbb{Z}_2 \)—horizontal edges and \( \| \tilde{P} \| = 2s_1 + s_2 \).

Also,

\[
\left| \tilde{\mathcal{J}}_{2s_1+s_2}^{2r_1+r_2} \right| = \sum_{0 \leq r_1 \leq k - s_1 - s_2, 0 \leq r_2 < k - s_1 - s_2} \text{index of } \text{St}^c \left( U_{(d,P)}^{(d,P)} \right) = J_{2s_1+s_2}^{2r_1+r_2}.
\]

(c) For \( 0 \leq r \leq k - s, 0 \leq s \leq k \), put \( J_{s}^{k} = \bigcup_{0 \leq r \leq k-s} \mathbb{J}_{s}^{r} \), where
\[ J^s_s = \left\{ R^d \in I^k_s \mid R^d = U^{R^d}_R, (R^d)^+ \text{ and } (R^d)^- \text{ are the same, } \#(U^{R^d}_R) = s, \right\} \]

Also, \([J^s_s] = \sum_{[\lambda_i]^1[\lambda_j]^2 \in \Omega^i_s} \text{index of } Sf \left( U^{R^d}_R \right) = f^r^s \text{ and } |J^s_s| = \sum_{0 \leq r \leq k-s} |J^r^s_s| .\]

**Definition 3.1.7.** (i) Let \( |J^{2k}_{2s_1+s_2}| = f_{2s_1+s_2} \left( |J^{2k}_{2s_1+s_2}| = \tilde{f}_{2s_1+s_2} \right) .\)

The diagrams in \( J^{2k}_{2s_1+s_2} \) are indexed as follows:

\[ \left\{ U^{(d_i,P_i)}_{(d_i,P_i)} \mid 1 \leq i \leq f_{2s_1+s_2} \right\}_{U^{(d_i,P_i)}_{(d_i,P_i)} \in J^{2k}_{2s_1+s_2}} \left( \left\{ U^{(d_i,P_i)}_{(d_i,P_i)} \mid 1 \leq i \leq \tilde{f}_{2s_1+s_2} \right\}_{U^{(d_i,P_i)}_{(d_i,P_i)} \in \tilde{J}^{2k}_{2s_1+s_2}} \right) .\]

(i) if \( 2r_1 + r_2 < 2r'_1 + r'_2 \)

(ii) if \( 2r_1 + r_2 = 2r'_1 + r'_2 \) and \( r_1 + r_2 < r'_1 + r'_2 \)

(iii) if \( 2r_1 + r_2 = 2r'_1 + r'_2 \) and \( r_1 + r_2 = r'_1 + r'_2 \)

then it can be indexed arbitrarily.

where \( r_1 \) is the number of pairs of \( \{e\} \)-horizontal edges in \( U^{(d_i,P_i)}_{(d_i,P_i)} \left( U^{(d_i,P_i)}_{(d_i,P_i)} \right) \); \( r'_1 \)

is the number of pairs of \( \{e\} \)-horizontal edges in \( U^{(d_i,P_i)}_{(d_i,P_i)} \left( U^{(d_i,P_i)}_{(d_i,P_i)} \right) \); \( r_2 \) is the number of \( \mathbb{Z}_2 \)-horizontal edges in \( U^{(d_i,P_i)}_{(d_i,P_i)} \left( U^{(d_i,P_i)}_{(d_i,P_i)} \right) \) and \( r'_2 \) is the number of \( \mathbb{Z}_2 \)-horizontal edges in \( U^{(d_i,P_i)}_{(d_i,P_i)} \left( U^{(d_i,P_i)}_{(d_i,P_i)} \right) \).

(ii) Let \( |J^k_s| = f_s \). The diagrams in \( J^k_s \) are indexed as follows:

\[ \left\{ U^{R^d_i}_{R^d_i} \mid 1 \leq i \leq f_s \right\}_{U^{R^d_i}_{R^d_i} \in J^k_s} \]

(i) if \( r < r' \)

(ii) if \( r = r' \) then it can be indexed arbitrarily

where \( r(r') \) is the number of horizontal edges in \( U^{R^d_i}_{R^d_i} \left( U^{R^d_j}_{R^d_j} \right) .\)

**Definition 3.1.8.** (a) For \( 0 \leq s_1, s_2 \leq k, 0 \leq s_1 + s_2 \leq k \), define \( G_{2s_1+s_2} \) (Gram matrices of the algebra of \( \mathbb{Z}_2 \)-relations) as follows:

\[ G_{2s_1+s_2} = \left( A_{2r_1+r_2, 2r'_1+r'_2} \right)_{0 \leq r_1 + r_2, r'_1 + r'_2 \leq k-s_1-s_2 \atop 0 \leq r_1, r'_1 \leq k-s_1-s_2, 0 \leq r_2, r'_2 \leq k-s_1-s_2} \]

where \( A_{2r_1+r_2, 2r'_1+r'_2} \) denotes the block matrix whose entries are \( a_{ij} \) with
\[ a_{ij} = x^{l(P_i \lor P_j)} \quad \text{if} \quad \# (U^{(d_i, P_i)} U^{(d_j, P_j)}) = 2s_1 + s_2 \]
\[ a_{ij} = 0 \quad \text{Otherwise i.e.,} \quad \# (U^{(d_i, P_i)} U^{(d_j, P_j)}) < 2s_1 + s_2, \]

\[ l(P_i \lor P_j) = l \left( U^{(d_i, P_i)} U^{(d_j, P_j)} \right) \text{ where } l(P_i \lor P_j) \text{ denotes the number of connected components in } d_i, d_j \text{ excluding the union of all the connected components of } P_i \text{ and } P_j \text{ or equivalently, } l \left( U^{(d_i, P_i)} U^{(d_j, P_j)} \right) \text{ is the number of loops which lie in the middle row when } U^{(d_i, P_i)} \text{ is multiplied with } U^{(d_j, P_j)}, U^{(d_i, P_i)} \in \mathbb{J}_{2s_1+s_2}^{2r_1+r_2} \text{ and } U^{(d_j, P_j)} \in \mathbb{J}_{2s_1+s_2}^{2r_1 + r_2} \text{ respectively.} \]

(b) For \( 0 \leq s_1 \leq k, 0 \leq s_2 \leq k - 1, 0 \leq s_1 + s_2 \leq k - 1 \), define \( G_{2s_1+s_2} \) Gram matrices of signed partition algebra as follows:

\[ \tilde{G}_{2s_1+s_2} = \left( \tilde{A}_{2r_1+r_2, 2r_1'+r_2'} \right)_{0 \leq r_1+r_2, r_1'+r_2' \leq k-1-s_1-s_2} \]
\[ \text{where } \tilde{A}_{2r_1+r_2, 2r_1'+r_2'} \text{ denotes the block matrix whose entries are } a_{ij} \text{ with} \]
\[ a_{ij} = x^{l(\tilde{P}_i \lor \tilde{P}_j)} \quad \text{if} \quad \# (U^{(\tilde{d}_i, \tilde{P}_i)} U^{(\tilde{d}_j, \tilde{P}_j)}) = 2s_1 + s_2 \]
\[ a_{ij} = 0 \quad \text{Otherwise i.e.,} \quad \# (U^{(\tilde{d}_i, \tilde{P}_i)} U^{(\tilde{d}_j, \tilde{P}_j)}) < 2s_1 + s_2, \]

\[ l(\tilde{P}_i \lor \tilde{P}_j) = l \left( U^{(\tilde{d}_i, \tilde{P}_i)} U^{(\tilde{d}_j, \tilde{P}_j)} \right) \text{ where } l(\tilde{P}_i \lor \tilde{P}_j) \text{ denotes the number of connected components in } \tilde{d}_i, \tilde{d}_j \text{ excluding the union of all the connected components of } \tilde{P}_i \text{ and } \tilde{P}_j \text{ or equivalently, } l \left( U^{(\tilde{d}_i, \tilde{P}_i)} U^{(\tilde{d}_j, \tilde{P}_j)} \right) \text{ is the number of loops which lie in the middle row when } U^{(\tilde{d}_i, \tilde{P}_i)} \text{ is multiplied with } U^{(\tilde{d}_j, \tilde{P}_j)}, U^{(\tilde{d}_i, \tilde{P}_i)} \in \mathbb{J}_{2s_1+s_2}^{2r_1+r_2} \text{ and } U^{(\tilde{d}_j, \tilde{P}_j)} \in \mathbb{J}_{2s_1+s_2}^{2r_1 + r_2} \text{ respectively.} \]

(c) For \( 0 \leq s \leq k \), define \( G_s \) (Gram matrices of partition algebra) as follows:

\[ G_s = (A_{r, r'})_{0 \leq r, r' \leq k-s} \]
\[ \text{where } A_{r, r'} \text{ denotes the block matrix whose entries are } a_{ij} \text{ with} \]
\[ a_{ij} = x^{l(R_{r} R_{r'})} \quad \text{if} \quad \# (U^{R_{r} R_{r'}}) = s \]
\[ a_{ij} = 0 \quad \text{Otherwise i.e.,} \quad \# (U^{R_{r} R_{r'}}) < s, \]

\[ l(R_{r} R_{r'}) = l \left( U^{R_{r} R_{r'}} \right) \text{ where } l(d_i d_j) \text{ denotes the number of connected components which lie in the middle row while multiplying } U^{R_{r_i}}, U^{R_{r_j}}, U^{R_{r_i}} \in \mathbb{J}_{r} \text{ and } U^{R_{r_j}} \in \mathbb{J}_{s'} \text{ respectively.} \]
Note 3.1.9. For the sake of simplicity, we shall write \( l(P_i \lor P_j), l(\bar{P}_i \lor \bar{P}_j) \) as \( l(d_i', d_j'), l(\bar{d}_i', \bar{d}_j') \) respectively where \( d_i' = U_{(d_i, P_i)}^{(d_i, P_i)} \) and \( d_j' = U_{(d_j, P_j)}^{(d_j, P_j)} \).\( \widetilde{d}_i' = U_{(\bar{d}_i, \bar{P}_i)}^{(\bar{d}_i, \bar{P}_i)} \) and \( \widetilde{d}_j' = U_{(\bar{d}_j, \bar{P}_j)}^{(\bar{d}_j, \bar{P}_j)} \).

Lemma 3.1.10. Let \( U_{(d_i, P_i)}^{(d_i, P_i)}, U_{(d_j, P_j)}^{(d_j, P_j)} \in J_{2s_1+2}^k \), \( U_{(\bar{d}_i, \bar{P}_i)}^{(\bar{d}_i, \bar{P}_i)} \), \( U_{(\bar{d}_j, \bar{P}_j)}^{(\bar{d}_j, \bar{P}_j)} \) be such that \( U_{R_i}^{R_i}, U_{R_i}^{R_j} \in J_s^k \) then

(i) \( l(d_i', d_j') \leq l(d_i, d_j), \forall j \leq i, \) where \( l(d_i', d_j') \) is the number of loops which lie in the middle row when \( U_{(d_i, P_i)}^{(d_i, P_i)} \) is multiplied with \( U_{(d_j, P_j)}^{(d_j, P_j)} \).

(ii) \( l(\bar{d}_i', \bar{d}_j') \leq l(\bar{d}_i, \bar{d}_j), \forall j \leq i, \) where \( l(\bar{d}_i', \bar{d}_j') \) is the number of loops which lie in the middle row when \( U_{(\bar{d}_i, \bar{P}_i)}^{(\bar{d}_i, \bar{P}_i)} \) is multiplied with \( U_{(\bar{d}_j, \bar{P}_j)}^{(\bar{d}_j, \bar{P}_j)} \).

(c) \( l(R_i R_j) \leq l(R_i R_j), \forall j \leq i, \) where \( l(R_i R_j) \) is the number of loops which lie in the middle row when \( U_{R_i}^{R_i} \) is multiplied with \( U_{R_j}^{R_j} \).

(ii) \( \det G_{2s_1+2}, \det \widetilde{G}_{2s_1+2} \) and \( \det G_s \) are non-zero polynomials with leading coefficient 1.

Proof.

Proof of (i)(a): First we shall prove for \( j < i \).

A loop consists of at least one horizontal edge from the bottom row of \( U_{(d_i, P_i)}^{(d_i, P_i)} \) and one from the top row of \( U_{(d_j, P_j)}^{(d_j, P_j)} \). Hence the number of loops in the middle component of the product \( U_{(d_i, P_i)}^{(d_i, P_i)} U_{(d_j, P_j)}^{(d_j, P_j)} \) is always less than the minimum of \( l(d_i', d_j') \) and \( l(d_j', d_i') \). Thus, \( l(d_i', d_j') \leq l(d_i', d_j') \), for all \( j \).

For if \( j < i \),

Case (i): \( l(d_i', d_j') \leq l(d_j') < l(d_i') \) when \( 2r_1 + r_2 < 2r_1' + r_2' \) where \( r_1(r_1') \) is the number of pairs of \( \{ e \} \) horizontal edges and \( r_2(r_2') \) is the number of \( \mathbb{Z}_2 \)-horizontal edges in \( U_{(d_i, P_i)}^{(d_i, P_i)} U_{(d_j, P_j)}^{(d_j, P_j)} \) respectively.

Case (ii): \( l(d_i', d_j') \leq l(d_j') = l(d_i') \) when \( 2r_1 + r_2 = 2r_1' + r_2' \) and \( r_1 + r_2 < r_1' + r_2' \).

Suppose that \( r_2 < r_2' \), i.e., at least two \( \mathbb{Z}_2 \) horizontal edges of \( U_{(d_i, P_i)}^{(d_i, P_i)} \) are contributed to make a loop or one \( \mathbb{Z}_2 \)-horizontal edge of \( U_{(d_j, P_j)}^{(d_j, P_j)} \) is connected to a \( \mathbb{Z}_2 \)-through class of \( U_{(d_i, P_i)}^{(d_i, P_i)} \) in the product \( U_{(d_i, P_i)}^{(d_i, P_i)} U_{(d_j, P_j)}^{(d_j, P_j)} \).
Therefore, the number of loops is strictly less than $2r'_1 + r'_2$, hence
\[ l(d'_i d'_j) \leq l(d'_i d'_i) = l(d'_j d'_j). \]

**Case (iii):** $l(d'_i d'_j) = l(d'_j d'_j) = 2r_1 + r_2 = 2r'_1 + r'_2$ and $r_1 + r_2 = r'_1 + r'_2$.

Every $\{e\}$-through class of $U^{(d_i, P_i)}$ is uniquely connected to a $\{e\}$-through class of $U^{(d_j, P_j)}$ and vice versa and if $l(d'_i d'_j) = l(d'_j d'_j) = l(d'_i)$ then every $\{e\}(\mathbb{Z}_2)$-horizontal edge of $U^{(d_i, P_i)}$ is connected uniquely to a $\{e\}(\mathbb{Z}_2)$-horizontal edge of $U^{(d_j, P_j)}$ and vice versa which implies that $U^{(d_i, P_i)} = U^{(d_j, P_j)}$.

Thus, if $U^{(d_i, P_i)}(d_j, P_j)$ and $2r_1 + r_2 = 2r'_1 + r'_2$ and $r_1 + r_2 = r'_1 + r'_2$ then $l(d'_i d'_j) < l(d'_j d'_j) = l(d'_i d'_i)$.

Proof of (i)(b) and (i)(c) is similar to the proof of (i)(a).

**Proof of (ii):** It follows from (i) of Lemma 3.1.10, that the degree of the monomial
\[ \prod_{\sigma \in \mathcal{F}_{2n+2}} a_{\sigma(i)}, \] is strictly less than the degree of the monomial
\[ \prod_{i=1}^{2n+2} a_{i}. \]

Thus, the determinant of the Gram matrix $G_{2n+2}$ of the algebra of $\mathbb{Z}_2$-relations is a non-zero monic polynomial with integer coefficients and the roots are all algebraic integers.

Similarly, we can prove for the determinant of the Gram matrices $\tilde{G}_{2n+2}$ and $G_s$ of signed partition algebras and partition algebras respectively.

\[ \square \]

**Lemma 3.1.11.** The Gram matrices $G_{2n+2}, \tilde{G}_{2n+2}$ and $G_s$ are symmetric.

**Proof.** The proof follows from the Definition 3.1.8, since the top and bottom rows of the diagrams in $J_{2n+2}^{2k}, \tilde{J}_{2n+2}^{2k}, J_s^k$ have the same number of horizontal edges. \[ \square \]

**Remark 3.1.12.** Every partition diagram can be represented as a set partition and in set partition we can talk about subsets. Thus a connected component of the diagram $U^{(d_j, P_j)}(d_i, P_i)$ is contained in a connected component of $U^{(d_j, P_j)}(d_i, P_i)$ if the corresponding set partition of $U^{(d_j, P_j)}(d_i, P_i)$ is contained in the set partition of $U^{(d_j, P_j)}(d_i, P_i)$.

**Definition 3.1.13.** Let $U^{(d_j, P_j)}(d_i, P_i) \in J_{2n+2}^{2k}, U^{(d_j, P_j)}(d_i, P_i) \in \tilde{J}_{2n+2}^{2k}$ and $U^{(d_j, P_j)}(d_i, P_i) \in J_s^k$ and $U^{(d_j, P_j)}(d_i, P_i) \in J_s^k$ where $J_{2n+2}^{2k}, J_s^k$ are as in Notation 3.1.6.
(a) Define a relation on $J_{2s_1 + s_2}^{2k}$ as follows:

$$U_{(d,P)}^{(d',P')} < U_{(d,P)}^{(d',P')} \left( U_{(d,P)}^{(d',P')} < U_{(d,P)}^{(d',P')} \right),$$

(i) if each $\{e\}$-through class of $U_{(d,P)}^{(d',P')} \left( U_{(d,P)}^{(d',P')} \right)$ is contained in a $\{e\}$-through class of $U_{(d,P)}^{(d',P')} \left( U_{(d,P)}^{(d',P')} \right)$;

(ii) every $\mathbb{Z}_2$-through class of $U_{(d,P)}^{(d',P')} \left( U_{(d,P)}^{(d',P')} \right)$ is contained in a $\mathbb{Z}_2$-through class of $U_{(d,P)}^{(d',P')} \left( U_{(d,P)}^{(d',P')} \right)$;

(iii) every $\{e\}$-horizontal edge of $U_{(d,P)}^{(d',P')} \left( U_{(d,P)}^{(d',P')} \right)$ is contained in a $\mathbb{Z}_2$-horizontal edge or $(\{e\}$ or $\mathbb{Z}_2$)-through class of $U_{(d,P)}^{(d',P')} \left( U_{(d,P)}^{(d',P')} \right)$ and

(iv) every $\mathbb{Z}_2$-horizontal edge of $U_{(d,P)}^{(d',P')} \left( U_{(d,P)}^{(d',P')} \right)$ is contained in a horizontal edge or $\mathbb{Z}_2$-through class of $U_{(d,P)}^{(d',P')} \left( U_{(d,P)}^{(d',P')} \right)$.

We say that $U_{(d,P)}^{(d',P')} \left( U_{(d,P)}^{(d',P')} \right)$ is a coarser diagram of $U_{(d,P)}^{(d',P')} \left( U_{(d,P)}^{(d',P')} \right)$.

(b) Define a relation on $J_s^k$ as follows: $U_{R^d}^{R^d} < U_{R^d}^{R^d}$,

(i) if each through class of $U_{R^d}^{R^d}$ is contained in a through class of $U_{R^d}^{R^d}$;

(ii) if each horizontal edge of $U_{R^d}^{R^d}$ is contained in a horizontal edge of $U_{R^d}^{R^d}$,

We say that $U_{R^d}^{R^d}$ is a coarser diagram of $U_{R^d}^{R^d}$. The relation $<$ holds for the diagrams in $J_s^k$.

3.1.1 Stirling Numbers of Second Kind of the Algebra of $\mathbb{Z}_2$-Relations, Signed Partition Algebras and Partition Algebras:

Lemma 3.1.14. (a) The number of diagrams having $2s_1 + s_2$ through classes and $2p_1 + p_2$ horizontal edges which lie above and coarser than the diagram

$U_{(d,P)}^{(d',P')} \in J_{2s_1 + s_2}^{2k} \left( U_{(d,P)}^{(d',P')} \in J_{2s_1 + s_2}^{2k} \right)$ whose underlying partition is $\lambda(\check{\lambda})$, where
\( \lambda \in \Omega_{r_1,r_2}^s \) \( \tilde{\lambda} \in \tilde{\Omega}_{r_1,r_2}^s \) is given by,

\[
\sum_{i=p_1}^{r_1} \binom{r_1}{i} 2^{i-p_1} S(i, p_1) \left[ \sum_{j=0}^{r_1-i} \binom{r_1-i}{j} (2s_1 + s_2)^{r_1-i-j} \sum_{l=p_2-j \leq l \leq r_2 \atop p_2-j \geq 0} \binom{r_2}{l} s_2^{r_2-l} S(l+j, p_2) \right]
\]

with \( p_1 \leq r_1 \) and \( r_1 - p_1 \geq p_2 - r_2 \) where \( S(i, p_1) \) and \( S(l+j, p_2) \) are the Stirling numbers of the second kind.

In particular,

(i) if \( r_1 = 0 \) then the number of diagrams having \( p_2 \) number of \( \mathbb{Z}_2 \)-horizontal edges which lie above and coarser than the given \( U^{(d,P)}_{(a,P)} \) \( U^{(d,P)}_{(a,P)} \) with \( r_2 \) number of \( \mathbb{Z}_2 \)-horizontal edges is given by

\[
\sum_{i=p_2}^{r_2} \binom{r_2}{i} s_2^{r_2-i} S(i, p_2).
\]

(ii) if \( s_1 = 0, s_2 = 0 \) then the number of diagrams having \( 2p_1 + p_2 \) horizontal edges which lie above and coarser than \( U^{(d,P)}_{(a,P)} \) \( U^{(d,P)}_{(a,P)} \) is given by

\[
\sum_{i=p_1}^{r_1} \binom{r_1}{i} 2^{i-p_1} S(i, p_1) S(r_2 + r_1 - i, p_2).
\]

(b) The number of diagrams having \( s \) through classes and \( p \) horizontal edges which lie above and coarser than the diagram \( U_{R_d}^{R_d} \in \mathbb{J}_s^r \) whose underlying partition is \( \lambda \), where \( \lambda = [\lambda_1]^1 [\lambda_2]^2 \) as in Definition 3.1.1(b) is given by,

\[
\sum_{i=p}^{r} \binom{r}{i} s^{r-i} S(i, p).
\]

with \( p \leq r \) where \( S(i, p) \) is the Stirling numbers of the second kind.

**Proof.**

Step 1: Reducing \( 2r_1 \) number of \( \{e\} \)-horizontal edges to \( 2p_1 \) number of \( \{e\} \)-horizontal edges.

Choose \( i \) pair of \( \{e\} \)-horizontal edges from \( r_1 \) pair of \( \{e\} \)-horizontal edges of \( U^{(d,P)}_{(a,P)} \) \( U^{(d,P)}_{(a,P)} \) such that \( i \geq p_1 \).

For given \( i \) pair of \( \{e\} \)-horizontal edges, the number of ways to partition a set of \( i \) pair of \( \{e\} \)-horizontal edges into \( p_1 \) pair of \( \{e\} \)-horizontal edges is given by the Stirling number of second kind \( S(i, p_1) \).
We know that two \( \{e\}\)-horizontal edges can be combined together in two ways. Thus, \( p_1 \) number of \( \{e\}\)-horizontal edges can be obtained in \( 2^{i-p_1} \) ways.

**Step 2:** Reducing \( 2(r_1 - i) \) number of \( \{e\}\)-horizontal edges together with \( r_2 \) number of \( \mathbb{Z}_2 \)-horizontal edges to obtain \( p_2 \) number of \( \mathbb{Z}_2 \)-horizontal edges.

Choose \( j \) pair of \( \{e\}\)-horizontal edges from \( r_1 - 1 \) pair of \( \{e\}\)-horizontal edges such that \( 0 \leq j \leq r_1 - i \).

Choose \( l \) number of \( \mathbb{Z}_2 \)-horizontal edges from \( r_2 \) number of horizontal edges such that \( l \geq r_2 - j \).

The number of ways to partition a set of \( j \) pair of \( \{e\}\)-horizontal edges together with \( l \) number of \( \mathbb{Z}_2 \)-horizontal edges into \( p_2 \) number of \( \mathbb{Z}_2 \)-horizontal edges are given by the Stirling number of the second kind \( S(l + j, p_2) \).

**Step 3:** Combining the remaining horizontal edges with through classes.

By combining the remaining \( r_1 - i - j \) pair of \( \{e\}\)-horizontal edges to any one of the through classes, we obtain \( (2s_1 + s_2)^{r_1-i-j} \) number of diagrams.

Also the remaining \( r_2 - l \) number of \( \mathbb{Z}_2 \)-horizontal edges can be combined only with \( \mathbb{Z}_2 \)-through classes which can be done in \( s_2^{r_2-l} \) ways.

Proof of (i), (ii) and (b) follows from proof of (a).

**Notation 3.1.15.**

(i) (a) The Stirling number of second kind of algebra of \( \mathbb{Z}_2 \) relations is denoted by \( B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} \) where

\[
B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} = \sum_{i=p_1}^{r_1} r_1 C_i \left( \sum_{j=0}^{r_1-i} r_{1-j} C_j \left( 2s_1 + s_2 \right)^{r_1-i-j} \sum_{p_2-j \leq l \leq r_2} \sum_{p_2-j \geq 0} r_2 C_l \left( 2s_2 \right)^{r_2-l} S(l + j, p_2) \right) \]

with \( p_1 \leq r_1, r_1 - p_1 \geq p_2 - r_2, 0 \leq s_1 \leq k, 0 \leq s_2 \leq k \) and \( r_1 + r_2 + s_1 + s_2 \leq k \), \( 2p_1 + p_2 < 2r_1 + r_2 \).

(b) The Stirling number of second kind of signed partition algebra is denoted by \( \tilde{B}_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} \) where

\[
\tilde{B}_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} = \]

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The Stirling number of second kind of partition algebra is denoted by $B_{r,p}^s$ where $B_{r,p}^s = \sum_{i=p}^r rC_i \ s_2^{r-i}$ with $p \leq r$, $r - p \geq p_2 - r_2$, $0 \leq s_1 \leq k - 1$, $0 \leq s_2 \leq k - 1$ and $r_1 + r_2 + s_1 + s_2 \leq k - 1$, $2p_1 + p_2 < 2r_1 + r_2$ and if $r_1 + r_2 + s_1 + s_2 = k$ then $s_1 = k$, or $r_1 \neq 0$.

(c) The Stirling number of second kind of partition algebra is denoted by $B_{r,p}^s$ where $B_{r,p}^s = \sum_{i=p}^r rC_i \ s_2^{r-i}$ with $p \leq r$ and $0 \leq s \leq k$.

(ii) $B_{2r_1+r_2,2r_1+r_2}^{s_1,s_2} = 1$, $\widetilde{B}_{2r_1+r_2,2r_1+r_2}^{s_1,s_2} = 1$ and $B_{r,r}^s = 1$.

(iii) $B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} = 0$, $\widetilde{B}_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} = 0$ and $B_{r,p}^s = 0$ otherwise.

Lemma 3.1.16. Let $B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2}$, $\widetilde{B}_{2r_1+r_2,2p_1+p_2}^{s_1,s_2}$ and $B_{r,p}^s$ be as in Notation 3.1.15. Then

(a) $B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} = B_{2r_1+r_2-2p_1+p_2}^{s_1,s_2} + (s_2 + p_2)B_{2r_1+r_2-2p_1+p_2}^{s_1,s_2}$, $\forall p_1 \leq r_1, r_2 \geq 1.$

In particular, if $r_1 = 0$ and $p_1 = 0$ then

$B_{0+r_2,0+p_2}^{s_1,s_2} = B_{0+r_2-1,0+p_2-1}^{s_1,s_2} + (s_2 + p_2)B_{0+r_2-1,0+p_2}^{s_1,s_2}.$

(b) $\widetilde{B}_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} = \widetilde{B}_{2r_1+r_2-2p_1+p_2}^{s_1,s_2} + (s_2 + p_2)\widetilde{B}_{2r_1+r_2-2p_1+p_2}^{s_1,s_2}$, $\forall p_1 \leq r_1, r_2 \geq 1.$

In particular, if $r_1 = 0$ and $p_1 = 0$ then

$\widetilde{B}_{0+r_2,0+p_2}^{s_1,s_2} = \widetilde{B}_{0+r_2-1,0+p_2-1}^{s_1,s_2} + (s_2 + p_2)\widetilde{B}_{0+r_2-1,0+p_2}^{s_1,s_2}.$

(c) $B_{r,r}^s = B_{r-1,p-1}^{s} + (s + p)B_{r-1,p}^{s}$, $p \leq r$.

Proof.

Proof of (a): Consider

$B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2}$

$$= \sum_{i=p_1}^{r_1} r_1C_i \ 2^{r_1-p_1} S(i,p_1) \left[ \sum_{j=0}^{r_1-i} r_1-iC_j \ (2s_1 + s_2)^{r_1-i-j} \left[ \sum_{p_2-j \leq l \leq r_2} \ r_2C_i \ s_2^{r_2-l} S(l+j,p_2) \right] \right]$$

By using the identities $r_2C_i = r_2-iC_i-1 + r_2-iC_i$ and

$S(l+j,p_2) = S(l+j-1,p_2-1) + p_2S(l+j-1,p_2)$ we have,
\[ B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} = B_{2r_1+r_2-1,2p_1+p_2-1}^{s_1,s_2} + \left( p_2 + s_2 \right) B_{2r_1+r_2-1,2p_1+p_2}^{s_1,s_2}, \quad p_1 \leq r_1 \text{ and } r_2 \geq 1. \]

Proof of (b) and (c) are same as that of proof of (a).  

By example 3.10.4 in [34], \( B_{r,p}^s \) will be called as Generalized Stirling numbers
and \( B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} \) will be called as \((s_1, s_2, r_1, r_2, p_1, p_2)\)-Stirling numbers of the second kind and it satisfy the following identity:

**Lemma 3.1.17.** Let \( B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} \) and \( \tilde{B}_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} \) be as in Notation 3.1.15.

(a) \[ B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} = B_{2(r_1-1)+r_2,2(p_1-1)+p_2}^{s_1,s_2} + B_{2(r_1-1)+r_2+1,2p_1+p_2}^{s_1,s_2} + \left( 2p_1 + 2s_1 \right) B_{2(r_1-1)+r_2,2p_1+p_2}^{s_1,s_2}, \]

with \( p_1 \leq r_1 - 1 \) and \((r_1 - 1) - p_1 \geq p_2 - r_2\).

In particular,

(i) if \( p_2 = 0 \) then
\[ B_{2r_1+r_2,2p_1}^{s_1,s_2} = B_{2(r_1-1)+r_2,2(p_1-1)+p_2}^{s_1,s_2} + \left( 2p_1 + 2s_1 + s_2 \right) B_{2(r_1-1)+r_2,2p_1+p_2}^{s_1,s_2}, \quad p_1 \leq r_1 - 1. \]

(ii) if \( r_2 = 0 \) and \( p_2 = 0 \) then
\[ B_{2r_1,2p_1}^{s_1,s_2} = B_{2(r_1-1)+r_2,2(p_1-1)+p_2}^{s_1,s_2} + \left( 2p_1 + 2s_1 + s_2 \right) B_{2(r_1-1),2p_1}^{s_1,s_2}, \quad p_1 \leq r_1 - 1. \]

(b) \[ \tilde{B}_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} = \tilde{B}_{2(r_1-1)+r_2,2(p_1-1)+p_2}^{s_1,s_2} + \tilde{B}_{2(r_1-1)+r_2+1,2p_1+p_2}^{s_1,s_2} + \left( 2p_1 + 2s_1 \right) \tilde{B}_{2(r_1-1)+r_2,2p_1+p_2}^{s_1,s_2}, \]

with \( p_1 \leq r_1 - 1 \) and \((r_1 - 1) - p_1 \geq p_2 - r_2\). In particular,

(i) if \( p_2 = 0 \) then
\[ \tilde{B}_{2r_1+r_2,2p_1}^{s_1,s_2} = \tilde{B}_{2(r_1-1)+r_2,2(p_1-1)+p_2}^{s_1,s_2} + \left( 2p_1 + 2s_1 + s_2 \right) \tilde{B}_{2(r_1-1)+r_2,2p_1+p_2}^{s_1,s_2}, \quad p_1 \leq r_1 - 1. \]

(ii) if \( r_2 = 0 \) and \( p_2 = 0 \) then
\[ \tilde{B}_{2r_1,2p_1}^{s_1,s_2} = \tilde{B}_{2(r_1-1)+r_2,2(p_1-1)+p_2}^{s_1,s_2} + \left( 2p_1 + 2s_1 + s_2 \right) \tilde{B}_{2(r_1-1),2p_1}^{s_1,s_2}, \quad p_1 \leq r_1 - 1. \]

Proof.

Proof of (a): Consider
\[ B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} \]

\[ = \sum_{i=p_1}^{r_1} r_1 C_i \ 2^{i-p_1} \ S(i,p_1) \left[ \sum_{j=0}^{r_1-i} r_1-j \ C_j (2s_1+s_2)^r_1-i-j \ \sum_{p_2-j \leq l \leq r_2} r_2 C_l \ s_2^{r_2-l} \ S(l+j,p_2) \right] \]

By using the identities \( r_1 C_i = r_1-1 C_{i-1} + r_1 C_i \), \( S(i,p_1) = S(i-1,p_1-1) + p_1 S(i-1,p_1) \) and Lemma 3.1.16 we have,

\[ B_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} = B_{2(r_1-1)+r_2,2(p_1-1)+p_2}^{s_1,s_2} + B_{2(r_1-1)+r_2+1,2p_1+p_2}^{s_1,s_2} + (2p_1 + 2s_1)B_{2(r_1-1)+r_2,2p_1+p_2}^{s_1,s_2} \]

with \( p_1 \leq r_1 - 1 \) and \( (r_1-1) - p_1 \geq p_2 - r_2 \).

Proof of (b) is similar to proof of (a). \( \square \)

**Example 3.1.18.** For fixed \( s_1, s_2 \), the table below gives the value of \( \tilde{B}_{2r_1+r_2,2p_1+p_2}^{s_1,s_2} \) as in Notation 3.1.15:

<table>
<thead>
<tr>
<th>( 2r_1 + r_2 )</th>
<th>( 2p_1 + p_2 )</th>
<th>2.1 + 2</th>
<th>2.2 + 0</th>
<th>0 + 3</th>
<th>2.1 + 1</th>
<th>2.1 + 0</th>
<th>0 + 2</th>
<th>0 + 1</th>
<th>0 + 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 + 2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2s_2</td>
<td>( s_2^2 )</td>
<td>2s_1 + 3s_2 + 3</td>
<td>4s_1s_2 + 3s_2^2 + 2s_1 + 3s_2 + 1</td>
<td>( 2s_1s_2 + s_2^2 )</td>
<td></td>
</tr>
<tr>
<td>2.2 + 0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2s_2</td>
<td>4s_1 + 2s_2 + 2</td>
<td>1</td>
<td>4s_1 + 2s_2 + 1</td>
<td>( (2s_1 + s_2)^2 )</td>
</tr>
<tr>
<td>0 + 3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3s_2 + 3</td>
<td>( 3s_2^2 + 3s_2 + 1 )</td>
<td>( s_2^2 )</td>
<td></td>
</tr>
<tr>
<td>2.1 + 1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( s_2 )</td>
<td>1</td>
<td>2s_1 + 2s_2 + 1</td>
<td>( (2s_1 + s_2)s_2 )</td>
</tr>
<tr>
<td>2.1 + 0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2s_2 + 1</td>
<td>( s_2^2 )</td>
</tr>
<tr>
<td>0 + 2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0 + 1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( s_2 )</td>
</tr>
</tbody>
</table>

### 3.1.2 Column Operations on the Gram Matrices of the Algebra of \( \mathbb{Z}_2 \)-Relations, Signed Partition Algebras and Partition Algebras

**Column Operations on the Gram matrices:**

Now, we apply column operations on the Gram matrices to eliminate the entries parallel to the diagonal entries having the same value. i.e., Suppose \( ij \)-entry of the Gram matrices \( G_{2s_1+s_2} \) is \( x^{2r_1+r_2} \) and \( ii \)-entry of \( G_{2s_1+s_2} \) is also \( x^{2r_1+r_2} \) such that \( ij \)-entry of \( G_{2s_1+s_2} \) lies outside the diagonal block \( A_{2r_1+r_2,2r_1+r_2} \) then by applying the column operations \( L_j \to L_j - L_i \) the
$ij$-entry of the Gram matrix $G_{2s_1+s_2} \left( \tilde{G}_{2s_1+s_2} \right)$ become zero. This column operation is done inductively on the Gram matrix $G_{2s_1+s_2} \left( \tilde{G}_{2s_1+s_2} \right)$. These are the only column operations we do on the Gram matrix $G_{2s_1+s_2} \left( \tilde{G}_{2s_1+s_2} \right)$.

Similar column operations are performed for the Gram matrix $G_s$ of partition algebra.

**Lemma 3.1.19.**

(a) Let $U^{(d_i, P_i)}_{(d_i, P_i)} \in \mathbb{F}^{2p_1+p_2}_{2s_1+s_2}$, $U^{(d_j, P_j)}_{(d_j, P_j)} \in \mathbb{F}^{2r_1+r_2}_{2s_1+s_2}$ with $2p_1 + p_2 < 2r_1 + r_2$ then $U^{(d_i, P_i)}_{(d_i, P_i)}$ is coarser than $U^{(d_j, P_j)}_{(d_j, P_j)}$ if and only if $l(d'_i d'_j) = x^{2p_1+p_2}$ where $\mathbb{F}^{2r_1+r_2}_{2s_1+s_2}$ is as in Notation 3.1.6.

(b) Let $U^{(\tilde{d}_i, \tilde{P}_i)}_{(\tilde{d}_i, \tilde{P}_i)} \in \mathbb{F}^{2p_1+p_2}_{2s_1+s_2}$, $U^{(\tilde{d}_j, \tilde{P}_j)}_{(\tilde{d}_j, \tilde{P}_j)} \in \mathbb{F}^{2r_1+r_2}_{2s_1+s_2}$ with $2p_1 + p_2 < 2r_1 + r_2$ then $U^{(\tilde{d}_i, \tilde{P}_i)}_{(\tilde{d}_i, \tilde{P}_i)}$ is coarser than $U^{(\tilde{d}_j, \tilde{P}_j)}_{(\tilde{d}_j, \tilde{P}_j)}$ if and only if $l(\tilde{d}'_i \tilde{d}'_j) = x^{2p_1+p_2}$ where $\mathbb{F}^{2r_1+r_2}_{2s_1+s_2}$ is as in Notation 3.1.6.

(c) Let $U^{R^{d_i}}_{R^{d_i}} \in \mathbb{F}^{p}_{s}$, $U^{R^{d_j}}_{R^{d_j}} \in \mathbb{F}^{r}_{s}$ with $p < r$ then $U^{R^{d_i}}_{R^{d_i}}$ is coarser than $U^{R^{d_j}}_{R^{d_j}}$ if and only if $l(R^{d_i} R^{d_j}) = x^p$ where $\mathbb{F}^{r}_{s}$ is as in Notation 3.1.6.

**Proof.**

**Proof of (a):** $U^{(d_i, P_i)}_{(d_i, P_i)}$ is coarser than $U^{(d_j, P_j)}_{(d_j, P_j)}$ if and only if every ${\{} e{\}}$-through class of $U^{(d_j, P_j)}_{(d_j, P_j)}$ is contained in a ${\{} e{\}}$-through class of $U^{(d_i, P_i)}_{(d_i, P_i)}$, every $\mathbb{Z}_2$-through class of $U^{(d_j, P_j)}_{(d_j, P_j)}$ is contained in a $\mathbb{Z}_2$-through class of $U^{(d_i, P_i)}_{(d_i, P_i)}$, every ${\{} e{\}}$-horizontal edge of $U^{(d_j, P_j)}_{(d_j, P_j)}$ is contained in a horizontal edge or ${\{} e{\}}$-through class of $U^{(d_i, P_i)}_{(d_i, P_i)}$ and every $\mathbb{Z}_2$-horizontal edge of $U^{(d_j, P_j)}_{(d_j, P_j)}$ is contained in a horizontal edge or $\mathbb{Z}_2$-through class of $U^{(d_i, P_i)}_{(d_i, P_i)}$.

Thus, the number of loops in the product $U^{(d_i, P_i)}_{(d_i, P_i)} U^{(d_j, P_j)}_{(d_j, P_j)}$, i.e., $l(R^{d_i} R^{d_j})$ is $x^{2p_1+p_2}$.

Proof of (b) and (c) are similar to the proof of (a).
Theorem 3.1.20.

(a) After applying the column operations the diagonal entry \( x^{2r_1+r_2} \) in the block matrix \( A_{2r_1+r_2,2r_1+r_2} \) for \( 0 \leq r_1, r_2 \leq k - s_1 - s_2 \), \( r_1 + r_2 \leq k - s_1 - s_2 \) and the block matrix \( \tilde{A}_{2r_1+r_2,2r_1+r_2} \) for \( 0 \leq r_1, r_2 \leq k - s_1 - s_2 - 1 \), \( 0 \leq r_1 + r_2 \leq k - s_1 - s_2 - 1 \) is replaced by

\[
\begin{align*}
(i) & \quad \prod_{j=0}^{r_1-1} [x^2 - x - 2(s_1 + j)] \prod_{l=0}^{r_2-1} [x - (s_2 + l)] & \text{if } r_1 \geq 1 \text{ and } r_2 \geq 1, \\
(ii) & \quad \prod_{j=0}^{r_2-1} [x - (s_2 + j)] & \text{if } r_1 = 0 \text{ and } r_2 \neq 0, \\
(iii) & \quad \prod_{j=0}^{r_1-1} [x^2 - x - 2(s_1 + j)] & \text{if } r_1 \neq 0 \text{ and } r_2 = 0.
\end{align*}
\]

Also, every diagonal element in the block matrix \( A_{2r_1+r_2,2r_1+r_2} \) and \( \tilde{A}_{2r_1+r_2,2r_1+r_2} \) are the same.

(b) After applying the column operations the diagonal entry \( x^r \) in the block matrix \( A_{r,r} \) for \( 0 \leq r + s \leq k \), \( 0 \leq r \leq k - s \) is replaced by

\[
\prod_{j=0}^{r-1} [x - (s + j)] \text{ if } r \geq 1 \text{ and } 1 \text{ if } r = 0.
\]

Also, every diagonal element in the block matrix \( A_{r,r} \) is the same.

Proof.

Proof of (a): The proof is by induction on the number of horizontal edges.

Let \( U^{(d,P)} ((d,P)) \) be any diagram corresponding to the diagonal entry \( x^{2r_1+r_2} \) in block matrix \( A_{2r_1+r_2,2r_1+r_2} \) \( \tilde{A}_{2r_1+r_2,2r_1+r_2} \) having \( 2s_1 + s_2 \) number of through classes and \( r_1 \) pairs of \( \{e\} \)-horizontal edges and \( r_2 \) number of \( \mathbb{Z}_2 \)-horizontal edges.

After applying column operations as mentioned earlier to eliminate the entries which lie above corresponding to the diagrams coarser than \( U^{(d,P)} ((d,P)) \), then by Lemma 3.1.19 and induction the diagonal entry \( x^{2r_1+r_2} \) is replaced as

\[
x^{2r_1+r_2} - \sum_{0 \leq j \leq r_1 \atop -r_2 \leq j' \leq -r_1, j,j' \neq 0 \atop -r_2 \leq j' < 0, j=0} (-1)^{1+j'} D_{2r_1,2r_1}^{r_1,s_2} \prod_{l=0}^{r_1-j-1} [x^2 - x - 2(s_1 + l)] \prod_{f=0}^{r_2+j'} [x - (s_2 + f)]
\]

(3.3)
where \( B^{s_1,s_2}_{2r_1+r_2,2p_1+p_2} \) gives the number of diagrams which has \( p_1 \) number of \( \{e\} \) horizontal edges and \( p_2 \) number of \( \mathbb{Z}_2 \) horizontal edges which lie above and coarser than 

\[
U^{(d,P)}_{(d,P)} \left( U^{(d,P)}_{(d,P)} \right).
\]

Fix \( s \) and put

\[
H_{2r_1+r_2,s} = \sum_{0 \leq j \leq d} (-1)^{1+j'} B^{s_1,s_2}_{2r_1+r_2,2|r_1-(d-j)|+r_2+d-j'} C_2|r_1-(d-j)|+r_2+d-j',s
\]

where \( C_{2r_1'+r_2',s} \) denote the coefficient of \( x^s \) in 

\[
\prod_{j=0}^{r_1'-1} [x^2 - x - 2(s_1 + j)] \prod_{l=0}^{r_2'-1} [x - (s_2 - l)]
\]

where \( d = 2r_1 + r_2 - s \).

We claim that,

\[
H_{2r_1+r_2,s} = (-1)^d C_{2r_1+r_2-1,s}.
\]

We shall prove this by using induction on \( 2r_1 + r_2 \).

By using Lemma 3.1.17 and induction hypothesis, the equation (3.4) becomes,

\[
H_{2r_1+r_2,s} = \sum_{0 \leq j \leq d} (-1)^{1+j'} \left\{ B^{s_1,s_2}_{2r_1+r_2-1,2|r_1-(d-j)|+r_2+d-j'-1} + (s_2 + r_2 + d - j') B^{s_1,s_2}_{2r_1+r_2-1,2|r_1-(d-j)|+r_2+d-j'} \right\} \left\{ C_2|r_1-(d-j)|+r_2+d-j'-1,s-1 + (s_2 + r_2 + d - j' - 1) C_2|r_1-(d-j)|+r_2+d-j'-1,s \right\}
\]

The equation (3.4) can be rewritten as follows:
Let \( Z \) operations inductively if the entry of the Gram matrix \( G \)

We perform the column operations inductively on the Gram matrix

where \( C \) following way.

\[
H_{2r_1 + r_2, s} = 
\sum_{0 \leq j \leq d}
\sum_{j' \leq d - j} \sum_{2j - j' \leq d - 1}
(-1)^{1 + j'} B_{2r_1 + r_2 - 1, 2[0 - (d - j)] + r_2 + d - j'} C_{2[0 - (d - j)] + r_2 + d - j' - 1, s - 1}
+ (-1)^d (s_2 + r_2 - 1) C_{2r_1 + r_2 - 1, s}
+ \sum_{0 \leq j \leq d}
\sum_{j' \leq d - j} \sum_{2j - j' \leq d - 1}
(-1)^{1 + j'} (s_2 + r_2 + d - j') B_{2r_1 + r_2 - 1, 2[0 - (d - j)] + r_2 + d - j'} C_{2[0 - (d - j)] + r_2 + d - j' - s}
+ \sum_{0 \leq j \leq d}
\sum_{j' \leq d - j} \sum_{2j - j' \leq d - 1}
(-1)^{1 + j'} (s_2 + r_2 + d - j' - 1) B_{2r_1 + r_2 - 1, 2[0 - (d - j)] + r_2 + d - j' - 1} C_{2[0 - (d - j)] + r_2 + d - j' - 1, s}
= C_{2r_1 + r_2 - 1, s - 1} + (-1)^d (s_2 + r_2 - 1) C_{2r_1 + r_2 - 1, s}
\]

(by induction and canceling the common terms)

Thus, equation (3.44) reduces to

\[
H_{2r_1 + r_2, s} = C_{2r_1 + r_2 - 1, s - 1} + (-1)^d (s_2 + r_2 - 1) C_{2r_1 + r_2 - 1, s} = (-1)^d C_{2r_1 + 2r_2 - 1, s}
\]

where \( C_{2r_1 + r_2, s} = C_{2r_1 + r_2 - 1, s - 1} + (s_2 + r_2 - 1) C_{2r_1 + r_2 - 1, s} \).

Proof of (b) is same as of proof of (a). \( \square \)

Lemma 3.1.21. Let \( U^{(d, P)} (d, P) \in \mathbb{J}_{2s_1 + s_2} \left( G_{2s_1 + s_2} \right) \). The \( ij \)-entry of the Gram matrix \( G_{2s_1 + s_2} \left( G_{2s_1 + s_2} \right) \) remains zero even after applying column operations inductively if the \( Z_2 \)-horizontal edge of \( U^{(d, P)} (d, P) \) \( U^{(d, P)} (d, P) \) coincides with the \( \{ e \} \)-through class of \( U^{(d, P)} (d, P) \) \( U^{(d, P)} (d, P) \) and vice versa.

Proof.

The proof follows from Definition 3.1.8 and there is no diagram in common which is coarser than both \( U^{(d, P)} (d, P) \) \( U^{(d, P)} (d, P) \) \( U^{(d, P)} (d, P) \) \( U^{(d, P)} (d, P) \) \( U^{(d, P)} (d, P) \). \( \square \)

We perform the column operations inductively on the Gram matrix \( G_{2s_1 + s_2} \) in the following way.

We perform the column operation \( L_k \rightarrow L_k - L_i \) whenever \( U^{(d, P)} (d, P) \) is coarser than \( U^{(d, P)} (d, P) \) \( \forall k > i, 1 \leq i \leq f_{2s_1 + s_2} \).
Let $i < j$ and $b_{ij}$ denote inductively the $ij$-entry of the matrix $G'_{2s_1+s_2}$ after the column operations are carried out. Then

$$b_{ij} = 0 - \sum_{U^{(d_k,P_k)}_{(d_k,P_k)} > U^{(d_j,P_j)}_{(d_j,P_j)}} b_{il}$$

The above can be rewritten as follows:

$$b_{ij} = - \sum_{U^{(d_k,P_k)}_{(d_k,P_k)} > U^{(d_j,P_j)}_{(d_j,P_j)}} b_{il} - \sum_{U^{(d_k,P_k)}_{(d_k,P_k)} \neq U^{(d_j,P_j)}_{(d_j,P_j)}} b_{il} \tag{3.5}$$

but $b_{il} = 0$, by induction, $b_{il} = 0$, since there is no diagram coarser than $U^{(d_k,P_k)}_{(d_k,P_k)}$ and $U^{(d_j,P_j)}_{(d_j,P_j)}$.

Similar column operations are performed for the Gram matrices $\tilde{G}_{2s_1+s_2}$ and $G_s$.

Lemma 3.1.22. Let $U^{(d_k,P_k)}_{(d_k,P_k)}, U^{(d_j,P_j)}_{(d_j,P_j)} \in \mathbb{J}_{2s_1+s_2}, U^{(d_k,P_k)}_{(d_k,P_k)}, U^{(d_j,P_j)}_{(d_j,P_j)} \in \mathbb{J}_{2s_1+s_2}$ and $U^{R^d}_{R^d}, U^{R^d}_{R^d} \in \mathbb{J}_s$ where $\mathbb{J}_{2s_1+s_2}, \mathbb{J}_{2s_1+s_2}$ and $\mathbb{J}_s$ are as in Notation 3.1.6.

(a) If $\# \left( U^{(d_k,P_k)}_{(d_k,P_k)} U^{(d_j,P_j)}_{(d_j,P_j)} \right) = 2s_1 + s_2$ then

$$b_{ij} = 0 \ \forall \ l(d'_k d'_j) \geq 0 \ \left( l(d'_k d'_j) \geq 0 \right), i \neq j, i < j,$$

(b) If $\# \left( U^{\tilde{d}_k, \tilde{P}_k}_{\tilde{d}_k, \tilde{P}_k} U^{\tilde{d}_j, \tilde{P}_j}_{\tilde{d}_j, \tilde{P}_j} \right) = 2s_1 + s_2$ then

$$b_{ij} = 0 \ \forall \ l(\tilde{d}'_k \tilde{d}'_j) \geq 0, i \neq j, i < j,$$

(c) If $\# \left( U^{R^d}_{R^d} U^{R^d}_{R^d} \right) = s$ then

$$b_{ij} = 0 \ \forall \ l \left( R^d R^d \right) \geq 0, i \neq j, i < j,$$

where $b_{ij}$ is as in (3.5).

Proof.

Proof of (a): Let $U^{(d_k,P_k)}_{(d_k,P_k)}$ be the unique diagram satisfying the following condition:

$U^{(d_k,P_k)}_{(d_k,P_k)}$ is the smallest diagram among the diagrams coarser than $U^{(d_i,P_i)}_{(d_i,P_i)}$ and $U^{(d_j,P_j)}_{(d_j,P_j)}$ satisfying

$$l(d'_k d'_j) = l(d'_k d'_j) = l(d'_k d'_j) \ \forall \ 1 \leq j \leq f_{2s_1+s_2}$$

where $d'_k = U^{(d_k,P_k)}_{(d_k,P_k)}, d'_j = U^{(d_j,P_j)}_{(d_j,P_j)}$ and $d'_i = U^{(d_i,P_i)}_{(d_i,P_i)}$. 
Proof of (c) is similar to proof of (a).

Subcase (i): \( l(d'_j d'_j) = 1 \) then there exists only one diagram lying above say \( U^{(d_k, p_k)} \).

Therefore

\[ b_{ij} = 1 - 1 = 0. \]

Thus, \( b_{ij} = 0 \) \( \forall \) \( l(d'_j d'_j) = 0, l(d'_j d'_j) = 1. \)

We prove the result by induction on \( j. \)

Assume that \( b_{ij} = 0 \) \( \forall \) \( l(d'_j d'_j) < l(d'_j d'_j). \)

\[ b_{ij} = a_{ij} - \sum_{U^{(d_k, p_k)} > U^{(d_j, p_j)} \atop U^{(d_k, p_k)} \atop U^{(d_j, p_j)}} b_{il} - \sum_{U^{(d_k, p_k)} > U^{(d_j, p_j)} \atop U^{(d_k, p_k)} \atop U^{(d_j, p_j)}} b_{li} \quad (3.6) \]

since \( U^{(d_k, p_k)} > U^{(d_j, p_j)}, l(d'_j d'_j) \leq l(d'_j d'_j) \) equation (3.6) becomes

\[ b_{ij} = a_{ij} - \sum_{U^{(d_k, p_k)} > U^{(d_j, p_j)} \atop U^{(d_k, p_k)} \atop U^{(d_j, p_j)}} b_{il} - b_{kk} \]

\[ = \left( a_{kj} - \sum_{U^{(d_k, p_k)} > U^{(d_j, p_j)} \atop U^{(d_k, p_k)} \atop U^{(d_j, p_j)}} b_{li} \right) - b_{kk} \]

\[ = b_{kk} - b_{kk} \quad (\text{by Theorem 3.1.20 and induction hypothesis}) \]

\[ = 0. \]

The same proof works for \( U^{(d_i, p_i)} U^{(d_i, p_i)} \in \mathbb{Z}^{2r_1 + s_2}_{2s_1 + s_2} \) where \( p^p \left( U^{(d_i, p_i)} U^{(d_i, p_i)} \right) = 2s_1 + s_2. \)

Proof of (c) is similar to proof of (a).

\[ \square \]

Notation 3.1.23.

(a) Let \( U^{(d_i, p_i)} U^{(d_i, p_i)} \in \mathbb{Z}^{2r_1 + s_2}_{2s_1 + s_2} \) such that \( p^p \left( U^{(d_i, p_i)} U^{(d_i, p_i)} \right) < 2s_1 + s_2, \) so that the \( ij \)-entry of the block matrix \( A_{2r_1 + r_2, 2r_1 + r_2} \) is zero and \( 0 \leq r_1 \leq k - s_1 - s_2, \)

\[ 0 \leq r_2 < k - s_1 - s_2, 2r_1 + r_2 \leq 2k - 2s_1 - s_2. \]
Put $U^{(d_i,P_i)}_{(d_i,P_j)} = U^{f_j}_{l_j} \otimes U^{d_i-f}_{d_i} \otimes U^{d_j-f}_{d_j}$ and $U^{(d_j,P_j)}_{(d_j,P_i)} = U^{f_j}_{l_j} \otimes U^{d_j-f}_{d_j}$ where $U^{f_j}_{l_j}$ is the sub diagram of $U^{(d_i,P_i)}_{(d_j,P_j)}$. $U^{f_j}_{l_j}$, $U^{d_j-f}_{d_j} \in \mathbb{J}_{2s_1+t_2}^{2s_1+s_2}$ and every $\{e\} (Z_2)$ through class of $U^{f_j}_{l_j}$ is replaced by a $\{e\} (Z_2)$-horizontal edge and vice versa.

(b) Let $U^{(d_i,P_i)}_{(d_i,P_j)}$, $U^{(d_j,P_j)}_{(d_j,P_i)} \in \mathbb{J}_{2s_1+s_2}^{2s_1+t_2}$ such that $\varphi\left(U^{(d_i,P_i)}_{(d_j,P_j)} \circ U^{(d_j,P_j)}_{(d_i,P_i)}\right) < 2s_1 + s_2$, so that the $ij$-entry of the block matrix $\tilde{A}_{2s_1+r_2,2s_1+s_2}$ is zero and $0 \leq r_1 \leq k-s_1-s_2-1$, $0 \leq r_2 < k-s_1-s_2-1,2r_1+r_2 \leq 2k-2s_1-s_2-1$.

Put $U^{(d_i,P_i)}_{(d_i,P_j)} = U^{f_j}_{l_j} \otimes U^{d_i-f}_{d_i}$ and $U^{(d_j,P_j)}_{(d_j,P_i)} = U^{f_j}_{l_j} \otimes U^{d_j-f}_{d_j}$ where $U^{f_j}_{l_j}$ is the sub diagram of $U^{(d_i,P_i)}_{(d_j,P_j)}$. $U^{f_j}_{l_j}$, $U^{d_j-f}_{d_j} \in \mathbb{J}_{2s_1+t_2}^{2s_1+s_2}$ and every $\{e\} (Z_2)$ through class of $U^{f_j}_{l_j}$ is replaced by a $\{e\} (Z_2)$-horizontal edge and vice versa.

(c) Let $UR^{d_i}_{R_i}$, $UR^{d_j}_{R_j} \in \mathbb{J}_s$ such that $\varphi\left(U^{d_i}_{R_i} \circ U^{d_j}_{R_j}\right) < s$, so that the $ij$-entry of the block matrix $A_{r,r}$ is zero and $0 \leq r \leq k-s$.

Put $UR^{d_i}_{R_i} = U^{f_i}_{l_i} \otimes U^{d_i-l_i}_{d_i}$ and $UR^{d_j}_{R_j} = U^{f_j}_{l_j} \otimes U^{d_j-l_j}_{d_j}$ where $U^{f_i}_{l_i}$ is the sub diagram of $UR^{d_i}_{R_i}$. $U^{f_i}_{l_i}, U^{f_j}_{l_j} \in \mathbb{J}_1$ and every through class of $U^{f_i}_{l_i}$ is replaced by a horizontal edge and vice versa.

Example 3.1.24. This example illustrates Notation 3.1.23.

<table>
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<th>s.no</th>
<th>$U^{(d_i,P_i)}_{(d_i,P_j)}$</th>
<th>$U^{(d_j,Q_j)}_{(d_j,Q_i)}$</th>
<th>$U^{f_j}_{l_j}$</th>
<th>$U^{f_i}_{l_i}$</th>
<th>$U^{d_i-f}<em>{d_i, -f} = U^{d_j-f}</em>{d_j, -f}$</th>
</tr>
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</tr>
<tr>
<td>2.</td>
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<td>$\square \square \square \square \square$</td>
<td>$\square \square \square \square \square$</td>
<td>$\square \square \square \square \square$</td>
<td>$\square \square \square \square \square$</td>
</tr>
</tbody>
</table>

Lemma 3.1.25.

(a) Let $U^{(d_i,P_i)}_{(d_i,P_j)} \in \mathbb{J}_{2s_1+s_2}^{2r_1+r_2}$ and $U^{(d_j,P_j)}_{(d_j,P_i)} \in \mathbb{J}_{2s_1+s_2}^{2r_1+r_2}$ such that

$\varphi\left(U^{(d_i,P_i)}_{(d_j,P_j)} \circ U^{(d_j,P_j)}_{(d_i,P_i)}\right) < 2s_1 + s_2$ with $U^{(d_i,P_i)}_{(d_i,P_i)} = U^{f_j}_{l_j} \otimes U^{d_i-f}_{d_i}$ and $U^{(d_j,P_j)}_{(d_j,P_j)} = U^{f_j}_{l_j} \otimes U^{d_j-f}_{d_j}$ where $U^{f_j}_{l_j} \in \mathbb{J}_{2s_1+t_2}^{2s_1+s_2}$, $U^{d_i-f}_{d_i} \in \mathbb{J}_{2s_1+s_2}^{2r_1+r_2}$ and $U^{d_j-f}_{d_j} \in \mathbb{J}_{2s_1+s_2}^{2r_1+r_2}$.
(b) Let \( U^{(d'_j, \vec{P}_j)}_{(d_i, \vec{P}_i)} \in \mathcal{P}_{2s_1+2s_2} \) and \( U^{(d''_j, \vec{P}'_j)}_{(d_i, \vec{P}_i)} \in \mathcal{P}_{2s'_1+2s'_2} \) such that \( \mathcal{P}(U^{(d'_j, \vec{P}_j)}_{(d_i, \vec{P}_i)}, U^{(d''_j, \vec{P}'_j)}_{(d_i, \vec{P}_i)}) < 2s_1 + 2s_2 \)

with \( U^{(d'_j, \vec{P}_j)}_{(d_i, \vec{P}_i)} = \tilde{U}^{(d'_j, \vec{P}_j)}_{(d_i, \vec{P}_i)} \otimes \tilde{U}^{d_i - f}_{d_i - f} \) and \( U^{(d''_j, \vec{P}'_j)}_{(d_i, \vec{P}_i)} = \tilde{U}^{(d''_j, \vec{P}'_j)}_{(d_i, \vec{P}_i)} \otimes \tilde{U}^{d_i - f}_{d_i - f} \) where \( \tilde{U}^{(d'_j, \vec{P}_j)}_{(d_i, \vec{P}_i)} \in \mathcal{P}_{2s_1+2s_2} \), \( \tilde{U}^{(d''_j, \vec{P}'_j)}_{(d_i, \vec{P}_i)} \in \mathcal{P}_{2s'_1+2s'_2} \), \( \tilde{U}^{d_i - f}_{d_i - f} \in \mathcal{P}_{2(s_1 - t_1) + 2s_2 - t_2} \) and \( \tilde{U}^{d_j - f}_{d_j - f} \in \mathcal{P}_{2(s_1 - t_1) + 2s_2 - t_2} \).

Then

\[ \tilde{b}_{ij} = 0, \text{ if } i < j \]

after all the column operations are carried out for the Gram matrices of the algebra of \( \mathbb{Z}_2 \)-relations and signed partition algebras if any one of the following conditions hold:

(i) \( r'_1 \neq r_1 \) or \( r'_2 \neq r_2 \) or

(ii) \( t''_1 \neq t_1 \) or \( t''_2 \neq t_2 \) or

(iii) \( U^{d_i - f}_{d_i - f} \neq U^{d_j - f}_{d_j - f} \left( \tilde{U}^{d_i - f}_{d_i - f} \neq \tilde{U}^{d_j - f}_{d_j - f} \right) \)

(c) Let \( U^{R_i} \in \mathcal{P}_s \) and \( U^{R'_j} \in \mathcal{P}_{s'} \) such that \( \mathcal{P}(U^{R_i}, U^{R'_j}) < s \) with \( U^{R_i} = U^{t_1 \otimes U^{d_i - l_1}} \) and \( U^{R'_j} = U^{t_2 \otimes U^{d_j - l_2}} \) where \( U^{t_1} \in \mathcal{P}_t \), \( U^{t_2} \in \mathcal{P}_{t'} \), \( U^{d_i - l_1} \in \mathcal{P}_{s - t} \) and \( U^{d_j - l_2} \in \mathcal{P}_{s' - t'} \).

Then

\[ \tilde{b}_{ij} = 0, \text{ if } i < j \]

after all the column operations are carried out for the Gram matrices of partition algebras if any one of the following conditions hold:

(i) \( r' \neq r \)

(ii) \( t'' \neq t \)

(iii) \( U^{d_i - l_1} \neq U^{d_i - l_2} \)

Proof.

Proof of (a): The proof is by induction on the conditions

(i) \( r'_1 \neq r_1 \) or \( r'_2 \neq r_2 \) or

(ii) \( t'_1 \neq t_1 \) or \( t'_2 \neq t_2 \) or
(iii) \( U^{d_i-f}_{(d_i,P_i)} \neq U^{d_j-f}_{(d_j,P_j)} \)

Since \( \sharp^p \left( U^{(d_i,P_i)}_{(d_i,P_i)} U^{(d_j,P_j)}_{(d_j,P_j)} \right) < 2s_1 + s_2 \) which implies that \( a_{ij} = 0 \).

After applying column operations inductively we get,

\[
b_{ij} = - \sum_{U^{(d_i,P_i)}_{(d_i,P_i)} > U^{(d_i,P_i)}_{(d_i,P_i)}} b_{il} - \sum_{U^{(d_j,P_j)}_{(d_j,P_j)} > U^{(d_j,P_j)}_{(d_j,P_j)}} b_{il}
\]  \hspace{1cm} (3.7)

Case (i): Suppose that \( \sharp^p \left( U^{(d_i,P_i)}_{(d_i,P_i)} U^{(d_i,P_i)}_{(d_i,P_i)} \right) = 2s_1 + s_2 \) then by Lemma 3.1.22 and induction hypothesis,

\[
b_{il} = 0.
\]

Case (ii): Suppose that \( \sharp^p \left( U^{(d_i,P_i)}_{(d_i,P_i)} U^{(d_i,P_i)}_{(d_i,P_i)} \right) < 2s_1 + s_2 \) then by using induction on \( i \) and \( l \)

\[
b_{il} = 0, \text{ if } i < l < j
\]

Thus, we shall consider only when \( l \geq i \)

Put, \( U^{(d_i,P_i)}_{(d_i,P_i)} = U^{p_i}_{l_j} \otimes U^{d_i-f}_{(d_i,P_i)} \) and \( U^{(d_j,P_j)}_{(d_j,P_j)} = U^{l_i}_{j_j} \otimes U^{d_j-f}_{(d_j,P_j)} \) where \( U^{p_i}_{l_j} \in \sum_{2t_1+t_2} \) which is coarser than \( U^{l_i}_{j_j} \).

Choose a diagram \( U^{(d_k,P_k)}_{(d_k,P_k)} \) which is the smallest among the diagrams coarser than both \( U^{(d_i,P_i)}_{(d_i,P_i)} \) and \( U^{(d_j,P_j)}_{(d_j,P_j)} \) and then \( U^{(d_k,P_k)}_{(d_k,P_k)} = U^{l_i}_{j_j} \otimes U^{d_k-f}_{(d_k,P_k)} \).

It is clear that, \( U^{(d_k,P_k)}_{(d_k,P_k)} \) is coarser than \( U^{(d_j,P_j)}_{(d_j,P_j)} \).

Thus after applying the column operations \( L_j \rightarrow L_j - L_k \) we get,

\[
b_{ij} = - \sum_{U^{(d_i,P_i)}_{(d_i,P_i)} > U^{(d_i,P_i)}_{(d_i,P_i)}} b_{il} - \sum_{U^{(d_j,P_j)}_{(d_j,P_j)} > U^{(d_j,P_j)}_{(d_j,P_j)}} b_{il} + b_{ik}
\]

\[
= b_{ik} - b_{ik}
\]

\[
= 0.
\]

Proof of (b) and (c) are same as that of proof of (a). \( \square \)
Notation 3.1.26. Put,

(i) \( \phi^{s_1,s_2}_{2r_1+r_2}(x) = \prod_{j=0}^{r_1-1} [x^2 - x - 2(s_1 + j)] \prod_{l=0}^{r_2-1} [x - (s_2 + l)], \quad r_1 \geq 1, r_2 \geq 1. \)

(ii) \( \phi^{s_1,s_2}_{2r_1+0}(x) = \prod_{j=0}^{r_1-1} [x^2 - x - 2(s_1 + j)] \quad r_2 = 0. \)

(iii) \( \phi^{s_1,s_2}_{2,0+r_2}(x) = \prod_{l=0}^{r_2-1} [x - (s_2 + l)], \quad r_1 = 0. \)

(iv) \( \phi^{s_1,s_2}_{0+0}(x) = 1 \) and \( \phi^{s_1,s_2}_{2r_1+r_2}(x) = 0 \) if any one of \( r_1, r_2 < 0. \)

(v) \( \phi^{r}(x) = \prod_{l=0}^{r-1} [x - (s + l)], \quad r \geq 1 \)

(vi) \( \phi^{0}(x) = 1 \) and \( \phi^{r}_{0} = 0 \) if \( r < 0. \)

Now, we derive the following relation between the polynomials which are needed in the following Lemmas:

Lemma 3.1.27.

(i) \( \phi^{s_1+s_2+t}_{2(r_1-t)+r_2}(x) = \phi^{s_1-s_2}_{2(r_1-t)+r_2}(x) - \sum_{m=1}^{2t} 2C_m \; r_1-tC_m \; 2^m \; m! \; \phi^{s_1-t,s_2+t}_{2(r_1-t-m)+r_2}(x). \)

(ii) \( \phi^{s_1,s_2+t}_{2r_1+r_2-t}(x) = \phi^{s_1,s_2-t}_{2r_1+r_2-t}(x) - \sum_{m=1}^{2t} 2C_m \; r_2-tC_m \; m! \; \phi^{s_1,s_2+t}_{2r_1+r_2-t-m}(x). \)

(iii) In general,

\[
\phi^{s_1+s_2+t}_{2(r_1-t_1)+r_2-t_2}(x) = \phi^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2-l}(x) \\
- \sum_{k=1}^{2t_1} 2t_1C_{k_1} \; (r_1-t_1-l)C_{k_1} \; k! \; \phi^{s_1-t_1,s_2-t_2}_{2(r_1-t_1-l-k)+r_2-t_2-l}(x) \\
- \sum_{k'=1}^{2t_2} 2t_2C_{k'} \; (r_2-t_2-l')C_{k'} \; k'! \; \phi^{s_1-t_1,s_2+t_2}_{2(r_1-t_1-l)+r_2-t_2-l'-k'}(x) \\
- \sum_{k=1}^{2t_1} \sum_{k'=1}^{2t_2} 2t_1C_{k_1} \; (r_1-t_1-l)C_{k_1} \; k! \; 2t_2C_{k'} \; (r_2-t_2-l')C_{k'} \; k'! \; \phi^{s_1+t_1,s_2+t_2}_{2(r_1-t_1-l-k)+r_2-t_2-l'-k'}(x)
\]

where \( \phi^{s_1+t_2}_{2(r_1-t)+r_2}(x) = \prod_{l=0}^{r_1-t-1} [x^2 - x - 2(s_1 + t + l)] \prod_{l'=0}^{r_2-1} [x - (s_2 + l')] \) and
\( \phi^{s_1,s_2+t}_{2r_1+r_2-t}(x) = \prod_{l=0}^{r_1-1} [x^2 - x - 2(s_1 + l)] \prod_{l'=0}^{r_2-t-1} [x - (s_2 + t + l')]. \)
Proof.

Proof of (i): We shall prove this by using induction on \( r_1 - t \) and \( r_2 \). Consider

\[
\phi_{2(r_1-t)}^{s_1-t,s_2}(x) - \sum_{m=1}^{2t} 2tC_m (r_1-t)C_m \, 2^m \, m! \phi_{2[r_1-t-m]+r_2}^{s_1+t,s_2}(x). \tag{3.8}
\]

\[
= \phi_{2(r_1-t-1)+r_2}^{s_1-t,s_2}(x)(x^2 - x - 2(s_1 + r_1 - 2t - 1))
- \sum_{m=1}^{2t} 2tC_m (r_1-t)C_m \, 2^m \, m! \phi_{2[r_1-t-m]+r_2}^{s_1+t,s_2}(x)
= \phi_{2(r_1-t-1)+r_2}^{s_1-t,s_2}(x)(x^2 - x - 2(s_1 + r_1 - 2t - 1))
+ \sum_{m=1}^{2t} 2tC_m (r_1-t-1)C_m \, 2^m \, m! \phi_{2[r_1-t-m-1]+r_2}^{s_1+t,s_2}(x)(x^2 - x - 2(s_1 + r_1 - 2t - 1))
- \sum_{m=1}^{2t} 2tC_m (r_1-t)C_m \, 2^m \, m! \phi_{2[r_1-t-m]+r_2}^{s_1+t,s_2}(x)
\quad \text{(by induction)}
= \phi_{2(r_1-t-1)+r_2}^{s_1-t,s_2}(x)(x^2 - x - 2(s_1 + r_1 - 2t - 1))
+ \sum_{m=1}^{2t} 2tC_m (r_1-t-1)C_m \, 2^m \, m! \phi_{2[r_1-t-m-1]+r_2}^{s_1+t,s_2}(x)(x^2 - x - 2(s_1 + r_1 - 2t - 1))
- \sum_{m=1}^{2t} 2tC_m (r_1-t-1)C_m + (r_1-t-1)C_m-1 \, 2^m \, m! \phi_{2[r_1-t-m-1]+r_2}^{s_1+t,s_2}(x)
\quad (x^2 - x - 2(s_1 + r_1 - m - 1))
\]

\[
= \phi_{2(r_1-t-1)+r_2}^{s_1+t,s_2}(x)(x^2 - x - 2(s_1 + r_1 - 2t - 1))
- \sum_{m=1}^{2t} 2tC_m (r_1-t-1)C_{m-1} \, 2^m \, m! \phi_{2[r_1-t-m]+r_2}^{s_1+t,s_2}(x)
+ \sum_{m=1}^{2t} 2tC_m (r_1-t-1)C_m \, 2^m \, m! \phi_{2[r_1-t-m-1]+r_2}^{s_1+t,s_2}(x)
\quad (by \text{canceling\ the\ common\ terms})
= \phi_{2(r_1-t-1)+r_2}^{s_1+t,s_2}(x)(x^2 - x - 2(s_1 + r_1 - 2t - 1)) - 4t\phi_{2(r_1-t-1)+r_2}^{s_1+t,s_2}(x)
= \phi_{2(r_1-t-1)+r_2}^{s_1+t,s_2}(x) (x^2 - x - 2(s_1 + r_1 - 1))
= \phi_{2(r_1-t)+r_2}^{s_1+t,s_2}(x)
\]

Proof of (ii) is similar to the proof of (i) and (iii) follows from (i) and (ii). \qed
Lemma 3.1.28. Let \( U^{(d_i,P_i)}_{(d_i,P_i)} U^{(d_j,P_j)}_{(d_j,P_j)} U^{(d_i',P_i')}_{(d_i',P_i')} U^{(d_j,P_j)}_{(d_j,P_j)} \) and \( U^{R_i'}_{R_i} U^{R_j'}_{R_j} \) are as in Notation 3.1.23.

(a) After performing the column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both \( U^{(d_i,P_i)}_{(d_i,P_i)} \) and \( U^{(d_j,P_j)}_{(d_j,P_j)} \), the zero in the \( ij \)-entry of the block matrix \( A_{2r_1+r_2,2r_1+r_2} \) for \( 0 \leq r_1, r_2, r_1 + r_2 \leq k - s_1 - s_2 \) is replaced by

\[
-2^t_1 t_1! t_2! x^{2(r_1-t_1)+r_2-t_2}.
\]

(b) After performing the column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both \( U^{(d_i,P_i)}_{(d_i,P_i)} \) and \( U^{(d_j,P_j)}_{(d_j,P_j)} \), the zero in the \( ij \)-entry of the block matrix \( A_{2r_1+r_2,2r_1+r_2} \) for \( 0 \leq r_1, r_2, r_1 + r_2 \leq k - s_1 - s_2 - 1 \) is replaced by

\[
-2^t_1 t_1! t_2! x^{2(r_1-t_1)+r_2-t_2}.
\]

(c) After performing the column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both \( U^{R_i'}_{R_i} \) and \( U^{R_j'}_{R_j} \), the zero in the \( ij \)-entry of the block matrix \( A_{r,r} \) for \( 0 \leq r \leq k - s \) is replaced by

\[
-t! x^{r-t}.
\]

Proof.

Proof of (a): We shall prove this by induction on \( t_1 \) and \( t_2 \).

Case (i): Let \( t_1 = 1 \) and \( t_2 = 1 \).

We know that the diagrams coarser than both \( U^{(d_i,P_i)}_{(d_i,P_i)} \) and \( U^{(d_j,P_j)}_{(d_j,P_j)} \) are obtained if and only if the pair of \( \{e\} \)-through classes and the pair of \( \{e\} \)-horizontal edges of \( U^{p_1}_{ij} \) or \( U^{p_2}_{ij} \) is connected by an \( \{e\} \)-horizontal edge which can be done in two ways and \( \mathbb{Z}_2 \)-horizontal edge and \( \mathbb{Z}_2 \)-through class of \( U^{p_1}_{ij} \) or \( U^{p_2}_{ij} \) is connected by a \( \mathbb{Z}_2 \)-edge which can be done in one way. Also \( U^{d_i-f}_{d_1-f} \) and \( U^{d_j-f}_{d_1-f} \) have 2(\( r_1-1 \)+\( r_2-1 \)) horizontal edges then after performing the column operations the zero in the \( ij \)-entry of the block matrix \( A_{2r_1+r_2,2r_1+r_2} \) is replaced by

\[
-2 \sum_{l=0}^{r_1-1} \sum_{l'=0}^{r_2-1+l} B^{a_1,a_2}_{2(r_1-1)+r_2-1,2(r_1-1-l)+r_2-1+l'} \phi^{a_1,a_2}_{2(r_1-1-l)+r_2-1+l'}(x)
\]

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which is equal to

\[-2\phi_{2[r_1-1]+r_2-1}^{s_1,s_2}(x) - 2 \sum_{l=1}^{r_1-1} \sum_{l'=1}^{r_2-1} B_{2(r_1-1)+r_2-2(r_1-l)+r_2-1+l'}^{s_1,s_2} \phi_{2[r_1-l]+r_2-1+l'}^{s_1,s_2}(x)\]

By Theorem 3.1.20 we know that,

\[
\phi_{2[r_1-1]+r_2-1}^{s_1,s_2}(x) = 2^{2(r_1-1)+r_2-1} \sum_{l=1}^{r_1-1} \sum_{l'=1}^{r_2-1} B_{2(r_1-1)+r_2-2(r_1-l)+r_2-1+l'}^{s_1,s_2} \phi_{2[r_1-l]+r_2-1+l'}^{s_1,s_2}(x)
\]  

(3.9)

Substituting equation (3.9) in the above expression and canceling the common terms we get,

\[-2x^{2(r_1-1)+r_2-1}.\]

In general, the diagrams coarser than both \(U^{(d_i,P_i)}_{(d_j,P_j)}\) and \(U^{(d_j,P_j)}_{(d_j,P_j)}\) are obtained if and only if \(t_1(t_2)\) pairs(numbers) of \(\{e\}\) \(Z_2\)-through classes and \(t_1(t_2)\) pairs(numbers) of \(\{e\}\) \(Z_2\)-horizontal edges of \(U^{(d_i,P_i)}_{t_f}\) or \(U^{(d_j,P_j)}_{t_f}\) is connected by an \(\{e\}\) \(Z_2\)-horizontal edges which can be done in \(2^t_1 t_1! t_2!\) ways. Also \(U^{d_i-f}_{d_j-f}\) and \(U^{d_j-f}_{d_j-f}\) have \(2(r_1-t_1) + r_2 - t_2\) horizontal edges then after performing the column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both \(U^{(d_i,P_i)}_{(d_i,P_i)}\) and \(U^{(d_j,P_j)}_{(d_j,P_j)}\) the zero in the \(ij\)-entry of the block matrix \(A_{2r_1+r_2,2r_1+r_2}\) is replaced by

\[-2^{t_1} t_1! t_2! \sum_{l=0}^{r_1-1} \sum_{l'=0}^{r_2-1} B_{2(r_1-t_1)+r_2-2(r_1-l)+r_2-2+l'}^{s_1,s_2} \phi_{2[r_1-l]+r_2-2+l'}^{s_1,s_2}(x)\]

which is equal to

\[-2^{t_1} t_1! t_2! \left(\phi_{2[r_1-t_1]+r_2-2}^{s_1,s_2}(x) - \sum_{l=1}^{r_1-1} \sum_{l'=1}^{r_2-1} B_{2(r_1-t_1)+r_2-2(r_1-l)+r_2-2+l'}^{s_1,s_2} \phi_{2[r_1-l]+r_2-2+l'}^{s_1,s_2}(x)\right)\]

Substituting equation (3.9) in the above expression and canceling the common terms we get,

\[-2^{t_1} t_1! t_2! x^{2(r_1-t_1)+r_2-2}.\]

Proof of (b) and (c) are similar to the proof of (a).

**Proposition 3.1.29.** Let \(U^{(d_i,P_i)}_{(d_i,P_i)}, U^{(d_j,P_j)}_{(d_j,P_j)}, U^{(d_i,P_i)}_{(d_i,P_i)}, U^{(d_j,P_j)}_{(d_j,P_j)}\) and \(U^{R_i}_{R_i}, U^{R_j}_{R_j}\) be as in Notation 3.1.23.

(a) After performing the column operations to eliminate the non-zero entries which lie above corresponding to the diagrams coarser than \(U^{(d_j,P_j)}_{(d_j,P_j)}\),
(b) After performing the column operations to eliminate the non-zero entries which lie above corresponding to the diagrams coarser than \( U_{(d_j,P_j)} \), then the \( ij \)-entry of the block matrix \( A_{2r_1+2r_2} \) for \( 0 \leq r_1 + r_2, r_1, r_2 \leq k-s_1-s_2 \) and the block matrix \( \tilde{A}_{2r_1+2r_2} \) for \( 0 \leq r_1+2r_2, r_1, r_2 \leq k-s_1-s_2-1 \) is replaced by

\[
(\text{i}) \quad (-1)^{t_1+t_2} (t_1)! (t_2)! 2^{\frac{1}{2}} \prod_{j=t_1}^{r_1-1} \left[ x^2 - x - 2(s_1 + j) \right] \prod_{l=t_2}^{r_2-1} \left[ x - (s_2 + l) \right] \quad \text{if } r_1 \geq 1 \text{ and } r_2 \geq 1,
\]

\[
(\text{ii}) \quad (-1)^{t_2} (t_2)! \prod_{l=t_2}^{r_2-1} \left[ x - (s_2 + l) \right] \quad \text{if } r_1 = 0 \text{ and } r_2 \neq 0,
\]

\[
(\text{iii}) \quad (-1)^{t_1} (t_1)! 2^{\frac{1}{2}} \prod_{j=t_1}^{r_1-1} \left[ x^2 - x - 2(s_1 + j) \right] \quad \text{if } r_1 \neq 0 \text{ and } r_2 = 0,
\]

(c) After performing the column operations to eliminate the non-zero entries which lie above corresponding to the diagrams coarser than \( U_{R_i} \), then the \( ij \)-entry is replaced by

\[
(-1)^{t} t! \prod_{j=t}^{r_1-1} \left[ x - (s + l) \right].
\]

**Proof.**

**Proof of (a):** We shall prove this by using induction on \( t_1, t_2 \) and the number of horizontal edges.

By Lemma 3.1.25 the \( ij \)-entry \( b_{ij} \) is given by

\[
b_{ij} = - \sum_{U_{(d_j,P_j)} > U_{(d_i,P_i)}} b_{il} - \sum_{U_{(d_i,P_i)} > U_{(d_j,P_j)}} b_{il} - \sum_{U_{(d_j,P_j)} > U_{(d_i,P_i)}} b_{il} \quad (3.10)
\]

**Case (i):** Let \( t_1 = 1, t_2 = 0 \) and \( U_{d_i-1}^{d_j} \) and \( U_{d_j-1}^{d_j-1} \) have \( 2(s_1 - 1) + s_2 \) through classes and no horizontal edge. After applying column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both \( U_{(d_i,P_i)}^{(d_j,P_j)} \) and \( U_{(d_j,P_j)}^{(d_j,P_j)} \) then by Lemma 3.1.28 and equation (3.10) the \( ij \)-entry \( b_{ij} \) of the block matrix \( A_{2s_1+s_2,2s_1+s_2} \) is given by

\[
b_{ij} = (-1) 2 1!.
\]
Since there is no diagram coarser than \(U^{(d_1, P_2)}\) alone.

**Case (ii):** Let \(t_1 = 0, t_2 = 1\) and \(U^{d_{1,-f}}_{d_1-f}\) and \(U^{d_{j,-f}}_{d_j-f}\) have \(2s_1 + s_2 - 1\) through classes and no horizontal edge. After applying column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both \(U^{(d_1, P_1)}\) and \(U^{(d_j, P_j)}\) then by Lemma 3.1.28 and equation (3.10) the \(ij\)-entry \(b_{ij}\) of the block matrix \(\tilde{A}_{2r_1+r_2,2r_1+r_2}\) is given by

\[
b_{ij} = (-1)^!.
\]

Since there is no diagram coarser than \(U^{(d_1, P_2)}\) alone.

In general, suppose that the diagrams \(U^{d_{1,-f}}_{d_1-f}\) and \(U^{d_{j,-f}}_{d_j-f}\) have \(2(s_1 - t_1) + s_2 - t_2\) through classes and have \(r_1 - t_1\) pair of \(\{e\}\)-horizontal edges and \(r_2 - t_2\) number of \(\mathbb{Z}_2\)-horizontal edges then after performing column operations to eliminate the coarser elements of \(U^{(d_1, P_1)}\) and \(U^{(d_j, P_j)}\) having \(t'\) pair of \(\{e\}\)-through classes (\(\{e\}\)-horizontal edges) with \(t' < t\), the 0 in the \(ij\)-entry \(b_{ij}\) of the block matrix \(\tilde{A}_{2r_1+r_2,2r_1+r_2}\) is replaced by \(- (t_1)! (t_2)! 2^{t_1} \cdot x^{2(t_1-t_1)+r_2-t_2}\) inductively.

For, \(0 \leq f' \leq t_1\) and \(0 \leq f'' \leq t_2\), the number of diagrams obtained by joining \(f'(f'')\) pairs(numbers) of \(\{e\}\) (\(\mathbb{Z}_2\)) through classes with \(f'(f'')\) pairs(numbers) of \(\{e\}\) (\(\mathbb{Z}_2\)) horizontal edges by \(\{e\}\) (\(\mathbb{Z}_2\)) in \(U^{f_1}_{f_1}\) can be done in \((t_1C_f)^2 (t_2C_{f''})^2 2^{f'f''}\) ways.\(f', f''\) which is coarser than \(U^{(d_1, P_2)}\) having \((r_1 - t_1 - l)\)-pairs of \(\{e\}\)-horizontal edges and \(r_2 - t_2 - l'\) number of \(\mathbb{Z}_2\)-horizontal edges is given by

\[
\sum_{m=0}^{2t_1-2f'} \sum_{m'=0}^{2t_2-2f''} (r_1 - t_1 - l + m)C_m (2t_1 - 2f')C_m 2^m m! (r_2 - t_2 - l' + m')C_{m'} (2t_2 - 2f'')C_{m'} (m')! (t_1C_{f'})^2 f'! 2^{f''} (t_2C_{f''} f'')! B_{2(r_1-t_1)r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'}^{(t_1,f'),(s_2,(t_2-f''))}
\]

(3.11) is obtained by choosing \(m(m')\) pairs(numbers) of \(\{e\}\) (\(\mathbb{Z}_2\))-horizontal edges from \(r_1-t_1-l+m (r_2-t_2-l' + m')\) pairs(numbers) of \(\{e\}\) (\(\mathbb{Z}_2\)) in \(U^{d_{1,-f}}_{d_1-f}\) and choose \(m(m')\) of \(\{e\}\) (\(\mathbb{Z}_2\))-connected components(\(\{e\}\) (\(\mathbb{Z}_2\))-horizontal edges or \(\{e\}\) (\(\mathbb{Z}_2\))-through classes) in \(U^{d_{j,-f}}_{d_j-f}\). We can connect these \(\{e\}\) (\(\mathbb{Z}_2\))-components by a \(\{e\}\) (\(\mathbb{Z}_2\))-horizontal edge in \(2^m m! (m')!\) ways. \(m\) and \(m'\) cannot exceed \(2t_1-2f'\) and \(2t_2-2f''\) respectively, since \(U^{d_{j,-f}}_{d_j-f}\) has \(2t_1-2f'\)-pairs of \(\{e\}\)-components and \(2t_2-f''\) number of \(\mathbb{Z}_2\)-components.
\[ b_{ij} = -2^{t_1} t_2! \frac{x^{2(r_1-t_1)+r_2-t_2}}{2^{t_1} \sum_{l=1}^{r_1-t_1} \sum_{t'=1}^{t_2-l} \sum_{m=0}^{2t_2} \sum_{m'=0}^{2t_1+t_1} 2t_1 C_m (r_1 - t_1 - l + m) C_m 2^m m! 2t_2 C_{m'} (r_2 - t_2 - l' + m') C_{m'} (m')! B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'}^{s_1-t_1,s_2-t_2} \phi_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \]

\[
- \left( \sum_{f'=1}^{t_1} \sum_{t=0}^{t_2} \sum_{m=0}^{m'} \sum_{m'=0}^{m} \sum_{t''=0}^{t} (t_1 C_{f'})^2 2^{t'} f^{t'}! (t_2 C_{f''})^2 f''^{t''}! (-1)^{t_1-f'} 2^{t_1-f'} (t_1 - f')! (-1)^{t_2-f''} (t_2 - f'')! 2(t_1 - f') C_m (r_1 - t_1 - l + m) C_m 2^m m! 2(t_2 - f'') C_{m'} (r_2 - t_2 - l' + m') C_{m'} (m')! B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'}^{s_1-t_1+f',s_2-t_2+f''} \phi_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \right)
\]

\[
= -2^{t_1} t_2! \frac{x^{2(r_1-t_1)+r_2-t_2}}{2^{t_1} \sum_{f'=1}^{t_1} \sum_{t'=-1}^{t_2} \frac{(-1)^{t_1-f'} (t_1 C_{f'})^2 2^{t'} f^{t'}! (t_2 C_{f''})^2 f''^{t''}! (t_2 - f'')! x^{2(r_1-t_1)+r_2-t_2}}{(-1)^{t_1} 2^{t_1} t_1! \sum_{f'=1}^{t_2} (-1)^{t_2-f''} (t_2 C_{f''})^2 f''^{t''}! (t_2 - f'')! x^{2(r_1-t_1)+r_2-t_2}} \]

\[
= -(-1)^{t_1} 2^{t_1} t_1! \sum_{f'=1}^{t_1} (t_1 C_{f'})^2 2^{t_1-f'} (t_1 - f')! (-1)^{t_2-f''} (t_2 C_{f''})^2 f''^{t''}! (t_2 - f'')! x^{2(r_1-t_1)+r_2-t_2}
\]

\[
= -(-1)^{t_1} 2^{t_1} \sum_{f'=1}^{t_2} (-1)^{t_2-f''} (t_2 C_{f''})^2 f''^{t''}! (t_2 - f'')! x^{2(r_1-t_1)+r_2-t_2}
\]

\[
= -(-1)^{t_1} (t_1)! (t_2)! 2^{t_1} \left\{ \sum_{l=1}^{r_1-t_1} \sum_{t'=1}^{t_2-l} \sum_{m=0}^{m'} \sum_{m'=0}^{m} 2t_1 C_m (r_1 - t_1 - l + m) C_m 2^m m! 2t_2 C_{m'} (r_2 - t_2 - l' + m') C_{m'} (m')! B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'}^{s_1-t_1+s_2-t_2} \phi_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \right\}
\]

\[
= -(-1)^{t_1} (t_1)! (t_2)! 2^{t_1} \frac{x^{2(r_1-t_1)+r_2-t_2}}{2^{t_1} \sum_{f'=1}^{t_1} \sum_{t'=1}^{t_2} \frac{(-1)^{t_1-f'} (t_1 C_{f'})^2 2^{t'} f^{t'}! (t_2 C_{f''})^2 f''^{t''}! (t_2 - f'')! x^{2(r_1-t_1)+r_2-t_2}}{-(-1)^{t_1} 2^{t_1} (t_1)! (t_2)! 2^{t_1} \frac{x^{2(r_1-t_1)+r_2-t_2}}{2^{t_1} \sum_{f'=1}^{t_2} (-1)^{t_2-f''} (t_2 C_{f''})^2 f''^{t''}! (t_2 - f'')! x^{2(r_1-t_1)+r_2-t_2}}}
\]

\[
= -(-1)^{t_1} (t_1)! (t_2)! 2^{t_1} x^{2(r_1-t_1)+r_2-t_2}
\]
expanding and using Lemma 3.1.28 we get,

\[
b_{ij} = (-1)^{t_1 + t_2} (t_1)! (t_2)! 2^{t_1} \left\{ x^{2(r_1-t_1)+r_2-t_2} - \sum_{l=1}^{r_1-t_1} \sum_{l'=1}^{r_2-t_2} \sum_{l''=1}^{r_2-t_2-l''} B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l}(x) \phi_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \right. \\
+ \sum_{l=1}^{r_1-t_1} \sum_{l'=1}^{r_2-t_2} \sum_{k'=1}^{2t_2} 2t_2 C_{k'} (r_2-t_2-l') C_{k'} k! B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'} \phi_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \\
- \sum_{l=1}^{r_1-t_1} \sum_{l'=1}^{r_2-t_2} \sum_{m'=1}^{2t_2} 2t_2 C_{m'} (r_2-t_2-l' + m') C_{m'} m'! \\
\left. B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'+m'} \phi_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \right. \\
\right. \\
+ \sum_{l=1}^{r_1-t_1} \sum_{l'=1}^{r_2-t_2} \sum_{k=1}^{2t_1} 2t_1 C_k (r_1-t_1-l) C_k 2^k k! B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'} \phi_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \\
\right. \\
- \sum_{l=1}^{r_1-t_1} \sum_{l'=1}^{r_2-t_2} \sum_{m=1}^{2t_1} 2t_1 C_m (r_1-t_1-l+m) C_m 2^m m! \\
\left. B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'+m'} \phi_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \right. \\
\right. \\
+ \sum_{l=1}^{r_1-t_1} \sum_{l'=1}^{r_2-t_2} \sum_{k'=1}^{2t_2} 2t_1 C_k (r_1-t_1-l) C_k 2^k k! 2t_2 C_{k'} (r_2-t_2-l') C_{k'} k'! \\
\left. B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'+k'} \phi_{2(r_1-t_1-l)+r_2-t_2-l'+k'}(x) \right. \\
\right. \\
+ \sum_{l=1}^{r_1-t_1} \sum_{l'=1}^{r_2-t_2} \sum_{m'=1}^{2t_1} 2t_1 C_m (r_1-t_1-l+m) C_m 2^m m! 2t_2 C_{m'} (r_2-t_2-l') C_{m'} m'! \\
\left. B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'} \phi_{2(r_1-t_1-l+m)+r_2-t_2-l'+m'}(x) \right. \\
\right. \\
+ \sum_{l=1}^{r_1-t_1} \sum_{l'=1}^{r_2-t_2} \sum_{k=1}^{2t_2} 2t_1 C_k (r_1-t_1-l) C_k 2^k k! 2t_2 C_{m'} (r_2-t_2-l' + m') C_{m'} m'! \\
\left. B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'+m'} \phi_{2(r_1-t_1-l)+r_2-t_2-l'+m'}(x) \right. \\
\right. \\
\left. - \sum_{l=1}^{r_1-t_1} \sum_{l'=1}^{r_2-t_2} \sum_{m=1}^{2t_1} 2t_1 C_m (r_1-t_1-l+m) C_m 2^m m! 2t_2 C_{m'} (r_2-t_2-l'+m') C_{m'} m'! \\
\left. B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'} \phi_{2(r_1-t_1-l+m)+r_2-t_2-l'+m'}(x) \right. \\
\right. \\
\left. (r_2-t_2-l' + m') C_{m'} m'! B_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'} \phi_{2(r_1-t_1-l+m)+r_2-t_2-l'+m'}(x) \right. \\
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Canceling the common terms by substituting suitably we get,
Proof (b) and (c) are similar to the proof of (a).

Corollary 3.1.30.

(a) Let $U^{(d_i, P_i)}_{(d_i, P_i)} U^{(d_j, P_j)}_{(d_j, P_j)} \in \mathbb{J}^{2r_1 + 2r_2}$ such that $\mathfrak{F}^{(d_i, P_i)} U^{(d_j, P_j)} < 2s_1 + s_2$ where $U^{(d_i, P_i)} = U^{(d_i, P_i)}_{1j} \otimes U^{(d_i, P_i)}_{d_i-f}$ and $U^{(d_j, P_j)} = U^{(d_j, P_j)}_{fj} \otimes U^{(d_j, P_j)}_{d_j-f}$ where $U^{(d_i, P_i)}_{1j} \otimes U^{(d_i, P_i)}_{fj}$ is the sub diagram of $U^{(d_i, P_i)}_{(d_i, P_i)} \left( U^{(d_i, P_i)}_{(d_i, P_i)} \right)$ having $t$ number of $\mathbb{Z}_2$-through classes (t pair of $\{e\}$-horizontal edges) with $f_1, f_2, \ldots, f_{2t}$ number of vertices respectively and t pair of $\{e\}$-horizontal edges (t number of $\mathbb{Z}_2$-through classes) with $f_1, f_2, \ldots, f_{2t}$ number of vertices respectively such that $f_1 + \cdots + f_1 + f_2 + \cdots + f_2 = f$ and $\mathfrak{F}^{(d_i, P_i)} U^{(d_i, P_i)}_{d_i-f} U^{(d_j, P_j)}_{d_j-f} = 2s_1 + s_2 - t$ then the zero in the $ij$-entry of the block matrix $A_{2r_1 + 2r_2}$ remains zero after all the column operations.

(b) Let $U^{(d_i, P_i)}_{(d_i, P_i)} U^{(d_j, P_j)}_{(d_j, P_j)} \in \mathbb{J}^{2r_1 + 2r_2}$ such that $\mathfrak{F}^{(d_i, P_i)} U^{(d_j, P_j)} < 2s_1 + s_2$ where $U^{(d_i, P_i)} = \tilde{U}^{(d_i, P_i)}_{1j} \otimes \tilde{U}^{(d_i, P_i)}_{d_i-f}$ and $U^{(d_j, P_j)} = \tilde{U}^{(d_j, P_j)}_{fj} \otimes \tilde{U}^{(d_j, P_j)}_{d_j-f}$ where $\tilde{U}^{(d_i, P_i)}_{1j} \otimes \tilde{U}^{(d_i, P_i)}_{fj}$ is the sub diagram of $U^{(d_i, P_i)}_{(d_i, P_i)} \left( U^{(d_i, P_i)}_{(d_i, P_i)} \right)$ having $t$ number of $\mathbb{Z}_2$-through classes (t pair of $\{e\}$-horizontal edges) with $f_1, f_2, \ldots, f_{2t}$ number of vertices respectively and t pair of $\{e\}$-horizontal edges (t number of $\mathbb{Z}_2$-through classes) with $f_1, f_2, \ldots, f_{2t}$ number of vertices respectively such that $f_1 + \cdots + f_1 + f_2 + \cdots + f_2 = f$ and $\mathfrak{F}^{(d_i, P_i)} \tilde{U}^{(d_i, P_i)}_{d_i-f} \tilde{U}^{(d_j, P_j)}_{d_j-f} = 2s_1 + s_2 - t$ then the zero in the $ij$-entry of the block matrix $A_{2r_1 + 2r_2}$ remains zero after all the column operations.
- horizontal edges) with \( f_{11}, f_{12}, \cdots, f_{tt} \) number of vertices respectively and \( t \)
pair of \( \{e\} \)-horizontal edges (\( t \) number of \( \mathbb{Z}_2 \)-through classes) with \( f_{21}, f_{22}, \cdots, f_{2t} \) number of vertices respectively such that \( f_{11} + \cdots + f_{1t} + f_{21} + \cdots + f_{2t} = f \)
and \( \sharp p \left( U_{d_i}^{d_j} - f U_{d_j}^{d_i} - f \right) = 2s_1 + s_2 - t \) then the zero in the \( ij \)-entry of the block
matrix \( \tilde{A}_{2r_1 + r_2, 2r_1 + r_2} \) remains zero after all the column operations.

Proof.
The proof is similar to the proof of Lemma 3.1.25. \( \square \)

Now, we have the main theorem of this section.

3.1.3 Main Theorem

Theorem 3.1.31.

(a) Let \( G'_{2s_1 + s_2} \) be the matrix similar to the Gram matrix \( G_{2s_1 + s_2} \) of the algebra of
\( \mathbb{Z}_2 \)-relations which is obtained after the column operations and the corresponding row operations on \( G_{2s_1 + s_2} \). Then

\[
G'_{2s_1 + s_2} = \begin{pmatrix}
\bigoplus_{0 \leq r_1 \leq k - s_1 - s_2} A'_{2r_1 + r_2, 2r_1 + r_2} & \\
& A'_{2r_1 + r_2, 2r_1 + r_2} & \\
& & \end{pmatrix}
\]

(b) Let \( \tilde{G}'_{2s_1 + s_2} \) be the matrix similar to the Gram matrix \( \tilde{G}_{2s_1 + s_2} \) of signed partition algebras which is obtained after the column operations and the corresponding row operations on \( \tilde{G}_{2s_1 + s_2} \). Then

\[
\tilde{G}'_{2s_1 + s_2} = \begin{pmatrix}
\bigoplus_{0 \leq r_1 \leq k - s_1 - s_2 - 1} \tilde{A}'_{2r_1 + r_2, 2r_1 + r_2} & \\
& \tilde{A}'_{2r_1 + r_2, 2r_1 + r_2} & \\
& & \end{pmatrix} \bigoplus \tilde{A}'_{r_i}
\]

where

(i) the diagonal element of \( A'_{2r_1 + r_2, 2r_1 + r_2} \) is given by
The entry
\[ A'_{2r_1+r_2,2r_1+r_2} = \prod_{j=0}^{r_1-1} [x^2 - x - 2(s_1 + j)] \prod_{l=0}^{r_2-1} [x - (s_2 + l)] \quad \text{if } r_1 \geq 1, r_2 \geq 1 \]
\[ A'_{2r_1+r_2,2r_1+r_2} = \prod_{j=0}^{r_1-1} [x^2 - x - 2(s_1 + j)] \quad \text{if } r_2 = 0 \]
\[ A'_{2r_1+r_2,2r_1+r_2} = \prod_{l=0}^{r_2-1} [x - (s_2 + l)] \quad \text{if } r_1 = 0 \]

(ii) The entry \( b_{ij} \) of the block matrix \( A'_{2r_1+r_2,2r_1+r_2} \) is replaced by
\[ (-1)^{t_1+t_2} 2^{t_1} (t_1)! (t_2)! \prod_{j=0}^{r_1-1} [x^2 - x - 2(s_1 + t_1 + j)] \prod_{l=0}^{r_2-t_2-1} [x - (s_2 + t_2 + l)] \quad \text{if } r_1 \geq 1, r_2 \geq 1 \]
\[ (-1)^{t_1} 2^{t_1} (t_1)! \prod_{j=0}^{r_1-1} [x^2 - x - 2(s_1 + t_1 + j)] \quad \text{if } r_2 = 0 \]
\[ (-1)^{t_2} (t_2)! \prod_{l=0}^{r_2-t_2-1} [x - (s_2 + t_2 + l)] \quad \text{if } r_1 = 0 \]
whenever \( U^{(d_i, P_i)} \) and \( U^{(d_i, P_j)} \) can be defined as in Notation 3.1.23 and Proposition 3.1.29.

(iii) All other entries in the block matrix \( A'_{2r_1+r_2,2r_1+r_2} \) are zero.

The underlying partitions of the diagrams corresponding to the entries of the block matrix \( \tilde{A}'_{2r_1+r_2,2r_1+r_2} \) are \( \lambda = [\lambda_1^2]^1[2\lambda_2^2][\lambda_3^2][2\lambda_4]^4 \) with \( \lambda_1^2 = (\lambda_{11}^2, \cdots, \lambda_{1s_1}^2) \), \( 2\lambda_2 = (2\lambda_{21}, \cdots, \lambda_{2s_2}) \), \( \lambda_3^2 = (\lambda_{31}^2, \cdots, \lambda_{3r_1}^2) \), \( 2\lambda_4 = (2\lambda_{41}, \cdots, 2\lambda_{4r_2}) \) such that at least one of \( \lambda_{1i}, \lambda_{2j}, \lambda_{3l}, \lambda_{4m} \) is greater than 1 for \( 1 \leq i \leq s_1, 1 \leq j \leq s_2, 1 \leq l \leq r_1 \) and \( 1 \leq m \leq r_2 \).

Since the diagrams corresponding to the partition \( \tilde{\lambda} = [\lambda_1^2]^1[2\lambda_2^2][\lambda_3^2][2\lambda_4]^4 \) with \( |\lambda_{1i}| = 1, \forall 1 \leq i \leq s_1, |\lambda_{2j}| = 1 \forall 1 \leq j \leq s_2, |\lambda_{3m}| = 0 \forall 1 \leq m \leq r_1 \) and \( |\lambda_{4l}| = 1 \forall 1 \leq l \leq r_2 \) does not belong to the signed partition algebra. Thus the block corresponding to the diagrams whose underlying partition is \( \tilde{\lambda} \) is studied separately.

(b') Let \( \tilde{A}'_{\lambda'} \) where the partition \( \lambda' \) is such that each \( \lambda_{1i}, \lambda_{2j}, \lambda_{3l}, \lambda_{4m} \) is equal to 1 for \( 1 \leq i \leq s_1, 1 \leq j \leq s_2, 1 \leq l \leq r_1 \) and \( 1 \leq m \leq r_2 \) and \( \tilde{A}'_{\lambda'} \) is the block submatrix corresponding to the diagrams whose underlying partition is \( \lambda \).

(i) The \( ii \)-entry \( x^{2r_1+r_2} \) of the matrix \( \tilde{A}'_{\lambda'} \) is replaced by
The zero in the $i,j$-entry is replaced by
\[
(-1)^{t_1+t_2} \cdot 2^{t_1} \cdot (t_1)! \cdot (t_2)! \prod_{j=0}^{r'_1-1} [x - 2(s_1 + j)] \prod_{l=0}^{r'_2-1} [x - 2(s_2 + l + t_2)] \\
+ \prod_{l=0}^{k-s_1-s_2-1} [x - (s_2 + l)]
\]
where $U_{(\bar{d},\bar{P})}$ and $U_{(\bar{d},\bar{P})}$ are as in Notation 3.1.23 and Proposition 3.1.29 with $1 \leq i, j \leq 2k - 2s_1 - 2s_2$ and $i \neq j$.

(iii) If $\not\equiv \left( U_{(\bar{d},\bar{P})} \cdot U_{(\bar{d},\bar{P})} \right) = 2s_1 + s_2$ then the $i,j$-entry is replaced by
\[
(-1)^{r_i+r_j} \cdot 2^{r_i-s_1-s_2-1} \prod_{l=0}^{k-s_1-s_2-1} [x - (s_2 + l)]
\]
where $U_{(\bar{d},\bar{P})} \in \mathbb{I}_{2s_1+s_2}$ and $U_{(\bar{d},\bar{P})} \in \mathbb{I}_{2s_1+s_2}$ with
\[1 \leq i, j \leq 2k - 2s_1 - 2s_2\] and $i \neq j$.

(iv) All other entries in $\tilde{A}'_s$ are zero.

(c) Let $G'_s$ be the matrix similar to the Gram matrix $G_s$ which is obtained after the column operations and the row operations on $G_s$. Then
\[
G'_s = \left( \bigoplus_{0 \leq l \leq k-s} A'_{r,r} \right)
\]
where

(i) the diagonal element of $A'_{r,r}$ is given by
\[
\prod_{l=0}^{r-1} [x - (s + l)]
\]
(ii) the entry $b_{ij}$ of the block matrix $A'_{r,r}$ is replaced by
\[
(-1)^{t} \cdot (t)! \prod_{j=t}^{r-1} [x - (s + l)]
\]
whenever $U_{r}^{e_{l}}$ and $U_{r}^{e_{j}}$ can be defined as in of Notation 3.1.23 and Proposition 3.1.29.

(iii) all other entries in the block matrix $A'_{r,r}$ are zero.
Proof.

Proof of (a): Every entry $x^{2r_1+r_2}$ in the sub block matrix $A'_{2r_1+r_2,2r_1+r_2}$ is also replaced by

$$\prod_{j=0}^{r_1-1} [x^2 - x - 2(s_1 + j)] \prod_{l=0}^{r_2-1} [x - (s_2 + l)]$$

We continue to do the column operations for the diagrams whose underlying partition is $\lambda$ and $\lambda = [\lambda_2^1]^1[2\lambda_2][\lambda_3^2][2\lambda_4]^4$ with $\lambda_2^2 = (\lambda_2^1, \cdots, \lambda_{1s_1}^2)$, $2\lambda_2 = (2\lambda_{21}, \cdots, \lambda_{2s_2})$, $\lambda_3^2 = (\lambda_3^2_{s_1}, \cdots, \lambda_{3r_1}^2)$, $2\lambda_4 = (2\lambda_{41}, \cdots, 2\lambda_{4r_2})$ such that at least one of $\lambda_{1i}, \lambda_{2j}, \lambda_{3l}, \lambda_{4m}$ is greater than 1 and hence the above entry gets eliminated.

Hence, it follows from Lemmas 3.1.19 and 3.1.25 the rectangular sub matrix $A'_{2r_1+r_2,2r_1+r_2}$ with $2r_1 + r_2 \neq 2r'_1 + r'_2$ becomes zero after the column operations are carried out.

After applying the row operations corresponding to the column operations performed in Lemmas 3.1.21, 3.1.25, Proposition 3.1.29, and Theorem 3.1.20, the Gram matrix $G'_{2s_1+s_2}$ which is similar to a matrix $G'_{2s_1+s_2}$ decomposes as a direct sum of block matrices

$$G'_{2s_1+s_2} = \left( \bigoplus_{0 \leq r_1 \leq k-s_1-s_2} A'_{2r_1+r_2,2r_1+r_2} \right)$$

where the diagonal element of $A'_{2r_1+r_2,2r_1+r_2}$ is given by

$$\prod_{j=0}^{r_1-1} [x^2 - x - 2(s_1 + j)] \prod_{l=0}^{r_2-1} [x - (s_2 + l)].$$

Result (i) follows from Theorem 3.1.20(a), result (ii) follows from Proposition 3.1.29(a) and result (iii) follow from Lemmas 3.1.21, 3.1.22(a) and 3.1.25(a) respectively.

Proof of (b)’:

The column operations corresponding to the diagrams whose underlying partition $[\lambda_2^1]^1[2\lambda_2][\lambda_3^2][2\lambda_4]^4$ where $|\lambda_{1i}| = 1$, $1 \leq i \leq s_1$, $|\lambda_{2j}| = 1$ $1 \leq j \leq s_2$, $|\lambda_{3m}| = 0$, $1 \leq m \leq r_1$ and $|\lambda_{4l}| = 1$, $1 \leq l \leq r_2$ such that $s_1 + s_2 + r_2 = k$ with $s_1 \leq k$ cannot be carried out for the block matrix $A'_{\lambda'}$, since the diagrams do not belong to the signed partition algebra.

Proof of (i): We prove the result by induction.

Case (i): Let $U_{(\tilde{d},\tilde{r})}$ be a diagram in $\mathcal{F}_{2s_1+s_2}$, after the column operations
the \( ii \)-entry corresponding to the diagram \( U^{(\tilde{d}, \tilde{P})}_{(d, \tilde{P})} \) will be replaced by

\[
\phi_{2j+k-s_1-s_2-j}^{s_1, s_2}(x) + \phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x)
\]

since the signed partition algebra do not contain diagrams with \( k-s_1-s_2 \) number of \( \mathbb{Z}_2 \)-horizontal edges.

**Case (ii):** Let \( U^{(\tilde{d}, \tilde{P})}_{(d, \tilde{P})} \) be a diagram in \( \mathbb{Z}_{2s_1+s_2}^{2j+k-s_1-s_2-2} \).

After applying the column operations \( L_i \to L_i - L_k \) for all \( U^{(\tilde{d}, \tilde{P})}_{(d, \tilde{P})} \) where \( U^{(\tilde{d}, \tilde{P})}_{(d, \tilde{P})} \in \mathbb{Z}_{2s_1+s_2}^{2j+k-s_1-s_2} \) with \( r_1 + r_2 + s_1 + s_2 \leq k - 1 \), the \( ii \)-entry will be replaced by

\[
\phi_{2j+1+k-s_1-s_2-2}^{s_1, s_2}(x) + 2\phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x) + \phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x) + \phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x)
\]

Again applying the column operations inside the block \( A'_{\lambda} \), the \( ii \)-entry is replaced as

\[
\phi_{2j+1+k-s_1-s_2-2}^{s_1, s_2}(x) + 2\phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x) + \phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x) - 2\phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x) - 2\phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x)
\]

After applying the row operations corresponding to the column operations which is performed to obtain the above \( ii \)-entry, the \( ii \)-entry is further replaced as follows:

\[
\phi_{2j+1+k-s_1-s_2-2}^{s_1, s_2}(x) - \phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x) + 2\phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x)
\]

In general, Let \( U^{(\tilde{d}, \tilde{P})}_{(d, \tilde{P})} \) be a diagram in \( \mathbb{Z}_{2s_1+s_2}^{2j+k-s_1-s_2-j} \).

After applying the column operations, by induction the \( ii \)-entry of the matrix \( A'_{\lambda} \) is replaced as

\[
\phi_{2j+k-s_1-s_2-j}^{s_1, s_2}(x) + \sum_{m=1}^{j-1} \sum_{m=1}^{j-1} C_m \phi_{2j+(j-m)+k-s_1-s_2-j+m}^{s_1, s_2}(x) + \phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x)
\]

After applying the row operations the \( ii \)-entry is further replaced as follows:

\[
\phi_{2j+k-s_1-s_2-j}^{s_1, s_2}(x) - \sum_{m=1}^{j-1} \sum_{m=1}^{j-1} C_m \phi_{2j+(j-m)+k-s_1-s_2-j+m}^{s_1, s_2}(x) + \phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x) + \phi_{2j+1+k-s_1-s_2}^{s_1, s_2}(x)
\]

Thus, for a diagram \( U^{(\tilde{d}, \tilde{P})}_{(d, \tilde{P})} \in \mathbb{Z}_{2s_1+s_2}^{2j+k-s_1-s_2-j} \) the \( ii \)-entry is replaced as
\[ \prod_{j=0}^{r'-1} [x^2 - x - 2(s_1 + j)] \prod_{l=0}^{k-s_1-s_2-r'_1-1} [x - (s_2 + l)] + \prod_{l=0}^{k-s_1-s_2-1} [x - (s_2 + l)]. \]

**Proof of (ii):** The proof follows from Proposition 3.1.29(b) and it is similar to the proof of Proof of (1), whenever \( U^{(d_i, \bar{P}_i)}_{(d_i, \bar{P}_i)} \) and \( U^{(d_j, \bar{P}_j)}_{(d_j, \bar{P}_j)} \) are as in Notation 3.1.23.

**Proof of (iii):**

**Case (i):** Let \( U^{(d_i, \bar{P}_i)}_{(d_i, \bar{P}_i)} \in \mathbb{J}^{2.1+k-s_1-s_2-1}_{2s_1+s_2} \) and \( U^{(d_j, \bar{P}_j)}_{(d_j, \bar{P}_j)} \in \mathbb{J}^{2.2+k-s_1-s_2-2}_{2s_1+s_2} \) such that 2 > 1 then \( l(d_i, \tilde{d}_j, \bar{P}_j) \leq 2 + k - s_1 - s_2 - 1 \).

There will be two diagrams say \( U^{(d_i, \bar{P}_i)}_{(d_i, \bar{P}_i)} \) and \( U^{(d_j, \bar{P}_j)}_{(d_j, \bar{P}_j)} \) coarser than \( U^{(d_j, \bar{P}_j)}_{(d_j, \bar{P}_j)} \).

**Subcase (i):** Suppose \( l(d_i, \tilde{d}_j, \bar{P}_j) = 2 + k - s_1 - s_2 - 1 \) then

\[ a_{ij} = \phi^{s_1,s_2}_{2.1+k-s_1-s_2-1}(x) + \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x). \]

Also, \( a_{ii} = \phi^{s_1,s_2}_{2.1+k-s_1-s_2-1}(x) + \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \) and \( a_{ii} = \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \)

and \( a_{ii} = \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \) and \( a_{ii} = \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \)

After applying the column operations the \( ij \)-entry in \( \tilde{A}_\lambda \) is replaced as

\[ b_{ij} = a_{ij} - a_{ii} - a_{ii} \]

\[ = -\phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \]

**Subcase (ii):** Suppose \( l(d_i, \tilde{d}_j, \bar{P}_j) < 2 + k - s_1 - s_2 - 1 \) then \( a_{ij} = \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \).

Also, \( a_{ii} = \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \) and \( a_{ii} = \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \)

After applying the column operations the \( ij \)-entry in \( \tilde{A}_\lambda \) is replaced as

\[ b_{ij} = a_{ij} - a_{ii} - a_{ii} \]

\[ = -\phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \]

In general, let \( U^{(d_i, \bar{P}_i)}_{(d_i, \bar{P}_i)} \in \mathbb{J}^{2r'_1+k-s_1-s_2-r'_1}_{2s_1+s_2} \) and \( U^{(d_j, \bar{P}_j)}_{(d_j, \bar{P}_j)} \in \mathbb{J}^{2r_1+k-s_1-s_2-r_1}_{2s_1+s_2} \) such that \( r_1 > r'_1 \) then \( l(d_i, \tilde{d}_j, \bar{P}_j) \leq 2r'_1 + k - s_1 - s_2 - r'_1 \).

After applying the column operations the \( ij \)-entry is replaced as

\[ b_{ij} = \left( \sum_{m=1}^{r_1-1} (-1)^{m-1} r_1 C_m - 1 \right) \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \]

\[ = (-1)^{r_1+1} \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \]

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After applying row operations the $ij$-entry is further replaced as follows:

$$b_{ij} = \left( \sum_{m=1}^{r_{i}^{\prime} - 1} (-1)^{m-1} r_{i}^{\prime} C_{m} - 1 \right) (-1)^{r_{i}^{\prime} + r_{j}} \phi_{2,0+k-s_{1}-s_{2}}^{s_{1},s_{2}}(x)$$

Thus, the $ij$-entry is replaced as

$$(-1)^{r_{i}^{\prime} + r_{j}} \phi_{2,0+k-s_{1}-s_{2}}^{s_{1},s_{2}}(x).$$

Proof of (b) and (c) are similar to proof of (a). 

Remark 3.1.32.

(a) $G'_{0+0} = \bigoplus_{0 \leq r_{1} \leq k} \bigoplus_{0 \leq r_{2} \leq k} A'_{2r_{1}+r_{2},2r_{1}+r_{2}}$

(b) $\widetilde{G}'_{0+0} = \bigoplus_{0 \leq r_{1} \leq k-1} \bigoplus_{0 \leq r_{2} \leq k-1} \bigoplus_{2r_{1}+r_{2} \leq 2k-1} \tilde{A}'_{2r_{1}+r_{2},2r_{1}+r_{2}} + \tilde{A}'_{\lambda'}$

where $A'_{2r_{1}+r_{2},2r_{1}+r_{2}}$ and $\tilde{A}'_{2r_{1}+r_{2},2r_{1}+r_{2}}$ are the diagonal block matrix whose diagonal entry is given by

(i) $\prod_{j=0}^{r_{1}-1} [x^{2} - x - 2j] \prod_{l=0}^{r_{2}-1} [x - l]$, $r_{1} \geq 1, r_{2} \geq 1$,

(ii) $\prod_{l=0}^{r_{1}-1} [x - l]$, $r_{1} = 0$,

(iii) $\prod_{j=0}^{r_{2}-1} [x^{2} - x - 2j]$, $r_{2} = 0$.

(b) Let $\tilde{A}'_{\lambda'}$ where the partition $\lambda'$ is such that $\lambda_{i} = \Phi, \lambda_{2j} = \Phi, \lambda_{3i} = 1, \lambda_{4m} = 1$ for $1 \leq i \leq s_{1}, 1 \leq j \leq s_{2}, 1 \leq l \leq r_{1}$ and $1 \leq m \leq r_{2}$ and $\tilde{A}'_{\lambda'}$ is the block sub matrix corresponding to the diagrams whose underlying partition is $\lambda'$.

The $ii$-entry $x^{2r_{i}^{\prime} + r_{i}^{2}}$ of the matrix $\tilde{A}'_{\lambda'}$ is replaced by

$$\prod_{j=0}^{r_{i}^{\prime}-1} [x^{2} - x - 2j] \prod_{l=0}^{r_{i}^{2}-1} [x - l] + \prod_{l=0}^{r_{i}^{\prime} + r_{i}^{2}-1} [x - l].$$

(c) $G'_{0} = \bigoplus_{0 \leq r \leq k} A'_{r,r}$. The $ii$-entry $x_{r}$ of the matrix $A'_{r,r}$ is replaced by

$$\prod_{l=0}^{r-1} [x - l].$$
3.2 Semisimplicity of Signed Partition Algebras

Semisimplicity of the algebra of $\mathbb{Z}_2$-relations and partition algebras are already discussed in [17] and [9] respectively but still we give an alternate approach.

Theorem 3.2.1.

(i) The algebra of $\mathbb{Z}_2$-relations $A_k^{\mathbb{Z}_2}(x)$, signed partition algebras $\tilde{A}_k^{\mathbb{Z}_2}(x)$ and partition algebras $A_k(x)$ are generically semisimple over a field of characteristic zero.

(ii) Suppose that the characteristic of the field $\mathbb{K}$ is 0, then

(a) the algebra of $\mathbb{Z}_2$-relations $A_k^{\mathbb{Z}_2}(x)$ is semisimple if and only if $q$ is not a root of the polynomial $f(x) = \prod_{\lambda, \mu: 2s_1 + s_2 = 0} \det G_{2s_1 + s_2}^{\lambda \mu}$.

(b) the signed partition algebra $\tilde{A}_k^{\mathbb{Z}_2}(q)$ is semisimple if and only if $q$ is not a root of the polynomial $f(x) = \prod_{\lambda, \mu: 2s_1 + s_2 = 0} \det \tilde{G}_{2s_1 + s_2}^{\lambda \mu}$.

(c) the partition algebra $A_k(x)$ is semisimple if and only if $q$ is not a root of the polynomial $f(x) = \prod_{\lambda, \mu: 0 \leq s_1 \leq k} \det G_{s_1 + s_2}^{\lambda \mu}$.

(iii) (a) In particular, $G_{2s_1 + s_2}^{\lambda \mu} \equiv G_{2s_1 + s_2}^{\lambda \mu}$ if

1. $\lambda = ([s_1], \Phi)$ and $\mu = [s_2]$ when $s_1, s_2 \neq 0$,
2. $\lambda = (\Phi, \Phi)$ and $\mu = [s_2]$ when $s_1 = 0, s_2 \neq 0$,
3. $\lambda = ([s_1], \Phi)$ and $\mu = \Phi$ when $s_1 \neq 0, s_2 = 0$
4. $\lambda = (\Phi, \Phi)$ and $\mu = \Phi$ when $s_1, s_2 = 0$.

for $0 \leq s_1 \leq k, 0 \leq s_2 \leq k, 0 \leq s_1 + s_2 \leq k$.

(b) In particular, $\tilde{G}_{2s_1 + s_2}^{\lambda \mu} \equiv \tilde{G}_{2s_1 + s_2}^{\lambda \mu}$ if

1. $\tilde{\lambda} = ([s_1], \Phi)$ and $\tilde{\mu} = [s_2]$ when $s_1, s_2 \neq 0$,
2. $\tilde{\lambda} = (\Phi, \Phi)$ and $\tilde{\mu} = [s_2]$ when $s_1 = 0, s_2 \neq 0$,
3. $\tilde{\lambda} = ([s_1], \Phi)$ and $\tilde{\mu} = \Phi$ when $s_1 \neq 0, s_2 = 0$.

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4. \( \tilde{\lambda} = (\Phi, \Phi) \) and \( \tilde{\mu} = \Phi \) when \( s_1, s_2 = 0 \),
for \( 0 \leq s_1 \leq k - 1, 0 \leq s_2 \leq k - 1, 0 \leq s_1 + s_2 \leq k - 1 \).

(c) \( G_s^\lambda \) coincides with \( G_s \) if
1. \( \lambda = s \) when \( s \neq 0 \),
2. \( \lambda = \Phi \) when \( s = 0 \)
for \( 0 \leq s \leq k \).

(iii) \( (a) \) If \( q \) is a root of the polynomial
\[
f(x) = \prod_{2s_1 + s_2 = 0}^{2k} \det G_{2s_1 + s_2}
\]
where \( \det G_{2s_1 + s_2} = \prod_{0 \leq r_1 < k - s_1 - s_2} \det A'_{2r_1 + r_2, 2r_1 + r_2} \)
\( A'_{2r_1 + r_2, 2r_1 + r_2} \) is as in Theorem 3.1.31 then the algebra \( A^Z_k(q) \) is not semisimple.

In particular, \( q \) is an integer such that \( 0 \leq q \leq k - 2 \) and \( q \) is a root of
the polynomial \( x^2 - x - 2r', 0 \leq r' \leq k - 2 \) then \( A^Z_k(q) \) is not semisimple.

(b) If \( q \) is a root of the polynomial
\[
f(x) = \prod_{2s_1 + s_2 = 0}^{2k} \det G_{2s_1 + s_2}
\]
where \( \det G_{2s_1 + s_2} = \prod_{0 \leq r_1 < k - s_1 - s_2} \det A'_{2r_1 + r_2, 2r_1 + r_2} \prod \det \tilde{A}'_{r,r}, \)
\( \tilde{A}'_{r_1 + r_2, 2r_1 + r_2} \) and \( \tilde{A}'_{r,r} \) are as in Theorem 3.1.31 then the algebra \( \tilde{A}^Z_k(q) \) is not semisimple.

In particular, \( q \) is an integer such that \( 0 \leq q \leq k - 2 \) and \( q \) is a root of
the polynomial \( x^2 - x - 2r', 0 \leq r' \leq k - 2 \) then \( \tilde{A}^Z_k(q) \) is not semisimple.

(c) If \( q \) is a root of the polynomial
\[
f(x) = \prod_{s=0}^{k} \det G_s
\]
where \( \det G_s = \prod_{0 \leq r \leq k - s} \det A'_{r,r} \), \( A'_{r,r} \) is as in Theorem 3.1.31 then the
algebra \( A_k(q) \) is not semisimple.
(iv) The algebra of $\mathbb{Z}_2$-relations, signed partition algebra and the partition algebra over a field of characteristics 0 are quasi-hereditary.

Proof.

Proof of (i): By Definition 6.3 in [14], we know that the matrix of the signed partition algebra

$$\tilde{G}_{2s_1+s_2}^{\lambda,\mu} = \Phi_1((S, s), (T, t))_{S, T \in \tilde{M}[(r, (s_1, s_2))]}
$$

The above matrix can be rewritten as follows:

$$\tilde{G}_{2s_1+s_2}^{\lambda,\mu} = (g_{ij})_{1 \leq i, j \leq 2s_1+s_2}
$$

where $g_{ij} = a_{ij}B_{\delta_1, \delta_2}^{\lambda,\mu}$,

$$a_{ij} = \begin{cases} 
\chi(P_i \vee P_j), & \text{if conditions (a) and (b) of Definition 2.2.6 in [14] are satisfied;} \\
0, & \text{Otherwise.}
\end{cases}
$$

$B_{\delta_1, \delta_2}^{\lambda,\mu} = 0$, if $a_{ij} = 0$ and $B_{\delta_1, \delta_2}^{\lambda,\mu} = B_{\delta_1}^{\lambda} \otimes B_{\delta_2}^{\mu}$ with $B_{\delta_1}^{\lambda} = \left(\phi_{\delta_1}(s_1, t_1)\right)$ and $B_{\delta_2}^{\mu} = \left(\phi_{\delta_2}(s_2, t_2)\right)$.

It follows from Theorem 3.8 in [5], $B_{\delta_1}^{\lambda} = R_{\delta_1}^{\lambda}A_1$ and $B_{\delta_2}^{\mu} = R_{\delta_2}^{\mu}A_2$ where

$A_1 = \left(\phi_1(s_1, t_1)\right)$ and $A_2 = \left(\phi_1(s_2, t_2)\right)$, are the bilinear forms associated to the cell module $W^{\lambda}$ and $W^{\mu}$ of the cellular algebras of $K[\mathbb{Z}_2 \wr S_{s_1}]$ and $K[\mathbb{S}_{s_2}]$ respectively.

Thus,

$$B_{\delta_1, \delta_2}^{\lambda,\mu} = R_{\delta_1}^{\lambda}A_1 \otimes R_{\delta_2}^{\mu}A_2 = \left(R_{\delta_1}^{\lambda} \otimes R_{\delta_2}^{\mu}\right)(A_1 \otimes A_2) = R_{\delta_1, \delta_2}^{\lambda,\mu}A
$$

where $A = A_1 \otimes A_2$ and $R_{\delta_1, \delta_2}^{\lambda,\mu} = R_{\delta_1}^{\lambda} \otimes R_{\delta_2}^{\mu}$.

Therefore, $g_{ij} = R_{\delta_1, \delta_2}^{\lambda,\mu}A$, when $i = j, R_{\delta_1, \delta_2}^{\lambda,\mu}A = Id$.

So that the Gram matrix becomes,

$$\tilde{G}_{2s_1+s_2}^{\lambda,\mu} = \left(a_{ij}R_{\delta_1, \delta_2}^{\lambda,\mu}\right)_{1 \leq i, j \leq 2s_1+s_2} = \left(a_{ij}\right)(R_{\delta_1, \delta_2}^{\lambda,\mu}A)_{1 \leq i, j \leq 2s_1+s_2} = H_{2s_1+s_2}^{\lambda,\mu}(Id \otimes A)
$$

where $H_{2s_1+s_2}^{\lambda,\mu} = \left(a_{ij}R_{\delta_1, \delta_2}^{\lambda,\mu}\right)_{1 \leq i, j \leq 2s_1+s_2}$ and if $i = j$ then $R_{\delta_1}^{\lambda} = Id$ and $R_{\delta_2}^{\mu} = Id$. 

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Thus, \( \det G_{2s_1+s_2}^{\lambda,\mu} = \det H_{2s_1+s_2}^{\lambda,\mu} \det (Id \otimes A) \).

Since \( \det(I \otimes A) \neq 0 \), over a field of characteristic zero and \( \det H_{2s_1+s_2}^{\lambda,\mu} \) is a non-zero monic polynomial, \( \det G_{2s_1+s_2}^{\lambda,\mu} \) is also a non-zero polynomial.

Therefore, the algebra \( \overline{\mathcal{A}}_{k}^{Z2}(x) \) is semisimple. The proof for the algebra of \( \mathbb{Z}_2 \)-relations and the partition algebras are similar to the above proof.

**Proof of (ii):**

By Theorem 3.8 in [5], \( \overline{\mathcal{A}}_{k}^{Z2}(x) \) is semisimple if and only if \( \det G_{2s_1+s_2}^{\lambda,\mu} \neq 0 \) for all \( s_1, s_2 \) and for all \( \lambda, \mu \), since

\[
\det G_{2s_1+s_2}^{\lambda,\mu} \neq 0 \text{ if and only if } \Phi \text{ is non-degenerate.}
\]

**Proof of (iii)(b):**

Now, \( \widetilde{G}_{2s_1+s_2}^{\lambda,\mu} = \widetilde{G}_{2s_1+s_2} \), if

1. \( \widetilde{\lambda} = ([s_1], \Phi) \) and \( \widetilde{\mu} = [s_2] \) when \( s_1, s_2 \neq 0 \),
2. \( \widetilde{\lambda} = (\Phi, \Phi) \) and \( \widetilde{\mu} = [s_2] \) when \( s_1 = 0, s_2 \neq 0 \),
3. \( \widetilde{\lambda} = ([s_1], \Phi) \) and \( \widetilde{\mu} = \Phi \) when \( s_1 \neq 0, s_2 = 0 \)

for \( 0 \leq s_1 \leq k - 1, 0 \leq s_2 \leq k - 1, 0 \leq s_1 + s_2 \leq k - 1, \text{since } A \text{ is the } 1 \times 1 \text{ identity matrix,} \)

if \( \lambda = (\Phi, \Phi) \) and \( \mu = \Phi \) when \( s_1, s_2 = 0 \), then \( \widetilde{G}_{2s_1+s_2}^{\Phi,\Phi} \) coincides with \( \widetilde{G}_{0+0} \).

**Proof of (iii)(b):** If \( q \) is a root of \( f(x) = \prod_{0 \leq r_1 \leq k-s_1-s_2-1} \det \overline{A}_{2r_1+r_2,2r_1+r_2} \prod_{0 \leq r_2 \leq k-s_1-s_2-1} \det A_{2r_1+r_2,2r_1+r_2} \).

Then \( \det \widetilde{G}_{2s_1+s_2} = 0 = \det \widetilde{G}_{2s_1+s_2}^{((s_1),\Phi),(s_2)} \).

Thus, the algebra \( \overline{\mathcal{A}}_{k}^{Z2}(x) \) is not semisimple.

In particular, by Remark 3.1.32 if \( q \) is an integer such that \( 0 \leq q \leq k - 2 \) and \( q \) is a root of polynomial \( x^2 - x - 2r' \), \( 0 \leq r' \leq k - 2 \) then the algebra \( \overline{\mathcal{A}}_{k}^{Z2}(x) \) is not semisimple.

Proof of (a) and (c) are similar to the proof of (b).

**Proof of (iv):** It follows from Remark 3.10 in [5] and Theorem 5.4 in [14].
3.3 Illustration

The following is an example of Gram matrix in $\mathbf{A}_3^D(x)$.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

After applying the column operations and by Theorem 3.1.31 the matrix $G_{2,1+0}$ reduces as follows:

\[
\begin{pmatrix}
d_5 & d_6 & d_7 & d_8 & d_9 & d_{10} & d_{11} & d_{12} & d_{13} \\
d_5 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_6 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
d_7 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\
d_8 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\
d_9 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\
d_{10} & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\
d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\
d_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\
d_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\
\end{pmatrix}
\]
\[
A_{2,2} \sim A'_{2,2} = \\
\begin{array}{cccccccccccc}
\text{Index} & d_{14} & d_{15} & d_{16} & d_{17} & d_{18} & d_{19} & d_{20} & d_{21} & d_{22} & d_{23} & d_{24} & d_{25} \\
\hline
14 & x^2 - x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
15 & 0 & x^2 - x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
16 & 0 & 0 & x^2 - x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
17 & 0 & 0 & 0 & x^2 - x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18 & 0 & 0 & 0 & 0 & x^2 - x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
19 & 0 & 0 & 0 & 0 & 0 & x^2 - x & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 - x & 0 & 0 & 0 & 0 & 0 \\
21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 - x & 0 & 0 & 0 & 0 \\
22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 - x & 0 & 0 & 0 \\
23 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 - x & 0 & 0 \\
24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 - x & 0 \\
25 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 - x \\
\end{array}
\]
$A' \sim A^{\alpha'} =$

<table>
<thead>
<tr>
<th></th>
<th>$d_{26}$</th>
<th>$d_{27}$</th>
<th>$d_{28}$</th>
<th>$d_{29}$</th>
<th>$d_{30}$</th>
<th>$d_{31}$</th>
<th>$d_{32}$</th>
<th>$d_{33}$</th>
<th>$d_{34}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{26}$</td>
<td>$x^3 - 3x$</td>
<td>$x^2 - x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-2x$</td>
<td>$x^3 - 3x$</td>
<td>0</td>
<td>$-2x$</td>
</tr>
<tr>
<td>$d_{27}$</td>
<td>$x^2 - x$</td>
<td>$x^3 - 3x$</td>
<td>0</td>
<td>$-2x$</td>
<td>0</td>
<td>$x^3 - 3x$</td>
<td>$-2x$</td>
<td>0</td>
<td>$x^3 - 3x$</td>
</tr>
<tr>
<td>$d_{28}$</td>
<td>0</td>
<td>0</td>
<td>$x^3 - 3x$</td>
<td>$x^2 - x$</td>
<td>$-2x$</td>
<td>0</td>
<td>$x^3 - 3x$</td>
<td>$-2x$</td>
<td>0</td>
</tr>
<tr>
<td>$d_{29}$</td>
<td>0</td>
<td>$-2x$</td>
<td>$x^2 - x$</td>
<td>$x^3 - 3x$</td>
<td>0</td>
<td>0</td>
<td>$-2x$</td>
<td>$x^3 - 3x$</td>
<td>0</td>
</tr>
<tr>
<td>$d_{30}$</td>
<td>0</td>
<td>0</td>
<td>$-2x$</td>
<td>0</td>
<td>$x^3 - 3x$</td>
<td>$x^2 - x$</td>
<td>0</td>
<td>$-2x$</td>
<td>$x^3 - 3x$</td>
</tr>
<tr>
<td>$d_{31}$</td>
<td>$-2x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$x^2 - x$</td>
<td>$x^3 - 3x$</td>
<td>$-2x$</td>
<td>0</td>
<td>$x^3 - 3x$</td>
</tr>
<tr>
<td>$d_{32}$</td>
<td>$x^3 - 3x$</td>
<td>$x^3 - 3x$</td>
<td>0</td>
<td>$-2x$</td>
<td>0</td>
<td>$-2x$</td>
<td>$x^4 - 6x^2 + x + 8$</td>
<td>$-2x^2 + 8$</td>
<td>$-2x^2 + 8$</td>
</tr>
<tr>
<td>$d_{33}$</td>
<td>0</td>
<td>$-2x$</td>
<td>$x^3 - 3x$</td>
<td>$x^3 - 3x$</td>
<td>$-2x$</td>
<td>0</td>
<td>$-2x^2 + 8$</td>
<td>$x^4 - 6x^2 + x + 8$</td>
<td>$-2x^2 + 8$</td>
</tr>
<tr>
<td>$d_{34}$</td>
<td>$-2x$</td>
<td>0</td>
<td>$-2x$</td>
<td>0</td>
<td>$x^3 - 3x$</td>
<td>$x^3 - 3x$</td>
<td>$-2x^2 + 8$</td>
<td>$-2x^2 + 8$</td>
<td>$x^4 - 6x^2 + x + 8$</td>
</tr>
</tbody>
</table>
After applying the row and column operations, the matrix $A_\lambda$ is reduced as follows:

<table>
<thead>
<tr>
<th></th>
<th>$d_{26}$</th>
<th>$d_{27}$</th>
<th>$d_{28}$</th>
<th>$d_{29}$</th>
<th>$d_{30}$</th>
<th>$d_{31}$</th>
<th>$d_{32}$</th>
<th>$d_{33}$</th>
<th>$d_{34}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$x^3 - 3x$</td>
<td>$x^2 - x$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-2x$</td>
<td>$-x^2 + x$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_{27}$</td>
<td>$x^2 - x$</td>
<td>$x^3 - 3x$</td>
<td>$0$</td>
<td>$-2x$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-x^2 + x$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_{28}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$x^3 - 3x$</td>
<td>$x^2 - x$</td>
<td>$-2x$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-x^2 + x$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_{29}$</td>
<td>$0$</td>
<td>$-2x$</td>
<td>$x^2 - x$</td>
<td>$x^3 - 3x$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-x^2 + x$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_{30}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-2x$</td>
<td>$0$</td>
<td>$x^3 - 3x$</td>
<td>$x^2 - x$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_{31}$</td>
<td>$-2x$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$x^2 - x$</td>
<td>$x^3 - 3x$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-x^2 + x$</td>
</tr>
<tr>
<td>$d_{32}$</td>
<td>$-x^2 + x$</td>
<td>$-x^2 + x$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$x^4 - 2x^3$</td>
<td>$-2x^2 + 2x + 8$</td>
<td>$-2x^2 + 2x + 8$</td>
</tr>
<tr>
<td>$d_{33}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-x^2 + x$</td>
<td>$-x^2 + x$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-2x^2 + 2x + 8$</td>
<td>$x^4 - 2x^3$</td>
<td>$-2x^2 + 2x + 8$</td>
</tr>
<tr>
<td>$d_{34}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-x^2 + x$</td>
<td>$-x^2 + x$</td>
<td>$-2x^2 + 2x + 8$</td>
<td>$-2x^2 + 2x + 8$</td>
<td>$x^4 - 2x^3$</td>
</tr>
</tbody>
</table>
The entry $x^2 - x$ in the above matrix cannot be eliminated while applying column operations since the following diagrams do not belong to $A_3^{x^2}(x)$.

\[
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\end{array}
\]

*****