Bivariate Mittag-Leffler Distribution

Chapter 2

Bivariate Mittag-Leffler Distribution

2.1 Introduction

Mittag-Leffler distribution has been studied extensively by many authors in the past decade (see Jayakumar and Pillai (1993), Kozubowski (1994, 1999), Lin (1998, 2001), Pillai (1990) and Weron and Kotulski (1996)). Pillai (1990) established that the Mittag-Leffler distribution is geometrically infinitely divisible. Various distributional properties of Mittag-Leffler are discussed in Jayakumar and Suresh (2003) and Lin (1998). Kozubowski (2001) estimated the parameters of Mittag-Leffler distribution using fractional moments. Even though a lot of investigations on Mittag-Leffler distribution were carried out, studies in the direction of its extensions to higher dimensions are not yet explored.
Mundassery and Jayakumar (2007a) introduced a bivariate Mittag-Leffler distribution.

**Definition 2.1.** A non negative random vector \((X, Y)\) is said to follow bivariate Mittag-Leffler distribution with parameters \(\mu_1, \mu_2, \alpha_1, \alpha_2\) and \(\theta\), denoted by BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\), if its Laplace transform is

\[
\phi(\lambda_1, \lambda_2) = \frac{1}{(1 + \mu_1 \lambda_1^{\alpha_1})(1 + \mu_2 \lambda_2^{\alpha_2}) - \theta \mu_1 \mu_2 \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}},
\]

\(\lambda_1, \lambda_2 \geq 0; \ 0 < \alpha_1, \alpha_2 \leq 1; \ \mu_1, \mu_2 > 0; \ 0 \leq \theta \leq 1.
\]

Note that

\[
\phi(\lambda_1, 0) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1}} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{1}{1 + \mu_2 \lambda_2^{\alpha_2}}.
\]

When \(\theta = 1\),

\[
\phi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}.
\]

and \(\theta = 0\) implies that \(X\) and \(Y\) are independent. When \(\alpha_1 = \alpha_2 = 1\), BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\) distribution gives a generalization of the MBE \((\mu_1, \mu_2, \theta)\) distribution discussed in (1.17).

Now, we define a bivariate positive stable distribution.

**Definition 2.2.** A non negative random vector \((W_1, W_2)\) is said to follow bivariate positive stable distribution if its Laplace transform is

\[
\varphi(\lambda_1, \lambda_2) = e^{-\mu_1 \lambda_1^{\alpha_1} - \mu_2 \lambda_2^{\alpha_2} - r \mu_1 \mu_2 \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}},
\]

\(0 < \alpha_1, \alpha_2 \leq 1, \ \mu_1, \mu_2 > 0, \ 0 \leq r \leq 1.
\]
When $W_1$ and $W_2$ are independent ($r = 0$),

$$\varphi(\lambda_1, \lambda_2) = e^{-\mu_1 \lambda_1^{\alpha_1} - \mu_2 \lambda_2^{\alpha_2}}. \quad (2.4)$$

In Section 2, we discuss various distributional properties of BML ($\mu_1, \mu_2, \alpha_1, \alpha_2, \theta$). The characterizations of the distribution using geometric compounding are obtained in Section 3, while in Section 4, we have the characterizations using bivariate geometric compounding. Estimates of the parameters of BML ($\mu_1, \mu_2, \alpha_1, \alpha_2, 1$) are obtained in Section 5. Autoregressive processes with BML ($\mu_1, \mu_2, \alpha_1, \alpha_2, 1$) marginals are developed in Section 6. Bivariate Mittag-Leffler distributions that generalize Marshall-Olkin's bivariate exponential, Hawkes' bivariate exponential and Paulson's bivariate exponential are introduced in Section 7. A bivariate semi Mittag-Leffler distribution is introduced and studied in Section 8.

### 2.2 Distributional Properties

Kozubowski et al. (2005) established the one to one correspondence between the operator stable and operator geometric stable distributions as

$$G(t) = \frac{1}{1 - \ln \psi(t)}$$

where $G(t)$ and $\psi(t)$ are the characteristic functions of the operator geometric stable and operator stable distributions respectively. In the light of this, we note that BML ($\mu_1, \mu_2, \alpha_1, \alpha_2, \theta$) distribution is the geometric stable form of the bivariate positive
stable distribution with Laplace transform given in (2.3).

Therefore,

$$\phi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2} + r\mu_1\mu_2 \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}}.$$

The following theorem gives a mixture representation of BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution.

**Theorem 2.1.** Let \((W_1, W_2)\) have positive stable distribution with Laplace transform in (2.4) and \(Z\), independent of \((W_1, W_2)\), have standard exponential distribution.

Then \((X, Y) = (Z^{1/\alpha_1} W_1, Z^{1/\alpha_2} W_2)\) has BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution.

**Proof.** The Laplace transform of \((X, Y)\) is

$$\phi(\lambda_1, \lambda_2) = \int_0^\infty E\left(e^{-\lambda_1 z^{1/\alpha_1} W_1 - \lambda_2 z^{1/\alpha_2} W_2}/Z\right) f_Z(z) dz$$

$$= \int_0^\infty E\left(e^{-\lambda_1 z^{1/\alpha_1} W_1 - \lambda_2 z^{1/\alpha_2} W_2}\right) e^{-z} dz$$

$$= \int_0^\infty e^{-z(\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})} e^{-z} dz$$

$$= \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}.$$

\(\square\)

Theorem 2.1 implies that a random vector \((X, Y)\) with BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution admits the representation

$$(X, Y) \sim (Z^{1/\alpha_1} W_1, Z^{1/\alpha_2} W_2). \quad (2.5)$$

Now, we obtain the distribution function of BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\).
From the representation given in (2.5), we have

\[
P(X \leq x, Y \leq y) = P(Z^{\frac{1}{\alpha_1}}W_1 \leq x, Z^{\frac{1}{\alpha_2}}W_2 \leq y)
\]

\[
= \int_0^\infty P(Z^{\frac{1}{\alpha_1}}W_1 \leq x, Z^{\frac{1}{\alpha_2}}W_2 \leq y/Z)f_Z(z)dz
\]

\[
= \int_0^\infty F_{W_1,W_2}(\frac{x}{z^{\frac{1}{\alpha_1}}}, \frac{y}{z^{\frac{1}{\alpha_2}}})e^{-z}dz
\]

where \(F_{W_1,W_2}(\cdot, \cdot)\) represents bivariate positive stable distribution function with Laplace transform in (2.3).

When \(W_1\) and \(W_2\) are independent \((r = 0)\),

\[
P(X \leq x, Y \leq y) = \int_0^\infty S_{\alpha_1}(\frac{x}{z^{\frac{1}{\alpha_1}}})S_{\alpha_2}(\frac{y}{z^{\frac{1}{\alpha_2}}})e^{-z}dz
\]

where \(S_{\alpha_i}(\cdot), (i = 1, 2, )\) represents the distribution function of a standard positive stable random variable Laplace transform, \(\phi(\lambda) = e^{-\lambda^\alpha}\).

The joint density function of BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\) distribution is

\[
h_{X,Y}(x, y) = \int_0^\infty f_{W_1,W_2}(\frac{x}{z^{\frac{1}{\alpha_1}}}, \frac{y}{z^{\frac{1}{\alpha_2}}})e^{-z}dz
\]

where \(f_{W_1,W_2}(\cdot, \cdot)\) is the joint density function of bivariate positive stable distribution. When \(W_1\) and \(W_2\) are independent,

\[
h_{X,Y}(x, y) = \int_0^\infty D_{\alpha_1}(\frac{x}{z^{\frac{1}{\alpha_1}}})D_{\alpha_2}(\frac{y}{z^{\frac{1}{\alpha_2}}})e^{-z}dz
\]

where \(D_{\alpha_i}(\cdot), (i=1,2, )\) represents the density function of positive stable random variable stated in (2.2).
We note that the Laplace transform in (2.2) is the geometric version of the bivariate positive stable distribution given in (2.4). Therefore, BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) is geometrically infinitely divisible (see Kozubowski (2005)).

The product moments of the distribution, \(E(X^{\delta_1}Y^{\delta_2})\), exists if and only if \(0 < \delta_1 < \alpha_1, 0 < \delta_2 < \alpha_2\) and

\[
E(X^{\delta_1}Y^{\delta_2}) = \frac{\Gamma\left(\frac{\delta_1}{\alpha_1} + \frac{\delta_2}{\alpha_2} + 1\right)\Gamma(1 - \frac{\delta_1}{\alpha_1})\Gamma(1 + \frac{\delta_1}{\alpha_1})\Gamma(1 - \frac{\delta_2}{\alpha_2})\Gamma(1 + \frac{\delta_2}{\alpha_2})}{\Gamma(1 - \delta_1)\Gamma(1 - \delta_2)}.
\]

In (2.5), assume that \(Z, W_1\) and \(W_2\) are independent. Then

\[
E(X^{\delta_1}Y^{\delta_2}) = E(Z^{\delta_1 + \delta_2})E(W_1^{\delta_1}W_2^{\delta_2}). \tag{2.6}
\]

But

\[
E(Z^{\delta_1 + \delta_2}) = \Gamma\left(\frac{\delta_1}{\alpha_1} + \frac{\delta_2}{\alpha_2} + 1\right).
\]
\[
E(W_i^{\delta_i}) = \frac{\Gamma(1 - \frac{\delta_i}{\alpha_i})\Gamma(1 + \frac{\delta_i}{\alpha_i})}{\Gamma(1 - \delta_i)} \text{ for } i = 1, 2.
\]

Substituting in (2.6), we get the required result.

The following theorem shows the attraction of BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution towards the bivariate positive stable distribution in (2.4).

**Theorem 2.2.** Let \(\{(X_i, Y_i), i \geq 1\}\) be a sequence of random vectors which are independently and identically distributed as BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\). Define

\[
U_n = n^{\frac{1}{\alpha_1}}(X_1 + X_2 + ... + X_n) \quad \text{and} \quad V_n = n^{\frac{1}{\alpha_2}}(Y_1 + Y_2 + ... + Y_n).
\]

Then \((U_n, V_n)\) is asymptotically distributed as bivariate positive stable law with Laplace transform in (2.4).
Proof. Suppose that \((X_i,Y_i), i \geq 1\) are distributed as BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\).

Therefore,

\[
\phi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}.
\]

The Laplace transform of \((U_n, V_n)\) is

\[
\varphi_{U_n,V_n}(\lambda_1, \lambda_2) = E(e^{-\lambda_1 \sum_{i=1}^{n} X_i - \lambda_2 \sum_{n=2}^{n} Y_i})
\]

\[
= \left[ E(e^{-\lambda_1 \sum_{i=1}^{n} X_i}) \cdot E(e^{-\lambda_2 \sum_{n=2}^{n} Y_i}) \right]^n
\]

\[
= \left[ \frac{1}{1 + \frac{1}{n}(\mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})} \right]^n.
\]

When \(n \to \infty\), we get

\[
\varphi_{U_n,V_n}(\lambda_1, \lambda_2) \to e^{-\mu_1 \lambda_1^{\alpha_1} - \mu_2 \lambda_2^{\alpha_2}}.
\]

\[\square\]

2.3 Characterization of BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\) through Geometric Compounding

Let \(\{(X_i, Y_i), i \geq 1\}\) be a sequence of independently and identically distributed random vectors with Laplace transform \(\psi(\lambda_1, \lambda_2)\). Define

\[
U_N = \sum_{i=1}^{N} X_i \quad V_N = \sum_{i=1}^{N} Y_i
\]  

(2.7)

where \(N\) follows geometric distribution such that

\[
P(N = n) = p(1 - p)^{n-1}, \quad n = 1, 2, 3, ..., 0 < p < 1
\]  

(2.8)
and is independent of \((X_i, Y_i), i \geq 1\). Then the Laplace transform of \((U_N, V_N)\) is

\[
\phi(\lambda_1, \lambda_2) = E(e^{-\lambda_1 U_N - \lambda_2 V_N})
\]

\[
= \sum_{n=1}^{\infty} E(e^{-\lambda_1 (X_1 + X_2 + \ldots + X_n) - \lambda_2 (Y_1 + Y_2 + \ldots + Y_n) / N = n}) P(N = n)
\]

\[
= \sum_{n=1}^{\infty} E(e^{-\lambda_1 (X_1 + X_2 + \ldots + X_n) - \lambda_2 (Y_1 + Y_2 + \ldots + Y_n)}) P(N = n)
\]

\[
= \sum_{n=1}^{\infty} [E(e^{-\lambda_1 X_i - \lambda_2 Y_i})]^n (1 - p)^{n-1}
\]

\[
= \frac{p \psi(\lambda_1, \lambda_2)}{1 - (1 - p) \psi(\lambda_1, \lambda_2)}. \quad (2.9)
\]

Now, we obtain a characterization of BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution.

**Theorem 2.3.** Let \([(X_i, Y_i), i \geq 1]\) be a sequence of independently and identically distributed random vectors and \(N\), independent of \((X_i, Y_i), i \geq 1\), follows geometric distribution in (2.8). Suppose that \(U_N\) and \(V_N\) are as defined in (2.7). Then \((p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)\) is distributed as BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) if and only if \((X_i, Y_i), i \geq 1\) follow BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution.

**Proof.** From (2.9), the Laplace transform of \((p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)\) is

\[
\phi(\lambda_1, \lambda_2) = \frac{p \psi(\frac{1}{\alpha_1} \lambda_1, \frac{1}{\alpha_2} \lambda_2)}{1 - (1 - p) \psi(\frac{1}{\alpha_1} \lambda_1, \frac{1}{\alpha_2} \lambda_2)}. \quad (2.10)
\]

where \(\psi(\lambda_1, \lambda_2)\) represents the Laplace transform of \((X_i, Y_i), i \geq 1\).

Taking

\[
\psi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}.
\]

Substituting \(\psi(\lambda_1, \lambda_2)\) in (2.10) and simplifying, we get

\[
\phi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}.
\]
Conversely, assume that \((p^{\frac{1}{2}}, U_N, p^{\frac{1}{2}} V_N)\) has BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution. From (2.10),

\[
\frac{1}{1 + \mu_1 \lambda_1 + \mu_2 \lambda_2^2} = \frac{p \psi(p^{\frac{1}{2}} \lambda_1, p^{\frac{1}{2}} \lambda_2)}{1 - (1 - p) \psi(p^{\frac{1}{2}} \lambda_1, p^{\frac{1}{2}} \lambda_2)}.
\]

Solving, we get

\[
\psi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1 + \mu_2 \lambda_2^2}.
\]

A characterization of BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution is obtained using repeated geometric compounding.

**Theorem 2.4.** Let \((U_{N_k}, V_{N_k})\) be defined as

\[
U_{N_k} = p^{\frac{1}{2}} \sum_{i=1}^{N_k-1} X_i, \quad V_{N_k} = p^{\frac{1}{2}} \sum_{i=1}^{N_k-1} Y_i
\]

where \(N_k\), independent of \((X_i, Y_i), i \geq 1\), follows geometric distribution such that

\[
P(N_k = n) = (1 - p_{k-1})^{n-1} p_{k-1}, \quad 0 < p_{k-1} < 1, \quad n = 1, 2, 3, ...
\]

\{\((X_i, Y_i), i \geq 1\)\} is a sequence of independently and identically distributed random vectors with distribution function \(F_{k-1}((., .))\) and Laplace transform \(\phi_{k-1}(\lambda_1, \lambda_2)\) \(k = 2, 3, ...

To start with, take \(F_1((., .)) = F((., .))\) and the corresponding Laplace transform as \(\phi(\lambda_1, \lambda_2)\). Then \((U_{N_k}, V_{N_k})\) is distributed as BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) if and only if \((X_i, Y_i), i \geq 1\) are distributed as BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\).

**Proof.** From (2.10), the Laplace transform of \((U_{N_k}, V_{N_k})\) is

\[
\phi_k(\lambda_1, \lambda_2) = \frac{p^{\frac{1}{2}} \phi_{k-1}(p^{\frac{1}{2}} \lambda_1, p^{\frac{1}{2}} \lambda_2)}{1 - (1 - p_{k-1}) \phi_{k-1}(p^{\frac{1}{2}} \lambda_1, p^{\frac{1}{2}} \lambda_2)}.
\]
Therefore,
\[ \phi_2(\lambda_1, \lambda_2) = \frac{p_1 \phi(p_1^{\alpha_1} \lambda_1, p_1^{\alpha_2} \lambda_2)}{1 - (1 - p_1) \phi(p_1^{\alpha_1} \lambda_1, p_1^{\alpha_2} \lambda_2)}, \]
since \( F_1(\ldots) = F(\ldots) \), \( \phi_1(\lambda_1, \lambda_2) = \phi(\lambda_1, \lambda_2) \).

Applying recursively (2.11),
\[ \phi_k(\lambda_1, \lambda_2) = \frac{\prod_{i=1}^{k-1} p_i \left[ \prod_{i=1}^{k-1} p_i^{\alpha_1} \lambda_1, \prod_{i=1}^{k-1} p_i^{\alpha_2} \lambda_2 \right]}{1 - \phi \left[ \prod_{i=1}^{k-1} p_i^{\alpha_1} \lambda_1, \prod_{i=1}^{k-1} p_i^{\alpha_2} \lambda_2 \right] + \prod_{i=1}^{k-1} p_i \phi \left[ \prod_{i=1}^{k-1} p_i^{\alpha_1} \lambda_1, \prod_{i=1}^{k-1} p_i^{\alpha_2} \lambda_2 \right]} \tag{2.12} \]

Suppose that \((X_i, Y_i), i \geq 1\) are distributed as BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\).

Therefore,
\[ \phi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}. \]

Substituting \( \phi(\lambda_1, \lambda_2) \) in (2.12) and simplifying, we get
\[ \phi_k(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}. \]

Conversely, assuming that \((U_{N_k}, V_{N_k})\) is distributed as BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\). Substituting \( \phi_k(\lambda_1, \lambda_2) \) in (2.12) and simplifying, we get
\[ \phi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}. \]

Now we introduce BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\) distribution by considering the geometric sum of a set of independently and identically distributed random variables. Let \( \{(X_i, Y_i), i \geq 1\} \) be a sequence of independently and identically distributed random
vectors such that the components $X_i$ and $Y_i$ are independently distributed as Mittag-Leffler.

Therefore,

$$\psi_{X_i}(\lambda_1, 0) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1}}, \quad \psi_{Y_i}(0, \lambda_2) = \frac{1}{1 + \mu_2 \lambda_2^{\alpha_2}} \quad \text{for} \quad i = 1, 2, 3, \ldots$$

The joint Laplace transform of $(X_i, Y_i), i \geq 1$ is

$$\psi(\lambda_1, \lambda_2) = \frac{1}{(1 + \mu_1 \lambda_1^{\alpha_1})(1 + \mu_2 \lambda_2^{\alpha_2})}. \quad (2.13)$$

From (2.10), the Laplace transform of $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ is

$$\phi(\lambda_1, \lambda_2) = \frac{p}{(1 + p \mu_1 \lambda_1^{\alpha_1})(1 + p \mu_2 \lambda_2^{\alpha_2}) - 1 + p}$$

$$= \frac{p}{1 + p \mu_1 \lambda_1^{\alpha_1} + p \mu_2 \lambda_2^{\alpha_2} + p^2 \mu_1 \mu_2 \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} - 1 + p}$$

$$= \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2} + p \mu_1 \mu_2 \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}}.$$

Comparing with (2.1), $\theta = 1 - p$.

Therefore, $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ has BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1 - p)$ distribution.

On the other hand suppose that $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ follows BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1 - p)$ distribution. From (2.10),

$$\frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2} + p \mu_1 \mu_2 \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}} = \frac{p \psi(\frac{1}{\alpha_1} \lambda_1, \frac{1}{\alpha_2} \lambda_2)}{1 - (1 - p) \psi(\frac{1}{\alpha_1} \lambda_1, \frac{1}{\alpha_2} \lambda_2)}.$$

Solving, we obtain

$$\psi(\lambda_1, \lambda_2) = \frac{1}{(1 + \mu_1 \lambda_1^{\alpha_1})(1 + \mu_2 \lambda_2^{\alpha_2})}.$$ 

Hence we have the following theorem.
Theorem 2.5. Consider a sequence of independently and identically distributed random vectors \( \{(X_i, Y_i), i \geq 1\} \) and \( N \) has geometric distribution given in (2.8). Assume that \( N \) is independent of \((X_i, Y_i), i \geq 1\). Then \((p_1^{1/N}U_N, p_2^{1/N}V_N)\) has BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1-p)\) distribution if and only if \(X_i\) and \(Y_i\) are independently distributed as Mittag-Leffler.

Hence Theorem 2.5 enables to generate BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\) distribution as a geometric compound of random vectors \((X_i, Y_i), i \geq 1\) such that \(X_i\) and \(Y_i, i \geq 1\) are independent Mittag-Leffler random variables.

The following theorem gives a characterization of geometric distribution.

Theorem 2.6. Suppose that \( \{(X_i, Y_i), i \geq 1\} \) is a sequence of independently and identically distributed BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) random vectors. Then \((p_1^{1/N}U_N, p_2^{1/N}V_N)\) and \((X_i, Y_i), i \geq 1\) are identically distributed if and only if \(N\) is geometric.

Proof. The proof of the 'if' part is omitted as it is presented in Theorem 2.3.

To prove the converse, assume that \((p_1^{1/N}U_N, p_2^{1/N}V_N)\) and \((X_i, Y_i), i \geq 1\) are identically distributed as BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\). Without loss of generality, we take \(\mu_1 = \mu_2 = 1\). The Laplace transform of \((p_1^{1/N}U_N, p_2^{1/N}V_N)\) is given by

\[
\phi(\lambda_1, \lambda_2) = \sum_{n=1}^{\infty} \left( \psi(p_1^{1/N} \lambda_1, p_2^{1/N} \lambda_2) \right)^n P(N = n),
\]

where \(\psi(\lambda_1, \lambda_2)\) represents the Laplace transform of \((X_i, Y_i), i \geq 1\). Therefore, from
the assumption it follows,
\[
\sum_{n=1}^{\infty} \left( \frac{1}{1 + p\lambda_1^{\alpha_1} + p\lambda_2^{\alpha_2}} \right)^n P(N = n) = \frac{1}{1 + \lambda_1^{\alpha_1} + \lambda_2^{\alpha_2}}.
\]
Expanding both sides,
\[
\sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j(j + n - 1)!}{j!(n-1)!} (\lambda_1^{\alpha_1} + \lambda_2^{\alpha_2})^j p^j P(N = n) = \sum_{j=0}^{\infty} (-1)^j(\lambda_1^{\alpha_1} + \lambda_2^{\alpha_2})^j.
\]
Comparing the coefficients of \((\lambda_1^{\alpha_1} + \lambda_2^{\alpha_2})^j\),
\[
\sum_{n=1}^{\infty} \frac{n(n+1)(n+2)\ldots(n+j-1)}{j!} p^j P(N = n) = 1, \text{ for } j = 1, 2, 3, \ldots
\]
Therefore,
\[
E(N) = \frac{1}{p}, \quad E(N(N + 1)) = \frac{2}{p^2} \text{ and so on.}
\]
Consider
\[
E(1 - t)^{-N} = 1 + \frac{t}{1!} E(N) + \frac{t^2}{2!} E(N(N + 1)) + \frac{t^3}{3!} E(N(N + 1)(N + 2)) + \ldots
\]
\[
= \frac{p}{p - t}
\]
\[
= \frac{p}{1 - p} \sum_{n=1}^{\infty} \left( \frac{1 - p}{1 - t} \right)^n
\]
\[
= p \sum_{n=1}^{\infty} (1 - t)^{-n} (1 - p)^{n-1}.
\]
Also
\[
E(1 - t)^{-N} = \sum_{n=1}^{\infty} (1 - t)^{-n} P(N = n).
\]
Therefore,
\[
\sum_{n=1}^{\infty} (1 - t)^{-n} P(N = n) = p \sum_{n=1}^{\infty} (1 - t)^{-n} (1 - p)^{n-1}.
\]
Comparing we get,

\[ P(N = n) = (1 - p)^{n-1}p, \quad n = 1, 2, 3, \ldots \]

\[
\square
\]

2.4 Characterization of BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\) through Bivariate Geometric Compounding

Block (1977) discussed the probability distributions of random sums of independently and identically distributed random vectors when the number of summands follow the bivariate geometric distribution in (1.15).

Let \(\{(X_i, Y_i), i \geq 1\}\) be a sequence of independently and identically distributed random vectors with Laplace transform \(\psi(\lambda_1, \lambda_2)\). Define

\[ U_{N_1} = \sum_{i=1}^{N_1} X_i \text{ and } V_{N_2} = \sum_{i=1}^{N_2} Y_i \]

(2.14)

where \((N_1, N_2)\) has the bivariate geometric distribution with p.g.f. in (1.16).

Block (1977) has given the expression for the Laplace transform of \((U_{N_1}, V_{N_2})\).

\[ \phi(\lambda_1, \lambda_2) = \psi(\lambda_1, \lambda_2)(\rho_{00} + \rho_{10}\psi(\lambda_1, 0) + \rho_{01}\phi(0, \lambda_2) + \rho_{11}\phi(\lambda_1, \lambda_2)). \]

(2.15)

Therefore,

\[ \phi(\lambda_1, \lambda_2) = \frac{\psi(\lambda_1, \lambda_2)}{1 - \rho_{11}\psi(\lambda_1, \lambda_2)} (\rho_{00} + \rho_{10}\phi(\lambda_1, 0) + \rho_{01}\phi(0, \lambda_2)). \]

(2.16)
Also, from (2.15)

\[
\phi(\lambda_1,0) = \frac{(p_{00} + p_{01})\psi(\lambda_1,0)}{1 - (p_{11} + p_{10})\psi(\lambda_1,0)} \quad \text{and} \quad \phi(0,\lambda_2) = \frac{(p_{00} + p_{10})\psi(0,\lambda_2)}{1 - (p_{11} + p_{01})\psi(0,\lambda_2)}.
\] (2.17)

By choosing appropriately \(\psi(\lambda_1, \lambda_2), p_{00}, p_{10}, p_{01}\) and \(p_{11}\) we obtain characterizations of BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\) distribution and also generate some other forms of bivariate Mittag-Leffler distribution.

**Theorem 2.7.** Consider a sequence \(\{(X_i, Y_i), i \geq 1\}\) of independently and identically distributed random vectors. Let \((N_i, N_2)\) be independent of \((X_i, Y_i), i \geq 1\) and have the bivariate geometric distribution with p.g.f. in (1.16) such that \(p_{00} = 0\) and \(p_{10} + p_{01} + p_{11} = 1\). Let \(U_{N_i} = \sum_{i=1}^{N_i} X_i\) and \(V_{N_2} = \sum_{i=1}^{N_2} Y_i\). Then \((p_{01} U_{N_1}, p_{10}^2 V_{N_2})\) follows bivariate Mittag-Leffler distribution with independent marginals where \((U_{N_1}, V_{N_2})\) are as stated in (2.14) if and only if \((X_i, Y_i), i \geq 1\) have BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution.

**Proof.** From (2.15), the Laplace transform of \((p_{01} U_{N_1}, p_{10}^2 V_{N_2})\) is

\[
\phi(\lambda_1, \lambda_2) = \psi(p_{01}^{\frac{1}{\alpha_1}} \lambda_1, p_{10}^{\frac{1}{\alpha_2}} \lambda_2)\psi(p_{00} + p_{10} \phi(\lambda_1, 0) + p_{01} \phi(0, \lambda_2) + p_{11} \phi(\lambda_1, \lambda_2)).
\] (2.18)

When \(p_{00} = 0\) and \(p_{10} + p_{01} + p_{11} = 1\),

\[
\phi(\lambda_1, \lambda_2) = \frac{\psi(p_{01}^{\frac{1}{\alpha_1}} \lambda_1, p_{10}^{\frac{1}{\alpha_2}} \lambda_2)}{1 - p_{11} \psi(p_{01}^{\frac{1}{\alpha_1}} \lambda_1, p_{10}^{\frac{1}{\alpha_2}} \lambda_2)} (p_{10} \phi(\lambda_1, 0) + p_{01} \phi(0, \lambda_2)).
\] (2.19)

Also, from (2.18)

\[
\phi(\lambda_1, 0) = \frac{p_{01} \psi(p_{01}^{\frac{1}{\alpha_1}} \lambda_1, 0)}{1 - (1 - p_{01}) \psi(p_{01}^{\frac{1}{\alpha_1}} \lambda_1, 0)} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{p_{10} \psi(0, p_{10}^{\frac{1}{\alpha_2}} \lambda_2)}{1 - (1 - p_{10}) \psi(0, p_{10}^{\frac{1}{\alpha_2}} \lambda_2)}.
\] (2.20)
If \((X_i, Y_i), i \geq 1\) have BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution, then from (2.20), we get

\[
\phi(\lambda_1, 0) = \frac{1}{1 + \mu_1 \lambda_1^\alpha_1} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{1}{1 + \mu_2 \lambda_2^\alpha_2}.
\]

Substituting \(\phi(\lambda_1, 0)\) and \(\phi(0, \lambda_2)\) in (2.19) and simplifying

\[
\phi(\lambda_1, \lambda_2) = \frac{1}{(1 + \mu_1 \lambda_1^\alpha_1)(1 + \mu_2 \lambda_2^\alpha_2)}.
\]

Conversely, suppose that \((\frac{1}{p_{01}} U_{N_1}, \frac{1}{p_{10}} V_{N_2})\) follows bivariate Mittag-Leffler distribution with independent marginals. Substituting the Laplace transform of \((\frac{1}{p_{01}} U_{N_1}, \frac{1}{p_{10}} V_{N_2})\) in (2.19),

\[
\frac{1}{(1 + \mu_1 \lambda_1^\alpha_1)(1 + \mu_2 \lambda_2^\alpha_2)} = \frac{\psi(\frac{1}{p_{01}} \lambda_1, \frac{1}{p_{10}} \lambda_2)}{1 - p_{11} \psi(\frac{1}{p_{01}} \lambda_1, \frac{1}{p_{10}} \lambda_2)} \left( p_{10} \left( 1 + \mu_1 \lambda_1^\alpha_1 + p_{01} \frac{1}{1 + \mu_2 \lambda_2^\alpha_2} \right) \right).
\]

Therefore,

\[
1 - p_{11} \psi(\frac{1}{p_{01}} \lambda_1, \frac{1}{p_{10}} \lambda_2) = \psi(\frac{1}{p_{01}} \lambda_1, \frac{1}{p_{10}} \lambda_2) \left[ p_{10} (1 + \mu_2 \lambda_2^\alpha_2) + p_{01} (1 + \mu_1 \lambda_1^\alpha_1) \right].
\]

On simplification, we get

\[
\psi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^\alpha_1 + \mu_2 \lambda_2^\alpha_2}.
\]

\[\square\]

**Theorem 2.8.** Let \(\{(X_i, Y_i), i \geq 1\}\) be a sequence of independently and identically distributed Mittag-Leffler random vectors with Laplace transform \(\psi(\lambda_1, \lambda_2)\). Suppose that \((N_1, N_2)\) is independent of \((X_i, Y_i), i \geq 1\) and follows bivariate geometric distribution with p.g.f. given in (1.16). Let \(p_{00} = 0, p_{10} + p_{01} + p_{11} = 1\). Then \(((1 - p_{11})^\frac{1}{p_{01}} U_{N_1}, (1 - p_{11})^\frac{1}{p_{10}} V_{N_2})\) has Laplace transform

\[
\phi(\lambda_1, \lambda_2) = \frac{1}{(1 + (1 - p_{11}) \lambda_1^\alpha_1)(1 + (1 - p_{11}) \lambda_2^\alpha_2)}.
\]
if and only if \( \psi(\lambda_1, \lambda_2) = \frac{1}{1 + p_{01}\lambda_1^{\alpha_1} + p_{10}\lambda_2^{\alpha_2}} \)

Proof. Suppose that

\[ \psi(\lambda_1, \lambda_2) = \frac{1}{1 + p_{01}\lambda_1^{\alpha_1} + p_{10}\lambda_2^{\alpha_2}}. \]

Using (2.17),

\[ \phi(\lambda_1, 0) = \frac{1}{1 + (1 - p_{11})\lambda_1^{\alpha_1}} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{1}{1 + (1 - p_{11})\lambda_2^{\alpha_2}}. \]

From (2.16) the Laplace transform of \((1 - p_{11})^{\frac{1}{\alpha_1}} U_{N_1}, (1 - p_{11})^{\frac{1}{\alpha_2}} V_{N_2}\) is

\[ \phi(\lambda_1, \lambda_2) = \frac{\psi((1 - p_{11})^{\frac{1}{\alpha_1}} \lambda_1, (1 - p_{11})^{\frac{1}{\alpha_2}} \lambda_2)}{1 - p_{11}\psi((1 - p_{11})^{\frac{1}{\alpha_1}} \lambda_1, (1 - p_{11})^{\frac{1}{\alpha_2}} \lambda_2)} \left( p_{10}\phi(\lambda_1, 0) + p_{01}\phi(0, \lambda_2) \right). \]

(2.21)

Substituting \( \phi(\lambda_1, 0) \) and \( \phi(0, \lambda_2) \) in (2.21) and simplifying, we get

\[ \phi(\lambda_1, \lambda_2) = \frac{1}{(1 + (1 - p_{11})\lambda_1^{\alpha_1})(1 + (1 - p_{11})\lambda_2^{\alpha_2})}. \]

In order to prove the converse, substituting \( \phi(\lambda_1, \lambda_2), \phi(\lambda_1, 0) \) and \( \phi(0, \lambda_2) \) in (2.21). On simplification, we get

\[ \psi(\lambda_1, \lambda_2) = \frac{1}{1 + p_{01}\lambda_1^{\alpha_1} + p_{10}\lambda_2^{\alpha_2}}. \]

Using the Laplace transform of bivariate compounding mentioned in (2.15), now we obtain BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)\) as the random sum distribution of independently and identically distributed random vectors in which the components have independent Mittag-Leffler.
**Theorem 2.9.** Suppose that \( \{(X_i, Y_i), i \geq 1\} \) are independently and identically distributed random vectors with Laplace transform

\[
\psi(\lambda_1, \lambda_2) = \left(1 + \frac{\mu_1 \lambda_1^{\alpha_1}}{1 + m}\right)^{-1} \left(1 + \frac{\mu_2 \lambda_2^{\alpha_2}}{1 + m}\right)^{-1}, \quad 0 < \alpha_1, \alpha_2 \leq 1. \tag{2.22}
\]

Take \( p_{00} = (1 + m)^{-1} \), \( p_{10} = p_{01} = 0 \) and \( p_{11} = m(1 + m)^{-1} \), \( \mu_1, \mu_2 > 0, \ m > 0 \). Then the random vector \((U_{N_1}, V_{N_2})\) has BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \frac{m}{1 + m})\) distribution if and only if \((X_i, Y_i), i \geq 1\) have the Laplace transform in (2.22).

**Proof.** Suppose that \((X_i, Y_i), i \geq 1\) have the Laplace transform in (2.22). From (2.15), the Laplace transform of \((U_{N_1}, V_{N_2})\) is

\[
\phi(\lambda_1, \lambda_2) = \frac{\phi(\lambda_1, \lambda_2)}{(1 + m)^{-1} \left(1 + \frac{\mu_1 \lambda_1^{\alpha_1}}{1 + m}\right) \left(1 + \frac{\mu_2 \lambda_2^{\alpha_2}}{1 + m}\right) - m}
\]

Comparing with (2.1), we get \((U_{N_1}, V_{N_2})\) has BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \frac{m}{1 + m})\) distribution.

Conversely, let \((U_{N_1}, V_{N_2})\) have BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, \frac{m}{1 + m})\) distribution. From (2.15), we get

\[
\psi(\lambda_1, \lambda_2) = \frac{(1 + m)\phi(\lambda_1, \lambda_2)}{1 + m\phi(\lambda_1, \lambda_2)}.
\]

Substituting \(\phi(\lambda_1, \lambda_2)\) and simplifying, we get (2.22). \(\square\)
2.5 Estimation of Parameters

In this Section, we obtain the log moment estimators of the parameters of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution. Kozubowski (2000a) showed that a random variable $X$ following Mittag-Leffler distribution with parameter $\alpha$ can be represented as $ZW^{1/\alpha}$ where $Z$ has standard exponential distribution and $W$ follows positive stable distribution given in (1.3). Suppose that a random vector $(X, Y)$ has BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution. Then $(X, Y)$ admits the relation

$$(X, Y) \overset{d}{=} (Z(\mu_1 W_1)^{\frac{1}{\alpha_1}}, Z(\mu_2 W_2)^{\frac{1}{\alpha_2}})$$

where $W_1$ and $W_2$ are independently distributed stable random variables and also independent of $Z$ which follows standard exponential distribution. Now let us consider the estimation of the parameter $\alpha_1$. We have

$$X \overset{d}{=} Z(\mu_1 W_1)^{\frac{1}{\alpha_1}}.$$

Then it follows that the random variables $\ln X$ and $\ln Z + \frac{1}{\alpha_1} \ln(\mu_1 W_1)$ have same distributions.

Therefore,

$$(\ln X)^2 \overset{d}{=} (\ln Z)^2 + \frac{1}{\alpha_1^2} (\ln \mu_1)^2 + \frac{1}{\alpha_1^2} (\ln W_1)^2$$

$$+ \frac{2}{\alpha_1} \ln Z \ln \mu_1 + \frac{2}{\alpha_1} \ln Z \ln W_1 + \frac{2}{\alpha_1^2} \ln W_1 \ln \mu_1.$$  \tag{2.23}
Taking expectations on both sides of (2.23)

\[
E(\ln X)^2 = E(\ln Z)^2 + \frac{1}{\alpha^2_1} (\ln \mu_1)^2 + \frac{1}{\alpha^2_1} E(\ln W_1)^2
\]
\[
+ \frac{2}{\alpha_1} \ln \mu_1 E(\ln Z) + \frac{2}{\alpha_1} E(\ln Z) E(\ln W_1)
\]
\[
+ \frac{2}{\alpha^2_1} \ln \mu_1 E(\ln W_1).
\]  

(2.24)

Let us consider first the moments of the random variable \(\ln Z\). Define the function \(h_k(m)\) as

\[
h_k(m) = \int_0^\infty (\ln x)^k e^{-mx} dx, \quad k = 1, 2.
\]

From Gradshteyn and Ryzhik (1996), we have

\[
h_1(m) = -\frac{1}{m} (\gamma + \ln m), \quad Re(m) > 0
\]

where \(\gamma\) is the Euler's constant.

Thus the first moment of the random variable \(\ln Z\) is

\[
E(\ln Z) = h_1(1) = -\gamma.
\]

Consider now the second moment of the random variable \(\ln Z\). From Gradshteyn and Ryzhik (1996), we have

\[
h_2(m) = \frac{1}{m} \left( \frac{\pi^2}{6} + (\gamma + \ln m)^2 \right), \quad Re(m) > 0.
\]

It follows that the second moment \(E(\ln Z)^2\) is

\[
E(\ln Z)^2 = h_2(1) = \gamma^2 + \frac{\pi^2}{6}.
\]
Now, we will consider the first and the second moment of the random variable \( \ln W_1 \).

Define the function \( g_1(t) \) as

\[
 g_1(t) = \int_0^\infty \frac{\ln x}{x^2 - 2x \cos t + 1} \, dx, \quad 0 < t < 2\pi.
\]

Then

\[
 \int_0^\infty \frac{\ln x}{x^2 - 2x \cos t + 1} = \int_0^1 \frac{\ln x}{x^2 - 2x \cos \alpha + 1} + \int_1^\infty \frac{\ln x}{x^2 - 2x \cos t + 1}.
\]

Consider the second integral on the right side. The change of variables \( x = 1/y \) gives

\[
 \int_1^\infty \frac{\ln x}{x^2 - 2x \cos t + 1} = -\int_0^1 \frac{\ln y}{y^2 - 2y \cos t + 1}.
\]

Thus we obtain that

\[
 g_1(t) = \int_0^1 \frac{\ln x}{x^2 - 2x \cos t + 1} - \int_0^1 \frac{\ln y}{y^2 - 2y \cos t + 1} = 0.
\]

The first moment of the random variable \( \ln W_1 \) is

\[
 E(\ln W_1) = \frac{\sin \pi \alpha}{\pi \alpha} \int_0^\infty \frac{\ln x}{x^2 + 2x \cos(\pi \alpha) + 1} = \frac{\sin \pi \alpha}{\pi \alpha} g_1(\pi + \alpha \pi) = 0.
\]

Finally, consider the second moment of the random variable \( \ln W_2 \). Define the function \( g_2(t) \) as

\[
 g_2(t) = \int_0^\infty \frac{(\ln x)^2}{x^2 - 2x \cos t + 1}.
\]

Taking the same argument as in the case of the function \( g_1(t) \), we obtain that

\[
 g_2(t) = \int_0^1 \frac{(\ln x)^2}{x^2 - 2x \cos t + 1} + \int_1^\infty \frac{(\ln x)^2}{x^2 - 2x \cos t + 1}
\]

\[
 = \int_0^1 \frac{(\ln x)^2}{x^2 - 2x \cos t + 1} + \frac{1}{x^2 - 2x \cos t + 1}.
\]

\[
 = 2 \int_0^1 \frac{(\ln x)^2}{x^2 - 2x \cos t + 1}.
\]
Now, using Prudnikov et al. (1981), we obtain
\[
g_2(t) = \frac{t(t - 2\pi)(t - \pi)}{3 \sin t}.
\]

Then the second moment of the random variable \(\ln W_1\) is
\[
E(\ln W_1)^2 = \frac{\sin \pi \alpha}{\pi \alpha} \int_0^{\infty} \frac{(\ln x)^2 dx}{x^2 + 2x \cos(\pi \alpha) + 1}.
\]

Using the above results, from (2.24), the second moment of \((\ln X)^2\) as
\[
E(\ln X)^2 = \gamma^2 + \frac{\pi^2}{6} + \frac{\pi^2(1 - \alpha^2)}{3\alpha_1^2} + \frac{1}{\alpha_1^2}(\ln \mu_1)^2 - \frac{2}{\alpha_1} \gamma \ln \mu_1
\]

Replacing \(E(\ln X)^2\) by the corresponding sample equivalent we get a quadratic equation in \(\alpha_1\),
\[
\left(\gamma^2 - \frac{\pi^2}{6} - \frac{1}{n} \sum_{i=1}^{n} (\ln X_i)^2\right) \alpha_1^2 - 2 \gamma \alpha_1 \ln \mu_1 + (\ln \mu_1)^2 + \frac{\pi^2}{3}.
\]

Then as an estimate of \(\alpha_1\), we have
\[
\hat{\alpha}_1 = \frac{\gamma \ln \mu_1 - \sqrt{\gamma^2(\ln \mu_1)^2 - \left(\gamma^2 - \frac{\pi^2}{6} - \frac{1}{n} \sum_{i=1}^{n} (\ln X_i)^2\right) \left((\ln \mu_1)^2 + \frac{\pi^2}{3}\right)}}{\gamma^2 - \frac{\pi^2}{6} - \frac{1}{n} \sum_{i=1}^{n} (\ln X_i)^2}.
\]

The first moment of \(\ln X\) is
\[
E(\ln X) = -\gamma + \frac{1}{\alpha_1} \ln \mu_1.
\]
Therefore, the estimate of $\mu_1$ is

$$\hat{\mu}_1 = e^{\tilde{\alpha}_1 \left( \gamma + \frac{1}{n} \sum_{i=1}^{n} (\ln X_i) \right)}.$$ 

Similarly,

$$\hat{\alpha}_2 = \frac{\gamma \ln \mu_2 - \sqrt{\gamma^2 (\ln \mu_2)^2 - \left( \gamma^2 - \frac{n^2}{6} + \frac{1}{n} \sum_{i=1}^{n} (\ln Y_i)^2 \right) \left( (\ln \mu_2)^2 + \frac{n^2}{3} \right)}}{\gamma^2 - \frac{n^2}{6} + \frac{1}{n} \sum_{i=1}^{n} (\ln Y_i)^2}$$

and

$$\hat{\mu}_2 = e^{\tilde{\alpha}_2 \left( \gamma + \frac{1}{n} \sum_{i=1}^{n} (\ln Y_i) \right)}.$$ 

As an illustration, we estimate the unknown parameters using Monte Carlo method. For different values of the parameters $\alpha_1, \alpha_2, \mu_1$ and $\mu_2$, we simulate 10 sequences of 1000, 5000, 10000 observations following BML ($\mu_1, \mu_2, \alpha_1, \alpha_2, 1$) distribution. In Table 2.1, we present the averages and standard deviations of these estimators.

### 2.6 Autoregressive Processes with BML ($\mu_1, \mu_2, \alpha_1, \alpha_2, 1$)

**Marginals**

Gaver and Lewis (1980) developed exponential autoregressive process (EAR(1)) as the solution of the first order autoregressive equation

$$X_n = \rho X_{n-1} + \epsilon_n.$$
Table 2.1. The estimators of the parameters for different values of $\alpha_1, \alpha_2, \mu_1$ and $\mu_2$.

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<th>$\alpha_2 = 0.4$</th>
<th>$\mu_2 = 4$</th>
<th>$\alpha_1 = 0.4$</th>
<th>$\mu_1 = 15$</th>
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<td>(5.0321)</td>
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<td>$\alpha_2 = 0.4$</td>
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where $0 \leq \rho < 1$, \(\{\epsilon_n, n \geq 1\}\) is a sequence of independently and identically distributed exponential random variables and \(X'_n\)'s have marginally exponential distribution. Later, Jayakumar and Pillai (1993) developed first order Mittag-Leffler autoregressive process (MLAR(1)) as a generalization of the EAR(1) process. The MLAR(1) process has structure,

$$X_n = \begin{cases} 
\rho X_{n-1} & \text{with probability } \rho^a \\
\rho X_{n-1} + \epsilon_n & \text{with probability } 1 - \rho^a
\end{cases}$$

where $0 < \alpha \leq 1$. Mundassery and Jayakumar (2007b) introduced a bivariate first order autoregressive process with BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) marginals.

**Theorem 2.10.** Consider a first order autoregressive process \(\{(X_n, Y_n), n \geq 1\}\) with structure

\[
(X_0, Y_0) \overset{d}{=} (\epsilon_1, \psi_1) \quad \text{and for } n = 1, 2, 3, \ldots
\]

\[
(X_n, Y_n) = \begin{cases} 
(\rho^{\frac{1}{\alpha_1}} X_{n-1}, \rho^{\frac{1}{\alpha_2}} Y_{n-1}) & \text{with probability } \rho \\
(\rho^{\frac{1}{\alpha_1}} X_{n-1} + \epsilon_n, \rho^{\frac{1}{\alpha_2}} Y_{n-1} + \psi_n) & \text{with probability } 1 - \rho
\end{cases} \tag{2.25}
\]

where $0 \leq \rho < 1$, $0 < \alpha_1, \alpha_2 \leq 1$ and \(\{(\epsilon_n, \psi_n), n \geq 1\}\) is a sequence of independently and identically distributed random vectors. Then \(\{(X_n, Y_n), n \geq 1\}\) represents a stationary first order autoregressive process with BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) marginals if and only if \(\{(\epsilon_n, \psi_n), n \geq 1\}\) are distributed according to BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\).

**Proof.** The Laplace transform of the process in (2.25) is

\[
\phi_{X_n,Y_n}(\lambda_1, \lambda_2) = \rho \phi_{X_{n-1},Y_{n-1}}(\rho^{\frac{1}{\alpha_1}} \lambda_1, \rho^{\frac{1}{\alpha_2}} \lambda_2) + (1 - \rho) \phi_{X_{n-1},Y_{n-1}}(\rho^{\frac{1}{\alpha_1}} \lambda_1, \rho^{\frac{1}{\alpha_2}} \lambda_2) \phi_{\epsilon_n,\psi_n}(\lambda_1, \lambda_2). \tag{2.26}
\]
When the process is stationary with BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) marginals, from (2.26) we have

\[
\frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}} = \frac{\rho}{1 + \rho\mu_1 \lambda_1^{\alpha_1} + \rho\mu_2 \lambda_2^{\alpha_2}} + \frac{1 - \rho}{1 + \rho\mu_1 \lambda_1^{\alpha_1} + \rho\mu_2 \lambda_2^{\alpha_2}} \phi_{\epsilon, \psi}(\lambda_1, \lambda_2).
\]

On simplification, we get

\[
\phi_{\epsilon, \psi}(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}.
\]

Conversely, suppose that \((\epsilon_n, \psi_n), \ n \geq 1\) are distributed according to BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) and \((X_0, Y_0) \overset{d}{=} (\epsilon_1, \psi_1)\). Choose \(n=1\). From (2.26), we have

\[
\phi_{X_1, Y_1}(\lambda_1, \lambda_2) = \rho \phi_{X_0, Y_0}(\frac{1}{\rho^{\alpha_1}} \lambda_1, \frac{1}{\rho^{\alpha_2}} \lambda_2) + (1 - \rho) \phi_{X_0, Y_0}(\frac{1}{\rho^{\alpha_1}} \lambda_1, \frac{1}{\rho^{\alpha_2}} \lambda_2) \phi_{\epsilon_1, \psi_1}(\lambda_1, \lambda_2).
\]

Substituting the Laplace transform of \((\epsilon_1, \psi_1)\)

\[
\phi_{X_1, Y_1}(\lambda_1, \lambda_2) = \frac{\rho}{1 + \rho\mu_1 \lambda_1^{\alpha_1} + \rho\mu_2 \lambda_2^{\alpha_2}} \frac{1 - \rho}{1 + \rho\mu_1 \lambda_1^{\alpha_1} + \rho\mu_2 \lambda_2^{\alpha_2}} \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}.
\]

Hence by mathematical induction, the process is stationary with BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) marginals.

Now we obtain a generalization of the first order autoregressive process developed in (2.25). Consider an autoregressive process with structure:

\[
(X_n, Y_n) = \begin{cases} 
(\epsilon_n, \psi_n), & \text{with probability } 1 - p \\
(p^{\frac{1}{\alpha_1}} X_{n-1}, p^{\frac{1}{\alpha_2}} Y_{n-1}), & \text{with probability } p(1 - q) \\
(p^{\frac{1}{\alpha_1}} X_{n-1} + \epsilon_n, p^{\frac{1}{\alpha_2}} Y_{n-1} + \psi_n), & \text{with probability } (1 - p)(1 - q) 
\end{cases}
\]

(2.27)
where \( \{(\epsilon_n, \psi_n), n \geq 1\} \) is a sequence of independently and identically distributed random vectors, \((X_{n-1}, Y_{n-1})\) and \((\epsilon_n, \psi_n)\) are independent random vectors and \(0 < p < 1, q = 1 - p\) and \(0 < \alpha_1, \alpha_2 \leq 1\). Note that for \(q = 0\), we get the first order autoregressive process discussed in (2.25).

The following theorem gives a necessary and sufficient condition for the stationarity of the process in (2.27).

**Theorem 2.11.** Let \( (X_0, Y_0) \overset{d}{=} (\epsilon_1, \psi_1) \). The process \( \{(X_n, Y_n), n \geq 1\} \) defined in (2.27) is stationary with BML\((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) marginals if and only if \( \{(\epsilon_n, \psi_n), n \geq 1\} \) is a sequence of independently and identically distributed random vectors according to BML\((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution.

**Proof.** The Laplace transform of (2.27) is

\[
\phi_{X,Y}(\lambda_1, \lambda_2) = q\phi_{\epsilon,\psi}(\lambda_1, \lambda_2) + p(1 - q)\phi_{X_{n-1},Y_{n-1}}(p^{\frac{1}{\alpha_1}} \lambda_1, p^{\frac{1}{\alpha_2}} \lambda_2) + (1 - p)(1 - q)\phi_{X_{n-1},Y_{n-1}}(p^{\frac{1}{\alpha_1}} \lambda_1, p^{\frac{1}{\alpha_2}} \lambda_2)\phi_{\epsilon,\psi}(\lambda_1, \lambda_2). \tag{2.28}
\]

When the process is stationary, we have

\[
\phi_{X,Y}(\lambda_1, \lambda_2) = q\phi_{\epsilon,\psi}(\lambda_1, \lambda_2) + p(1 - q)\phi_{X,Y}(p^{\frac{1}{\alpha_1}} \lambda_1, p^{\frac{1}{\alpha_2}} \lambda_2) + \frac{(1 - p)(1 - q)}{\phi_{X,Y}(p^{\frac{1}{\alpha_1}} \lambda_1, p^{\frac{1}{\alpha_2}} \lambda_2)\phi_{\epsilon,\psi}(\lambda_1, \lambda_2)}.
\]

Assume that \(\phi_{X,Y}(\lambda_1, \lambda_2)\) corresponds to BML\((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution. Substituting and simplifying, we get

\[
\phi_{\epsilon,\psi}(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}.
\]
Proof of the converse is obtained by mathematical induction. Suppose that 
\((\epsilon_n, \psi_n), n \geq 1\) follow BML\((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) distribution and \((X_0, Y_0) \overset{d}{=} (\epsilon_1, \psi_1)\). Put 
\(n = 1\) in (2.28) we obtain

\[
\phi_{X_1,Y_1}(\lambda_1, \lambda_2) = q\phi_{\epsilon_1,\psi_1}(\lambda_1, \lambda_2) + p(1-q)\phi_{X_0,Y_0}(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2) \\
+ (1-p)(1-q)\phi_{X_0,Y_0}(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)\phi_{\epsilon_1,\psi_1}(\lambda_1, \lambda_2).
\]

Under the assumption,

\[
\phi_{X_1,Y_1}(\lambda_1, \lambda_2) = \frac{q}{1 + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}} + \frac{p(1-q)}{1 + p(\mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2})} \\
+ \frac{(1-p)(1-q)}{1 + p(\mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2})} \frac{1}{1 + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}}.
\]

On simplification, we get

\[
\phi_{X_1,Y_1}(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}}.
\]

Hence by mathematical induction we get that the process is stationary with BML 
\((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) marginals.

\[\square\]

As a remark, we obtain a bivariate first order autoregressive process with MBE 
\((\mu_1, \mu_2, 1)\) marginals.

**Remark 2.1.** Consider a first order autoregressive process \((X_n, Y_n), n \geq 1\) with following structure:

\[
(X_0, Y_0) \overset{d}{=} (\epsilon_1, \psi_1) \text{ and for } n = 1, 2, 3, ...
\]

\[
(X_n, Y_n) = \begin{cases} 
(\epsilon_n, \psi_n), & \text{with probability } q \\
(pX_{n-1}, pY_{n-1}), & \text{with probability } (1-q)p \\
(pX_{n-1} + \epsilon_n, pY_{n-1} + \psi_n), & \text{with probability } (1-p)(1-q)
\end{cases}
\]
Then the process \( \{(X_n, Y_n), n \geq 1 \} \) is stationary with MBE \((\mu_1, \mu_2, 1)\) marginals if and only if the innovations \((\epsilon_n, \psi_n), n \geq 1\) follow MBE\((\mu_1, \mu_2, 1)\).

Proof of the Remark 2.1 is omitted since we can easily deduce from Theorem 2.11.

A first order autoregressive process \( \{X_n, n \geq 1\} \), called TEAR(1), is introduced in Lawrance and Lewis (1980). Its structure is

\[
X_n = \begin{cases} 
\rho \epsilon_n & \text{with probability } \rho \\
X_{n-1} + \rho \epsilon_n & \text{with probability } 1 - \rho 
\end{cases}
\]  

\hspace{1cm} (2.29)

where \(\{\epsilon_n, n \geq 1\} \) is a sequence of independently and identically distributed random variables and \(0 \leq \rho < 1\). Using this model, in the following theorem we develop a first order autoregressive process having BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) marginals.

**Theorem 2.12.** Let a first order autoregressive process \( \{(X_n, Y_n), n \geq 1\} \) have the structure

\[
(X_n, Y_n) = \begin{cases} 
(\rho^{n+1} \epsilon_n, \rho^{n+2} \psi_n) & \text{with probability } \rho \\
(X_{n-1} + \rho^{n+1} \epsilon_n, Y_{n-1} + \rho^{n+2} \psi_n) & \text{with probability } 1 - \rho 
\end{cases}
\]  

\hspace{1cm} (2.30)

where \(\{(\epsilon_n, \psi_n), n \geq 1\} \) is a sequence of independently and identically distributed random vectors. Then \(\{(X_n, Y_n), n \geq 1\} \) is a first order stationary autoregressive process with BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) marginals if and only if \((\epsilon_n, \psi_n), n \geq 1\) are distributed according to BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\), provided \((X_0, Y_0) \overset{d}{=} (\epsilon_1, \psi_1)\).
Proof. The Laplace transform of (2.30) is

\[ \phi_{X_n,Y_n}(\lambda_1, \lambda_2) = \rho \phi_{\epsilon_n,\psi_n}(\rho^{1\lambda_1}, \rho^{1\lambda_2}) + (1 - \rho)\phi_{X_{n-1},Y_{n-1}}(\lambda_1, \lambda_2) \phi_{\epsilon_n,\psi_n}(\rho^{1\lambda_1}, \rho^{1\lambda_2}). \]

(2.31)

When the process is stationary,

\[ \phi_{X,Y}(\lambda_1, \lambda_2) = \rho \phi_{\epsilon,\psi}(\rho^{1\lambda_1}, \rho^{1\lambda_2}) + (1 - \rho)\phi_{X,Y}(\lambda_1, \lambda_2) \phi_{\epsilon,\psi}(\rho^{1\lambda_1}, \rho^{1\lambda_2}). \]

Suppose that the process has BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) marginals. Then, we get

\[ \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}} = \rho \phi_{\epsilon,\psi}(\rho^{1\lambda_1}, \rho^{1\lambda_2}) + \frac{1 - \rho}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}} \phi_{\epsilon,\psi}(\rho^{1\lambda_1}, \rho^{1\lambda_2}). \]

On simplification, we get

\[ \phi_{\epsilon,\psi}(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}. \]

To prove the converse we use induction method. Suppose that \((\epsilon_n, \psi_n), n \geq 1\) are distributed according to BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) and \((X_0, Y_0) \overset{d}{=} (\epsilon_1, \psi_1)\).

Put \(n=1\) in (2.31), we get

\[ \phi_{X_1,Y_1}(\lambda_1, \lambda_2) = \rho \phi_{\epsilon_1,\psi_1}(\rho^{1\lambda_1}, \rho^{1\lambda_2}) + (1 - \rho)\phi_{X_0,Y_0}(\lambda_1, \lambda_2) \phi_{\epsilon_1,\psi_1}(\rho^{1\lambda_1}, \rho^{1\lambda_2}). \]

Substituting the Laplace transform of \((\epsilon_1, \psi_1)\)

\[ \phi_{X_1,Y_1}(\lambda_1, \lambda_2) = \frac{\rho}{1 + \rho \mu_1 \lambda_1^{\alpha_1} + \rho \mu_2 \lambda_2^{\alpha_2}} + \frac{1 - \rho}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}} \frac{1}{1 + \rho \mu_1 \lambda_1^{\alpha_1} + \rho \mu_2 \lambda_2^{\alpha_2}} \]

\[ = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}. \]

Hence by mathematical induction, we get the process \(\{(X_n, Y_n), n \geq 1\}\) is stationary with BML \((\mu_1, \mu_2, \alpha_1, \alpha_2, 1)\) marginals.
Lawrance and Lewis (1981) developed a first order new exponential autoregressive process (NEAR(1)) that will generalize both the EAR(1) and TEAR(1) processes. The structure of the NEAR(1) process is

$$X_n = \epsilon_n + \begin{cases} \beta X_{n-1} & \text{with probability } \rho \\ 0 & \text{with probability } 1 - \rho \end{cases}$$

where $0 \leq \beta \leq 1$ and $0 \leq \rho \leq 1$. \{\epsilon_n, n \geq 1\} is a sequence of independently and identically distributed random variables and could be generated as follows:

$$\epsilon_n = \begin{cases} E_n & \text{with probability } \frac{1-\beta}{1-(1-\rho)\beta} \\ (1-\rho)\beta E_n & \text{with probability } \frac{\rho \beta}{1-(1-\rho)\beta} \end{cases}$$

Note that for $\beta=1$, we get the EAR(1) process given in (1.20) and for $\beta = 1$, we have the TEAR(1) process given in (2.29).

Now, consider a first order autoregressive process with the following structure.

$$(X_n, Y_n) = (\epsilon_n, \psi_n) + \begin{cases} (\beta_{\epsilon n} X_{n-1}, \beta_{\psi n} Y_{n-1}) & \text{with probability } \rho \\ 0 & \text{with probability } 1 - \rho \end{cases}$$

where \{(\epsilon_n, \psi_n), n \geq 1\} is a sequence of random vectors distributed according to BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$. \{(\epsilon_n, \psi_n, n \geq 1\} is defined as follows:

$$(\epsilon_n, \psi_n) = \begin{cases} (E_n, F_n) & \text{with probability } \frac{1-\beta}{1-(1-\rho)\beta} \\ (1-\rho)\beta E_n, (1-\rho)\beta F_n & \text{with probability } \frac{\rho \beta}{1-(1-\rho)\beta} \end{cases}$$

When $\rho=1$ we get bivariate Mittag-Leffler autoregressive process developed in (2.25) and while $\beta=1$, it is the bivariate Mittag-Leffler autoregressive process given in (2.30).
2.7 Bivariate Mittag-Leffler Distributions Generated through Bivariate Geometric Compounding

In this Section, we introduce the bivariate Mittag-Leffler forms of some important bivariate exponential distributions (see Jayakumar and Mundassery (2006)). Marshall-Olkin (1967) obtained a bivariate exponential distribution which can be treated as a shock model. Consider a two component system which are subjected to fatal shocks. These shocks follow independent Poisson process with parameters $\delta_1, \delta_2$ and $\delta_{12}$ according as the shocks applied to component 1 only, component 2 only or both components respectively. Then the joint survival function of the life times of the components denoted by $(X_1, X_2)$ is

$$F(x_1, x_2) = e^{-\delta_1 x_1 - \delta_2 x_2 - \delta_{12} \max(x_1, x_2)} \quad x_1, x_2 > 0, \quad \delta_1, \delta_2, \delta_{12} > 0.$$  

The Laplace transform of Marshall-Olkin’s bivariate exponential distribution is

$$\phi(\lambda_1, \lambda_2) = \frac{(\delta + \lambda_1 + \lambda_2)(\delta_1 + \delta_2)(\delta_1 + \delta_{12}) + \lambda_1 \lambda_2 \delta_{12}}{(\delta + \lambda_1 + \lambda_2)(\delta_1 + \delta_{12} + \lambda_1)(\delta_2 + \delta_{12} + \lambda_2)} \quad (2.32)$$

where $\delta = \delta_1 + \delta_2 + \delta_{12}$.

We obtain a generalization of Marshall-Olkin’s bivariate exponential distribution.

**Theorem 2.13.** Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ random vectors. Let $U_{N_1} = \sum_{i=1}^{N_1} X_i$ and $V_{N_2} = \sum_{i=1}^{N_2} Y_i$, where $(N_1, N_2)$ has bivariate geometric distribution in (1.15) and independent of $(X_i, Y_i), i \geq 1$. Choose $p_{00} = \delta_{12}, p_{10} = \delta_2, p_{01} = \delta_1$ and $p_{11} = 1 - \delta$ where
\[ \delta = \delta_1 + \delta_2 + \delta_{12}. \] Then the distribution of \((U_{N_1}, V_{N_2})\) is the bivariate Mittag-Leffler generalization of the Marshall-Olkin's bivariate exponential distribution.

**Proof.** Assume that \((X_i, Y_i), i \geq 1\) have the Laplace transform

\[ \psi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}}. \]

Substituting \(p_{00}, p_{10}, p_{01}, p_{11}\) and \(\psi(\lambda_1, \lambda_2)\) in (2.17), we get

\[ \phi(\lambda_1, 0) = \frac{\delta_{12} + \delta_1}{\delta_{12} + \delta_1 + \mu_1 \lambda_1^{\alpha_1}} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{\delta_{12} + \delta_2}{\delta_{12} + \delta_2 + \mu_2 \lambda_2^{\alpha_2}}. \]

The Laplace transform of \((U_{N_1}, V_{N_2})\) is obtained by substituting \(\phi(\lambda_1, 0)\) and \(\phi(0, \lambda_2)\) in (2.16),

\[ \phi(\lambda_1, \lambda_2) = \frac{1}{\delta + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2}} \left( \frac{\delta_2(\delta_{12} + \delta_1)}{\delta_{12} + \delta_1 + \mu_1 \lambda_1^{\alpha_1}} + \frac{\delta_1(\delta_{12} + \delta_2)}{\delta_{12} + \delta_2 + \mu_2 \lambda_2^{\alpha_2}} \right). \]

On simplification, we get

\[ \phi(\lambda_1, \lambda_2) = \frac{(\delta + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})(\delta_1 + \delta_{12})(\delta_2 + \delta_{12}) + \mu_1 \mu_2 \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \delta_{12}}{(\delta + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2})(\delta_1 + \delta_{12} + \mu_1 \lambda_1^{\alpha_1})(\delta_2 + \delta_{12} + \mu_2 \lambda_2^{\alpha_2})}. \]

When \(\alpha_1 = \alpha_2 = 1\), \(\phi(\lambda_1, \lambda_2)\) coincides with the Laplace transform of Marshall-Olkin's bivariate exponential distribution in (2.32).

Hawkes (1972) obtained a bivariate exponential distribution which describes the failure time of a system having two components. Suppose that the two components are subjected to non fatal shocks occurring in independent Poisson fashion with parameters \(\delta_1\) and \(\delta_2\) \((\delta_1, \delta_2 > 0)\). Let the number of shocks needed to cause failure of these components have the bivariate geometric distribution with p.g.f. \(P(s_1, s_2)\).
Then the waiting time for failure of the components, denoted by $(X, Y)$, is given by the random sum

$$(X, Y) = \left( \sum_{j=1}^{N_1} X_{1j}, \sum_{j=1}^{N_2} X_{2j} \right)$$

(2.34)

where $X_{i,j}, j=1,2,3,...$ denote the inter arrival time of the $i^{th}$ process, $i=1,2$. Then $(X, Y)$ has the Laplace transform

$$\phi(\lambda_1, \lambda_2) = P \left( \frac{\delta_1}{\delta_1 + \lambda_1}, \frac{\delta_2}{\delta_2 + \lambda_2} \right).$$

Hawkes (1972) considered the p.g.f.

$$P(s_1, s_2) = \frac{s_1 s_2}{1 - p_{00} s_1 s_2} \left( \frac{p_{11} + p_{10} p_2 s_2}{1 - Q_2 s_2} + \frac{p_{01} p_1 s_1}{1 - Q_1 s_1} \right)$$

(2.35)

where $P_1 = p_{11} + p_{10}, \ P_2 = p_{11} + p_{01}, \ Q_1 = p_{00} + p_{01}, \ Q_2 = p_{00} + p_{10}$ and obtained a bivariate exponential distribution as bivariate geometric sum of independently and identically distributed random vectors. The Laplace transform of Hawkes' bivariate exponential distribution is

$$\phi(\lambda_1, \lambda_2) = \frac{m_1 m_2}{(m_1 + \lambda_1)(m_2 + \lambda_2)} \left( 1 + \frac{[p_{00} - (1 - P_1)(1 - P_2)]\lambda_1 \lambda_2}{(m_1 + P_1 \lambda_1)(m_2 + P_2 \lambda_2) - p_{00} m_1 m_2} \right), \quad (2.36)$$

$$m_i = \delta_i P_i, \quad i=1,2.$$

In the following theorem we derive the bivariate Mittag-Leffler form of (2.36).

**Theorem 2.14.** Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors such that each component has Mittag-Leffler distribution with Laplace transforms $\psi_{X_i}(\lambda_1) = \frac{\delta_1}{\delta_1 + \lambda_1^{\alpha_i}}$ and $\psi_{Y_i}(\lambda_2) = \frac{\delta_2}{\delta_2 + \lambda_2^{\beta_i}}, \ \delta_1, \delta_2 > 0, \ i=1,2,...$ respectively. Then the distribution of $(X, Y)$ defined in (2.34) gives a generalization to (2.36) when $(N_1, N_2)$ has the bivariate geometric distribution with p.g.f. in (2.35).
Proof. Suppose that \((X_i, Y_i), i \geq 1\) are independently and identically distributed such that

\[
\psi_{X_i}(\lambda_1) = \frac{\delta_1}{\delta_1 + \lambda_1^{\alpha_1}} \quad \text{and} \quad \psi_{Y_i}(\lambda_2) = \frac{\delta_2}{\delta_2 + \lambda_2^{\alpha_2}}, \quad i = 1, 2, \ldots
\]

Let the joint Laplace transform of \((X_i, Y_i), i \geq 1\) be \(\psi(\lambda_1, \lambda_2)\). From (2.35) the Laplace transform of \((X, Y)\) is

\[
\phi(\lambda_1, \lambda_2) = \frac{\psi(\lambda_1, \lambda_2)}{1 - \psi(\lambda_1, \lambda_2)} \left( p_{11} + \frac{p_{10}p_2\psi_{Y_1}(\lambda_2)}{1 - Q_2\psi_{Y_1}(\lambda_2)} + \frac{p_{01}p_1\psi_{X_1}(\lambda_1)}{1 - Q_1\psi_{X_1}(\lambda_1)} \right).
\]

Substituting \(\psi_{X_1}(\lambda_1), \ \psi_{Y_1}(\lambda_2)\) and \(\psi(\lambda_1, \lambda_2)\)

\[
\phi(\lambda_1, \lambda_2) = \frac{\delta_1\delta_2}{(\delta_1 + \lambda_1^{\alpha_1})(\delta_2 + \lambda_2^{\alpha_2}) - \delta_1\delta_2p_{00}} \left( p_{11} + \frac{p_{10}p_2\delta_2}{P_2\delta_2 + \lambda_2^{\alpha_2}} + \frac{p_{01}p_1\delta_1}{P_1\delta_1 + \lambda_1^{\alpha_1}} \right)
\]

\[
= \frac{m_1m_2}{(m_1 + \lambda_1^{\alpha_1})(m_2 + \lambda_2^{\alpha_2}) - \delta_1\delta_2p_{00}} \left( p_{11} \frac{m_1m_2}{m_1 + \lambda_1^{\alpha_1}} + \frac{p_{01}m_1}{m_1 + \lambda_1^{\alpha_1}} \right)
\]

where \(m_i = \delta_i\lambda_i, \ i=1,2.\)

\[
\phi(\lambda_1, \lambda_2) = \frac{m_1m_2}{(m_1 + \lambda_1^{\alpha_1})(m_2 + \lambda_2^{\alpha_2})} \left( p_{11} \frac{m_1m_2}{m_1 + \lambda_1^{\alpha_1}} + \frac{p_{01}m_1}{m_1 + \lambda_1^{\alpha_1}} \right)
\]

On simplification,

\[
\phi(\lambda_1, \lambda_2) = \frac{m_1m_2}{(m_1 + \lambda_1^{\alpha_1})(m_2 + \lambda_2^{\alpha_2})} \left( 1 + \frac{[p_{00} - (1 - P_2)(1 - P_2)]\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}}{(m_1 + \lambda_1^{\alpha_1})(m_2 + \lambda_2^{\alpha_2}) - \delta_1\delta_2p_{00}} \right).
\]

(2.37)

Hence when \(\alpha_1 = \alpha_2 = 1\) we get (2.36)

Paulson (1973) obtained a bivariate exponential distribution using the bivariate geometric compounding given in (2.15). Choosing \(p_{00} = a, \ p_{10} = b, \ p_{01} = c, \ p_{11} = d\)
and
\[ \psi(\lambda_1, \lambda_2) = \frac{1}{(1 + \theta_1 \lambda_1)(1 + \theta_2 \lambda_2)}. \]
\[ \theta_1, \theta_2 > 0, \quad 0 < a, b, c, d < 1, \quad a + b + c + d = 1, b + d < 1, \quad c + d < 1. \]

From (2.15),
\[ \phi(\lambda_1, 0) = \frac{1}{1 + \delta_1 \lambda_1} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{1}{1 + \delta_2 \lambda_2} \]
where \( \delta_1 = \frac{\theta_1}{a + c} \) and \( \delta_2 = \frac{\theta_2}{a + b} \). This implies that marginal distributions have exponential distribution. Hence \( \phi(\lambda_1, \lambda_2) \) represents the Laplace transform of a bivariate exponential distribution. We obtain a bivariate Mittag-Leffler distribution that generalizes the Paulson's (1973) bivariate exponential distribution.

Choose \( p_{00}, p_{10}, p_{01}, \) and \( p_{11} \) as before and
\[ \psi(\lambda_1, \lambda_2) = \frac{1}{(1 + \mu_1 \lambda_1^{\alpha_1})(1 + \mu_2 \lambda_2^{\alpha_2})}. \]

From (2.15), we get
\[ \phi(\lambda_1, \lambda_2) = \frac{1}{(1 + \mu_1 \lambda_1^{\alpha_1})(1 + \mu_2 \lambda_2^{\alpha_2})} \left( a + \frac{b(a + c)}{a + c + \mu_1 \lambda_1^{\alpha_1}} + \frac{c(b + d)}{b + d + \mu_2 \lambda_2^{\alpha_2}} + d \phi(\lambda_1, \lambda_2) \right). \]

Hence
\[ \phi(\lambda_1, 0) = \frac{1}{1 + \delta_1 \lambda_1^{\alpha_1}} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{1}{1 + \delta_2 \lambda_2^{\alpha_2}} \]
where \( \delta_1 = \frac{\mu_1}{a + c} \) and \( \delta_2 = \frac{\mu_2}{a + b} \).
2.8 Bivariate Semi Mittag-Leffler Distribution

In the following definition we introduce a bivariate semi Mittag-Leffler distribution (see Mundassery and Jayakumar (2007c)).

**Definition 2.3.** A non-negative random vector \((X, Y)\) is said to follow bivariate semi Mittag-Leffler distribution, denoted by BSML \((\alpha_1, \alpha_2, p)\), if its Laplace transform is

\[
\phi(\lambda_1, \lambda_2) = \frac{1}{1 + \xi(\lambda_1, \lambda_2)}
\]

(2.38)

where \(\xi(\lambda_1, \lambda_2)\) satisfies the functional equation

\[
\xi(\lambda_1, \lambda_2) = \frac{1}{p} \xi(\frac{1}{p\alpha_1} \lambda_1, \frac{1}{p\alpha_2} \lambda_2), \ 0 < p < 1, \ 0 < \alpha_1, \alpha_2 \leq 1.
\]

A solution of this functional equation is given by

\[
\xi(\lambda_1, \lambda_2) = \lambda_1^{\alpha_1} h_1(\lambda_1) + \lambda_2^{\alpha_2} h_2(\lambda_2).
\]

(2.39)

When \(h_i(\lambda_i) = 1\) for \(i = 1, 2\), we get BML \((1.1, \alpha_1, \alpha_2, 1)\) distribution.

Using the geometric compounding stated in (2.9), we now obtain a characterization of BSML \((\alpha_1, \alpha_2, p)\).

**Theorem 2.15.** Consider a sequence \(\{(X_i, Y_i), i \geq 1\}\) of independently and identically distributed random vectors. Define \((U_N, V_N)\) as \(U_N = \sum_{i=1}^{N} X_i\) and \(V_N = \sum_{i=1}^{N} Y_i\) where \(N\) is independent of \((X_i, Y_i), i \geq 1\) and has the geometric distribution such that \(P(N = n) = (1 - p)^{n-1}p, \ 0 < p < 1, \ n = 1, 2, 3, ...\). Then \((p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)\) is distributed as BSML \((\alpha_1, \alpha_2, p)\) if and only if \((X_i, Y_i), i \geq 1\) follow BSML \((\alpha_1, \alpha_2, p)\) distribution.
Proof. Suppose that \((X_i, Y_i), i \geq 1\) follows BSML \((\alpha_1, \alpha_2, \rho)\). Substituting its Laplace transform \(\psi(\lambda_1, \lambda_2)\), in (2.10) and on simplification the Laplace transform of \((p_{\alpha_1}^{\frac{1}{\alpha_1}} U_N, p_{\alpha_2}^{\frac{1}{\alpha_2}} V_N)\) becomes

\[
\phi(\lambda_1, \lambda_2) = \frac{1}{1 + \xi(\lambda_1, \lambda_2)}.
\]

Conversely, suppose that \((p_{\alpha_1}^{\frac{1}{\alpha_1}} U_N, p_{\alpha_2}^{\frac{1}{\alpha_2}} V_N)\) follows BSML \((\alpha_1, \alpha_2, \rho)\) distribution.

From (2.10),

\[
\frac{1}{1 + \xi(\lambda_1, \lambda_2)} = \frac{p \psi(p_{\alpha_1}^{\frac{1}{\alpha_1}} \lambda_1, p_{\alpha_2}^{\frac{1}{\alpha_2}} \lambda_2)}{1 - (1 - p) \psi(p_{\alpha_1}^{\frac{1}{\alpha_1}} \lambda_1, p_{\alpha_2}^{\frac{1}{\alpha_2}} \lambda_2)}.
\]

Solving, we get

\[
\psi(\lambda_1, \lambda_2) = \frac{1}{1 + \xi(\lambda_1, \lambda_2)}.
\]

Now by repeated geometric compounding, we have the characterization of BSML \((\alpha_1, \alpha_2, \rho)\).

**Theorem 2.16.** Let \(\{(X_i, Y_i), i \geq 1\}\) be a sequence of independently and identically distributed random vectors with distribution function \(F(x, y)\) and Laplace transform \(\phi(\lambda_1, \lambda_2)\). Suppose that \(N_{k-1}\) follows geometric distribution such that

\[
P(N_{k-1} = n) = (1 - p_{k-1})^{n-1} p_{k-1}, \quad 0 < p_{k-1} < 1, \quad n = 1, 2, 3, ...
\]

and \(N_{k-1}\) is independent of \((X_i, Y_i), i \geq 1\). Define \((U_{N_k}, V_{N_k})\) as

\[
U_{N_k} = p_{k-1}^{\frac{1}{\alpha_1}} \sum_{i=1}^{N_{k-1}} X_i \quad \text{and} \quad V_{N_k} = p_{k-1}^{\frac{1}{\alpha_2}} \sum_{i=1}^{N_{k-1}} Y_i
\]
where \((X_i, Y_i), i \geq 1\) are the summands of the \((k - 1)^{th}\) stage of compounding and independently and identically distributed according to \(F_{k-1}(., .), k = 2, 3, \ldots\). For the initial stage, choose \(F_1(., .) = F(., .)\). Then \(F_k(., .)\), the distribution of \((U_{N_k}, V_{N_k})\), and \(F(., .)\) are BSML \((\alpha_1, \alpha_2, \rho)\).

Proof of the theorem easily follows by the recursive application of (2.11).

Now, we obtain the bivariate extension of the semi Mittag-Leffler process developed in Jayakumar and Pillai (1993).

**Theorem 2.17.** Let \(\{(X_n, Y_n), n \geq 1\}\) constitute a first order autoregressive process with structure

\[
(X_n, Y_n) = \begin{cases} 
(\rho \alpha_1 X_{n-1}, \rho \alpha_2 Y_{n-1}) & \text{with probability } \rho \\
(\rho \alpha_1 X_{n-1} + \epsilon_n, \rho \alpha_2 Y_{n-1} + \psi_n) & \text{with probability } 1 - \rho
\end{cases}
\]

(2.40)

where \(\{(\epsilon_n, \psi_n), n \geq 1\}\) is a sequence of independently and identically distributed random vectors. Then \(\{(X_n, Y_n), n \geq 1\}\) defines a stationary first order autoregressive process with BSML \((\alpha_1, \alpha_2, \rho)\) marginals if and only if \((\epsilon_n, \psi_n), n \geq 1\) are distributed as BSML \((\alpha_1, \alpha_2, \rho)\), provided \((X_0, Y_0) = \begin{pmatrix} \epsilon_1 \\ \psi_1 \end{pmatrix}\).

Proof of the theorem can be obtained by proceeding with arguments similar to that of Theorem 2.10.

Now, we develop a first order stationary autoregressive process with BSML \((\alpha_1, \alpha_2, \rho)\) marginals that generalizes the process stated in (2.40).
Theorem 2.18. Suppose that a bivariate first order autoregressive process 
\((X_n, Y_n), n \geq 1\) have the following structure:

\[
(X_0, Y_0) \overset{d}{=} (\epsilon_1, \psi_1) \quad \text{and for } n = 1, 2, 3, \ldots
\]
\[
(X_n, Y_n) = \begin{cases} 
(\epsilon_n, \psi_n), & \text{with probability } q \\
(p^{\alpha_1} X_{n-1}, p^{\alpha_2} Y_{n-1}), & \text{with probability } (1 - q)p \\
(p^{\alpha_1} X_{n-1} + \epsilon_n, p^{\alpha_2} Y_{n-1} + \psi_n), & \text{with probability } (1 - p)(1 - q)
\end{cases} \quad (2.41)
\]

Then the process given in (2.41), is stationary with marginals BSML \((\alpha_1, \alpha_2, p)\) distribution if and only if \(\{(\epsilon_n, \psi_n), n \geq 1\}\) is a sequence of independently and identically distributed random vectors according to BSML \((\alpha_1, \alpha_2, p)\).

Proof of the theorem is omitted as it is obvious.

In the following theorem we develop a first order autoregressive process with BSML \((\alpha_1, \alpha_2, p)\) marginals, along the lines of the TEAR(1) process that discussed in Lawrance and Lewis (1980).

Theorem 2.19. Consider a first order autoregressive process \(\{(X_n, Y_n), n \geq 1\}\) with structure

\[
(X_0, Y_0) \overset{d}{=} (\epsilon_1, \psi_1)
\]
\[
(X_n, Y_n) = \begin{cases} 
(\epsilon_n, \psi_n), & \text{with probability } p \\
(X_{n-1} + \epsilon_n, Y_{n-1} + \psi_n), & \text{with probability } 1 - p
\end{cases} \quad (2.42)
\]

where \(\{(\epsilon_n, \psi_n), n \geq 1\}\) is a sequence of independently and identically distributed random vectors. Then \(\{(X_n, Y_n), n \geq 1\}\) defines a stationary first order autoregressive...
process with BSML ($\alpha_1, \alpha_2, \rho$) marginals if and only if \{($\epsilon_n, \psi_n$), $n \geq 1$\} are distributed as BSML ($\alpha_1, \alpha_2, \rho$).

Proof of the theorem follows easily.

A first order autoregressive process that will generalize the processes mentioned in Theorem 2.17 and Theorem 2.19, could be developed along the lines of the NEAR(1) process discussed in Lawrance and Lewis (1981).