Genealogical research can be significantly enhanced through real-time collaboration. Such collaboration reduces redundant research and provides timely information that can better guide one to discover new knowledge. In real-time collaboration, information recorded by one participant is nearly instantaneously and automatically broadcast to all other people interested in that information. Real-time collaboration requires a communication infrastructure that is fast, reliable and scalable.

Peer-to-peer systems appear to be faster, more reliable, scalable and cheaper: they are cheaper because there is no central server to purchase and maintain. Peers need only pay for local hardware and connections to the internet which they would also pay for in a client/server-based
system. To determine which is best we compare the broadcast speed, reliability, and scalability of several different peer-to-peer topologies with the client/server topology. The different peer-to-peer topologies include: random trees, random graphs, gnutella-like topologies, and hypercube networks. The reliability results were a little more mixed. The effectiveness of parallel computers is often determined by its communication network. The interconnection network is an important component of a parallel processing system.

A good interconnection network should have less topological network cost and meanwhile keep the network diameter as short as possible [22]. In general, the reliability for random graphs and hypercubes were excellent with random graphs being a little better than hypercubes in small networks and hypercubes being better than random graphs for large networks. The hypercube network is found to be more suitable for real-time genealogical collaboration networks. It is the fastest, one of the most reliable, and the most scalable topology.

### 3.1 Hypercube Networks

The hypercube is a very popular, versatile and vertex-transitive interconnection network. When the dimension of hypercube increases, the cardinality of its vertex set increases exponentially.
Definition 3.1.1. Let $Q^r$ denote the graph of an $r$-dimensional hypercube, $r \geq 1$. The vertex set $V(Q^r) = \{(x_0x_1\ldots x_{r-1}) : x_i = 0 \text{ or } 1, 0 \leq i \leq r - 1\}$. Two vertices $(x_0x_1\ldots x_{r-1})$ and $(y_0y_1\ldots y_{r-1})$ are adjacent if and only if they differ exactly in one position.

See Figure 3.1. The hypercube $Q^r$ has $2^r$ vertices and $r2^{r-1}$ edges. It is $r$-regular and its diameter is $r$. Further it is bipartite, Hamiltonian if $r \neq 1$ and Eulerian if $r$ is even [79]. It is has been proved in [6] that $\beta(Q^r) \leq r$. The bound is tight for $r \leq 4$, and it is not tight for $r = 5$. A laborious calculation verifies that $Q^5$ is resolved by the 4-vertex set $\{00000, 00011, 00101, 01001\}$. Caceres et al. [6] have determined $\beta(Q^r)$ for small values of $r$ by computer search; the values are shown

\begin{figure}
\centering
\begin{tabular}{ccc}
\hspace{1cm} a : \\
\includegraphics[width=0.5\textwidth]{fig1.png}
\end{tabular}
\caption{(a) Binary representation (b) Decimal representation}
\end{figure}
3.1.1 Star Resolving Number

In this section we introduce a new parameter called the star resolving number.

**Definition 3.1.2.** An $r$-dimensional star, denoted by $S_r$, is a graph with one vertex of degree $r - 1$ and $r - 1$ vertices of degree 1. The vertex of degree $r - 1$ is called the hub of $S_r$.

See Figure 3.2.

**Definition 3.1.3.** A set $W \subset V$ is said to be a star resolving set if $W$ resolves $G$ and if it induces a star. The minimum cardinality of $W$ is called the star resolving number and is denoted by $sr(G)$.

**Remark 3.1.1.** It is clear that $1 \leq sr(G) \leq \Delta(G) + 1$ for any graph $G$. In a star resolving set the maximum distance between any two vertices is 2.
We now proceed to identify a star resolving set in a hypercube network $Q^r$. It is clear that there are four copies of $Q^{r-2}$ in $Q^r$. We denote them as $Q^{r-2}_0$, $Q^{r-2}_{1,1}$, $Q^{r-2}_{1,2}$ and $Q^{r-2}_2$. Figure 3.3 exhibits the four copies of $Q^3$ in $Q^5$.

Let $x \in V(Q^{r-2}_0)$. A vertex $x' \in V(Q^{r-2}_{1,1})$ or $V(Q^{r-2}_{1,2})$ is called the image of $x$ if $d(x, x') = 1$. Note that vertices in $Q^{r-2}_0$, at distance 1 from $x$ are not considered as images of $x$. If $x'$ is the image of $x$ then $x$ is called the pre-image of $x'$.

The next result which we state as Lemma is crucial to our work. We omit the proof as this result has been proved in [55] for enhanced hypercubes.
Lemma 3.1.1. Let $x \in V(Q^r_0)$. Let $x' \in V(Q^r_{1,1})$ be the image of $x$. Let $w$ be any vertex in $Q^r_0$. Then $d(x', w) = 1 + d(x, w)$.

Lemma 3.1.2. Let $x \in V(Q^r_0)$. Let $x'_1 \in V(Q^r_{1,1})$ and $x'_2 \in V(Q^r_{1,2})$ be the images of $x$. Then $x'_1$ and $x'_2$ are equidistant from every vertex of $Q^r_0$.

Proof. Since the shortest paths from $x'_1$ and $x'_2$ to any vertex of $Q^r_0$ pass through $x$, the conclusion follows. \hfill \qedsymbol

Lemma 3.1.3. Let $G$ be $Q^r$, $r \geq 1$. Then $sr(G) \geq r$.

Proof. The subcube $Q^r_0$ of $Q^r$ is $r - 2$ regular and hence up to isomorphism it contains only one star $S_{r-1}$. Now there exist vertices $u \in Q^r_{1,1}$ and $v \in Q^r_{1,2}$ such that $u, v$ are equidistant from every vertex of $Q^r_0$ and in particular from every vertex of $S_{r-1}$. This implies that $sr(G) > r - 1$. \hfill \qedsymbol

Lemma 3.1.4. Let $G$ be $Q^r$. Then $sr(G) \leq r$.

Proof. We prove the theorem by induction on $r$.

Base Case: Let $G$ be $Q^3$ and $W_1 = \{w_0, w_1, w_2\}$, where $w_0 = 0, w_1 = 1$ and $w_2 = 2$. See Figure 3.4(a). It follows from the definition of hypercube edges that $w_0$ is adjacent to both $w_1$ and $w_2$. Further it is easy to check that $W_1$ is a resolving set for $Q^3$. Figure 3.4(b) shows
the distinct codes of vertices in $Q^3$, with respect to $W_1 = \{w_0, w_1, w_2\}$.

Since $W_1$ induces $S_3$, it is a star resolving set for $Q^3$.

Now assume that the result is true for the hypercube $Q^{r-1}, r \geq 4$. Let $W_1 = \{w_0\} \cup \{w_i : 1 \leq i \leq r-2\}$, where $w_i = 2^{i-1}$, be a star resolving set for $Q^{r-1}$. Here $w_0$ is the hub and it is adjacent to all $w_i, 1 \leq i \leq r-2$. Moreover $W_1 \subset V(Q_0^{r-2})$. Divide $Q^r$ into four copies $Q_0^{r-2}, Q_{1,1}^{r-2}, Q_{1,2}^{r-2}$ and $Q_2^{r-2}$. Let $x \in V(Q_0^{r-2})$ and let $x'_1 \in V(Q_{1,1}^{r-2})$, $x'_2 \in V(Q_{1,2}^{r-2})$ be the images of $x$. Then $x'_1$ and $x'_2$ are equidistant from every vertex of $V(Q_0^{r-2})$, by Lemma 3.1.2. Consequently they are equidistant from every vertex of $W_1$. So $W_1$ cannot resolve $Q^r$.

We now exhibit a resolving set for $Q^r$. Define $W = W_1 \cup \{w_{r-1}\}$ where $w_{r-1} = 2^{r-2}$ is a vertex either in $Q_{1,1}^{r-2}$ or $Q_{1,2}^{r-2}$. Clearly $w_{r-1}$ is adjacent to $w_0$. We claim that $W$ is a resolving set for $Q^r$.

**Case 1:** $x, y \in V(Q_0^{r-2})$ or $V(Q_{1,1}^{r-2})$ or $V(Q_{1,2}^{r-2})$

Since $W_1 \subset V(Q_0^{r-2})$ and since $Q_0^{r-2} \cup Q_{1,1}^{r-2}$ and $Q_0^{r-2} \cup Q_{1,2}^{r-2}$ are isomorphic to $Q^{r-1}$, by induction hypothesis $W_1$ resolves $x$ and $y$. The same argument applies to the following cases.

a) $x \in V(Q_0^{r-2})$ and $y \in V(Q_{1,1}^{r-2})$

b) $x \in V(Q_0^{r-2})$ and $y \in V(Q_{1,2}^{r-2})$
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Figure 3.4: (a) Resolving set $W_1 = \{w_0, w_1, w_2\}$ in $Q_3$ (b) Codes of vertices of $Q_3$ with respect to $W_1$

**Case 2:** $x \in V(Q_{r-2}^{1,1})$ and $y \in V(Q_{r-2}^{1,2})$

We need to prove that $d(x, w) \neq d(y, w)$ for some $w$ in $W = \{w_0, w_1, w_2...w_{r-2}\} \cup \{w_{r-1}\}$. Let $x', y' \in V(Q_0^{r-2})$ be the images of $x$ and $y$ respectively.

**Case 2.1:** $x' = y'$

In this case $x$ and $y$ are equidistant from every vertex of $Q_0^{r-2}$ and in particular every vertex of $W_1$. See Figure 3.5(a). Now $w_{r-1}$ belongs to either $Q_{1,1}^{r-2}$ or $Q_{1,2}^{r-2}$. Without loss of generality let $w_{r-1} \in Q_{1,1}^{r-2}$.

Then

$$d(y, w_{r-1}) = d(y, y') + d(y', w_{r-1})$$
$$= 1 + d(x', w_{r-1})$$
$$= 1 + 1 + d(x, w_{r-1})$$
$$= 2 + d(x, w_{r-1})$$
$$\neq d(x, w_{r-1})$$
Case 2.2: $x' \neq y'$

Now $x'$ and $y'$ are resolved by some $w$ in $W_1$. See Figure 3.5(b).

Hence $d(x', w) \neq d(y', w)$ and consequently $d(x, w) \neq d(y, w)$.

Case 3: $x \in V(Q_{1,1}^{r-2})$ and $y \in V(Q_{2}^{r-2})$

Let $x', y'$ be the images of $x$ and $y$ in $Q_{1,1}^{r-2}$

Case 3.1: $x' = y'$

In this case $x$ and $y$ are equidistant from every vertex of $Q_{1,1}^{r-2}$, in particular from $w_{r-1} \in Q_{1,1}^{r-2}$. Therefore $w_{r-1}$ will not resolve $x$ and $y$. See Figure 3.6. Now choose some $w \in W_1$. Let $d(x, w) = s$. Then
Figure 3.6: Illustration for case 3.1

\[
\begin{align*}
d(y, w) &= d(y', w) + 1 \\
&= d(x', w) + 1 \\
&= 1 + d(x, w) + 1 \\
&= 2 + d(x, w) \\
&\neq d(x, w)
\end{align*}
\]

Case 3.2: \( x' \neq y' \)

Case 3.2.1: \( d(x', w_{r-1}) \neq d(y', w_{r-1}) \)

In this case \( d(x, w_{r-1}) \neq d(y, w_{r-1}) \). That is \( w_{r-1} \) resolves \( x \) and \( y \).

Case 3.2.2: \( d(x', w_{r-1}) = d(y', w_{r-1}) \)

In this case \( w_{r-1} \) does not resolve \( x \) and \( y \). Choose some \( w \in W_1 \).

Let \( w' \in Q_{1,1}^{r-2} \) be the image of \( w \) and \( w'' \in Q_2^{r-2} \) be the image of \( w' \).

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Suppose that $y'' \in Q_r^{-2}$ be the image of $y' \in Q_{1,1}^{-2}$ and $x'' \in Q_2^{-2}$ be the image of $x' \in Q_{1,1}^{-2}$. See Figure 3.7. Let $P_1$ and $P_2$ be shortest paths from $w$ to $x$ and $w$ to $y''$ respectively. Let $P'_1$ and $P'_2$ be the images of $P_1$ and $P_2$ lying in $Q_{1,1}^{-2}$. Let $P''_1$ and $P''_2$ be the images of $P'_1$ and $P'_2$ lying in $Q_2^{-2}$. Without loss of generality let $|P_1| \leq |P_2|$. 

\[
d(x, w) = |P_1| - 1 \leq |P_2| - 1 = d(y'', w) = d(y', w) - 1 = d(y, w) - 2 \neq d(y, w) \]
Case 4: \( x, y \in V(Q_2^{r-2}) \)

Let \( x' \) and \( y' \) be the images of \( x \) and \( y \) respectively. There are three possibilities \( x', y' \in V(Q_{1,1}^{r-2}) \) or \( V(Q_{1,2}^{r-2}) \) or \( x' \in V(Q_{1,1}^{r-2}) \) and \( y' \in V(Q_{1,2}^{r-2}) \). The conclusions then follow by Case 1 or Case 2.

Case 5: \( x \in V(Q_{1,1}^{r-2}) \) and \( y \in V(Q_2^{r-2}) \)

Let \( x' \in V(Q_0^{r-2}) \) and \( y' \in V(Q_{1,2}^{r-2}) \) be the images of \( x \) and \( y \) respectively. Since \( Q_0^{r-2} \cup Q_{1,2}^{r-2} \) is resolved by \( W_1 \), there exist a \( w \in W_1 \) such that \( d(x', w) \neq d(y', w) \). This implies that \( d(x, w) \neq d(y, w) \). \( \square \)

Lemmas 3.1.3 and 3.1.4 imply the following result.

Theorem 3.1.1. Let \( G \) be \( Q^r \), \( r \geq 1 \). Then \( sr(G) = r. \)

3.1.2 Path Resolving Number

In this section we determine the path resolving number for hypercube networks.

Definition 3.1.4. [58] A resolving set \( W \) of \( G \) is a path resolving set for \( G \) if the graph induced by \( W \) is a path. The minimum cardinality of \( W \) is called path resolving number and is denoted by \( pr(G) \).

Lemma 3.1.5. Let \( G \) be \( Q^r \), \( r \geq 1 \). Then \( pr(G) \geq r. \)
Proof. Let $P$ be the path in $Q_{r}^{r-2}$. Now $P$ cannot resolve $Q^r$ as there are vertices $x \in Q_{1,1}^{r-2}$ and $y \in Q_{1,2}^{r-2}$ such that they are equidistant from every vertex of $Q_{r}^{r-2}$ in particular from every vertex of $P$. Since there exist a path $P_r$ in $Q_{0}^{r-2}$, $pr(G) \geq r$. \qed

Lemma 3.1.6. Let $G$ be $Q^r$, $r \geq 1$. Then $pr(G) \leq r$.

Proof. Proceeding as in Lemma 3.1.4 we conclude that $W = \{2^i - 1, 1 \leq i \leq r\}$ is a path resolving set for $Q^r$. \qed

Lemma 3.1.5 and Lemma 3.1.6 imply the following result.

Theorem 3.1.2. Let $G$ be $Q^r$, $r \geq 1$. Then $pr(G) = r$.

3.2 Enhanced Hypercube Networks

The hypercube has received considerable attention mainly due to its regular structure, small diameter, and good connection with a relatively small node degree [79]. Many variations of the hypercube have been suggested to improve the performance [15]. One of the variations is the enhancement [77] of the hypercube with same number of vertices. The enhanced hypercubes are much more attractive than the normal hypercubes due to their potentially nice topological properties.

The enhanced hypercube $Q^{r,k}$, $0 \leq k \leq r - 1$, is a graph with vertex set $V(Q^{r,k}) = V(Q^r)$ and edge set $E(Q^{r,k}) = E(Q^r) \cup \{(x_0 x_1 \ldots x_{k-2} x_{k-1} x_k$, 

... went on
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Figure 3.8: An enhanced hypercube $Q^{4,2}$ with binary representation $\ldots x_{r-1}, x_0 x_1 \ldots x_{k-2} \overline{x}_{k-1} \overline{x}_k \ldots \overline{x}_{r-1} : 0 \leq k \leq r - 1$. The edges of $Q^r$ in $Q^{r,k}$ are called the hypercube edges and the remaining edges are called complementary edges or skips [77]. See Figure 3.8. The enhanced hypercube, $Q^{r,k}, 0 \leq k \leq r - 1$ is $(r + 1)$-regular with $2^r$ vertices and $(r + 1)2^{r-1}$ edges. It is bipartite if and only if $r$ and $k$ have the same parity [30, 77].

3.2.1 One Size Resolving Number

In this section we determine a bound for the one size resolving number of enhanced hypercube networks.

Definition 3.2.1. A set $W$ of $G$ is a one size resolving set if the size of subgraph induced by $W$ is one and if distinct vertices of $G$ have distinct codes with respect to $W$. The minimum cardinality of a one size resolving set in $G$ is the one size resolving number and is denoted by $or(G)$.

A one size resolving set of cardinality $or(G)$ is called an or-set of
G. If \( G \) is a connected graph of order \( n \) containing an or-set, then it is clear that \( 2 \leq \text{or}(G) \leq n - 1 \).

We now proceed to identify a one size resolving set in an enhanced hypercube network \( Q^{r,2} \). It is clear that there are four copies of \( Q^{r-2,2} \) in \( Q^{r,2} \). We denote them as \( Q_0^{r-2,2}, Q_1^{r-2,2}, Q_1^{r-2,2}, \) and \( Q_2^{r-2,2} \). Figure 3.9 exhibits the four copies of \( Q^{3,2} \) in \( Q^{5,2} \). Let \( x \in V(Q_0^{r-2,2}) \). A vertex \( x' \in V(Q_{1,1}^{r-2,2}) \) or \( V(Q_{1,2}^{r-2,2}) \) is called an image of \( x \) if \( d(x, x') = 1 \). Note that vertices in \( Q_0^{r-2,2} \), at distance 1 from \( x \) are not considered as images of \( x \). If \( x' \) is the image of \( x \) in \( Q_{1,1}^{r-2,2} \) then \( x \) is called the pre-image of \( x' \). Let \( P = u_0u_1...u_n \) be a path in \( Q_0^{r-2,2} \). Then the path \( P' = u_0'u_1'...u_n' \) where \( u_i' \) is the image of \( u_i \) in \( Q_{1,1}^{r-2,2}(Q_{1,2}^{r-2,2}) \) is called
the image of \( P \) in \( Q_{1,1}^{r-2,2}(Q_{1,2}^{r-2,2}) \) and \( P \) is called the pre-image of \( P' \).

See Figure 3.10. We use the following result of Rajan et al. [55].

**Lemma 3.2.1.** [55] Let \( x \in V(Q_0^{r-2,2}) \) and let \( x' \in V(Q_{1,1}^{r-2,2}) \) be the image of \( x \). Let \( w \) be any vertex in \( Q_0^{r-2,2} \). Then \( d(x', w) = 1 + d(x, w) \).

**Lemma 3.2.2.** [55] Let \( x \in V(Q_0^{r-2,2}) \). Let \( x'_1 \in V(Q_{1,1}^{r-2,2}) \) and \( x'_2 \in V(Q_{1,2}^{r-2,2}) \) be the images of \( x \). Then \( x'_1 \) and \( x'_2 \) are equidistant from every vertex of \( Q_0^{r-2,2} \).

*Proof.* Since the shortest paths from \( x'_1 \) and \( x'_2 \) to any vertex of \( Q_0^{r-2,2} \) pass through \( x \), the conclusion follows. \( \square \)

**Theorem 3.2.1.** Let \( G \) be \( Q^{r,2} \). Then \( or(G) \leq r - 1, \ r > 3. \)

*Proof.* We prove the theorem by induction on \( r \).

**Base Case:** \( r = 4 \)

Let \( G \) be \( Q^{4,2} \) and let \( W_1 = \{w_0, w_1, w_2\} \), where \( w_0 = 0001, w_1 = 0011 \) and \( w_2 = 0110 \). It follows from the definition of hypercube edges that \( w_0 \) is adjacent to \( w_1 \) and that \( w_2 \) is non-adjacent to \( w_0 \) and \( w_1 \). It is easy to check that \( W_1 \) is a resolving set for \( Q^{4,2} \). Figure 3.11 shows the distinct codes of vertices in \( Q^{4,2} \), with respect to \( W_1 = \{w_0, w_1, w_2\} \). Thus \( W_1 \) is a one size resolving set for \( G \).
Now assume that the result is true for the enhanced hypercube $Q^{r-1,2}$, $r > 5$. Let $W_1 = \{w_0, w_1 \ldots w_{r-3}\}$ where $w_0 = \overbrace{0000\ldots01}^{(r-1)-bit}$ and 

$$w_i = x_0x_1\ldots x_{r-i-2}\overbrace{x_{r-i-1} \ldots x_{r-2}}^{(r-3)-bit}, 1 \leq i \leq r-3, x_s = 0, 0 \leq s \leq r-2$$

be a one size resolving set for $Q^{r-1,2}$. Here $w_0w_1 \in E$. Since $w_j, 0 \leq j \leq k$, and $w_{k+1}, 1 \leq k \leq r-4$ differ in two bits they are not adjacent in $Q^{r-1,2}$. This justifies the fact that the size of $W_1$ is one. Moreover $W_1 \subset V(Q^{r-2,2}_0)$. Divide $Q^{r,2}$ into four copies of $Q^{r-2,2}_0, Q^{r-2,2}_{1,1}, Q^{r-2,2}_{1,2}$ and $Q^{r-2,2}_2$. There exist vertices $x \in Q^{r-2,2}_{1,1}$ and $y \in Q^{r-2,2}_{1,2}$ having the same code with respect to every vertex of $Q^{r-2,2}_0$ and in particular with respect to every vertex of $W_1$. Hence $W_1$ cannot resolve $x$ and $y$. We exhibit a resolving set for $Q^{r,2}$. Now define $W = \{u_i : 0 \leq i \leq r-2\}$ where $u_0 = 0w_0 = x_0x_1\ldots x_{r-2}\overbrace{x_{r-1}}^{(r-1)-bit}$ and $u_i = 0w_i, 1 \leq i \leq r-3$ and $u_{r-2} = x_0\overbrace{x_1\ldots x_3}^{(r-3)-bit} \overbrace{x_{r-1} \ldots x_{r-2}}^{(r-3)-bit}$ is obtained by appending a 0 to each element of $W_1$ and including the additional vertex $x_0\overbrace{x_1\ldots x_3}^{(r-2)-bit}00$. Clearly the size of $W$ is one. We claim that $W$ is a resolving set of $Q^{r,2}$.
Case 1: \( x, y \in V(Q_{r-2,2}^r) \) or \( V(Q_{1,1}^{r-2,2}) \) or \( V(Q_{1,2}^{r-2,2}) \)

Since \( W_1 \subseteq V(Q_{r-2,2}^r) \) and since \( Q_{r-2,2}^r \cup Q_{1,1}^{r-2,2} \) and \( Q_{r-2,2}^r \cup Q_{1,2}^{r-2,2} \)
are isomorphic to \( Q_{r-1,2}^r \), by induction hypothesis \( W_1 \) resolves \( x \) and \( y \).

The same argument applies to the following cases.

a) \( x \in V(Q_{r-2,2}^0) \) and \( y \in V(Q_{1,1}^{r-2,2}) \)

b) \( x \in V(Q_{r-2,2}^0) \) and \( y \in V(Q_{1,2}^{r-2,2}) \)

Case 2: \( x \in V(Q_{1,1}^{r-2,2}) \) and \( y \in V(Q_{1,2}^{r-2,2}) \)

We need to prove that \( d(x, w) \neq d(y, w) \) for some \( w \) in \( W = \{w_0, w_1, w_2...w_{r-3}\} \cup \{w_{r-2}\} \). Let \( x', y' \in V(Q_{r-2,2}^r) \) be the images of \( x \) and \( y \) respectively.

Case 2.1: \( x' = y' \)

In this case \( d(y, w_{r-2}) = d(y, y') + d(y', w_{r-2}) = 1 + d(x', w_{r-2}) = 1 + 1 + d(x, w_{r-2}) \neq d(x, w_{r-2}) \).
Case 2.2: \( x' \neq y' \)

Now \( x' \) and \( y' \) are resolved by some \( w \) in \( W_1 \). Hence \( d(x', w) \neq d(y', w) \) and consequently \( d(x, w) \neq d(y, w) \).

Case 3: \( x \in V(Q_r^{r-2,2}) \) and \( y \in V(Q_r^{r-2,2}) \)

Let \( x' \) and \( y' \) be the images of \( x \) and \( y \) in \( Q_{1,1}^{r-2,2} \).

Case 3.1: \( x' = y' \)

In this case \( x \) and \( y \) are equidistant from every vertex of \( Q_{1,1}^{r-2,2} \), in particular from \( w_{r-1} \in Q_{1,1}^{r-2,2} \). Therefore \( w_{r-1} \) will not resolve \( x \) and \( y \). Now choose some \( w \in W_1 \). Let \( d(x, w) = s \). Then
\[
\begin{align*}
d(y, w) &= d(y', w) + 1 \\
&= d(x', w) + 1 \\
&= 1 + d(x, w) + 1 \\
&= 2 + d(x, w) \\
&\neq d(x, w)
\end{align*}
\]

Case 3.2: \( x' \neq y' \)

As proceeded in Lemma 3.1.4 we conclude that \( d(x, w) \neq d(y, w) \).

Case 4: \( x, y \in V(Q_2^{r-2,2}) \)
As before let \( x' \) and \( y' \) be the images of \( x \) and \( y \) respectively. There are two possibilities \( x', y' \in V(Q_{r-2}^{r-2,2}) \) or \( V(Q_{r-2}^{r-2,2}) \). Then \( d(x', w) \neq d(y', w) \) for some \( w \in W_1 \) by Case 1.

**Case 5:** \( x \in V(Q_{r,2}^{r-2}) \) and \( y \in V(Q_{r,2}^{r-2}) \)

Let \( x' \in V(Q_{0,2}^{r-2}) \) and \( y' \in V(Q_{1,2}^{r-2}) \) be the images of \( x \) and \( y \) respectively. Since \( Q_{0,2}^{r-2} \cup Q_{1,2}^{r-2} \) is resolved by \( W_1 \), there exist a \( w \in W_1 \) such that \( d(x', w) \neq d(y', w) \). This implies that \( d(x, w) \neq d(y, w) \).

**3.2.2 One Factor Resolving Number**

In this section we determine a bound for the one factor resolving number of enhanced hypercube networks.

**Definition 3.2.2.** [58] A resolving set \( W \) of \( G \) is a one factor resolving set for \( G \) if \( G[W] \cong tK_2 \), for some positive integer \( t \). The minimum \( t \) for which \( G[W] \cong tK_2 \) is called the one factor resolving number of \( G \) and it is denoted by \( onef(G) \).

**Theorem 3.2.2.** Let \( G \) be \( Q^{r,2} \). Then \( onef(G) \leq \frac{r-1}{2} \), \( r \) odd and \( r > 3 \).

**Proof.** We prove the theorem by induction on \( r \).

Now assume that the result is true for the hypercube \( Q^{r-2,2} \). Let \( W_1 = \)
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\{|\{x_0x_1\ldots \overline{x}_{r-2i-3}\ldots x_{r-3}\}, (x_0x_1\ldots \overline{x}_{r-2i-4}\overline{x}_{r-2i-3}\ldots x_{r-3}\}|, 0 \leq i \leq \frac{r-5}{2}\} be a one factor resolving set where w_j and w_{j+1} are adjacent, 0 \leq j \leq r-5, j even. Clearly \(W_1 \subset V(Q_0^{r-2,2})\). Divide \(Q^{r,2}\) into eight copies of \(Q^{r-3,2}\), namely \(Q_0^{r-3,2}, Q_{1,1}^{r-3,2}, Q_{1,2}^{r-3,2}, Q_{1,3}^{r-3,2}, Q_{2,1}^{r-3,2}, Q_{2,2}^{r-3,2}, Q_{2,3}^{r-3,2}\) and \(Q_3^{r-3,2}\). See Figure 3.12. There are many more subcubes of \(Q^{r,2}\) isomorphic to \(Q^{r-2,2}\). But we consider the subcubes \(Q_0^{r-3,2} \cup Q_{1,1}^{r-3,2}, Q_0^{r-3,2} \cup Q_{1,2}^{r-3,2}\) and \(Q_0^{r-3,2} \cup Q_{1,3}^{r-3,2}\). Each is isomorphic to \(Q^{r-2,2}\). Since \(W_1 \subset V(Q_0^{r-3,2})\), \(W_1\) resolves the above subcubes by assumption. Now there exist vertices \(x \in Q_{1,1}^{r-3,2}, y \in Q_{1,2}^{r-3,2}\) and \(z \in Q_{1,3}^{r-3,2}\) having the same code with respect to every vertex of \(Q_0^{r-3,2}\) and in particular with respect to every vertex of \(W_1\). Similarly there exist vertices one each in \(Q_{2,1}^{r-3,2}, Q_{2,2}^{r-3,2}\) and \(Q_{2,3}^{r-3,2}\) having the same code with respect to \(W_1\). So we need to augment \(W_1\). A cube \(Q^{3,2}\) in \(Q^{r,2}\), \(r > 3\) is said to be an s-neighbor cube if \(d(Q^{3,2}_0, Q^{3,2}) = s\). If a cube is resolved by some \(W_1 \subset V(Q_0^{r-3,2})\) then the s-neighbor cube is also resolved by the same resolving set \(W_1\) where \(1 \leq s \leq 3\). Therefore it is enough to resolve \(Q_{1,1}^{r-3,2}, Q_{1,2}^{r-3,2}, Q_{1,3}^{r-3,2}\). Let \(w_{r-3} \in V(Q_{1,1}^{r-3,2})\). Now \(W_1 \cup \{w_{r-3}\} \subset V(Q_0^{r-3,2} \cup Q_{1,1}^{r-3,2})\). This means that there are vertices one in each of \(Q_{1,2}^{r-3,2} \cup Q_{2,1}^{r-3,2}\) and \(Q_{1,3}^{r-3,2} \cup Q_{2,2}^{r-3,2}\) having same code as they are at distance 1 from \(Q_0^{r-3,2} \cup Q_{1,1}^{r-3,2}\). Now choose \(w_{r-2} \in W_1\) and augment
Figure 3.12: Eight copies of $Q^{3,2}$ in $Q^{6,2}$. 
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$V(Q_{1,2}^{r-3.2} \cup Q_{2,1}^{r-3.2})$ preferably $w_{r-2} \in V(Q_{2,1}^{r-3.2})$ so that $w_{r-2}$ and $w_{r-3}$ are adjacent. Therefore the augmented $W = \{w_0, w_1...w_{r-2}\}$, more precisely $W = \{(x_0x_1...x_{r-2i-1}x_{r-1}), (x_0x_1...x_{r-2i-2}x_{r-2i-1}...x_{r-1})\}$, $0 \leq i \leq \frac{r-3}{2}$) is a one factor resolving set.

3.3 Conclusion

In this Chapter we have introduced a new resolving parameter called the *star resolving number*. We have determined the star resolving number and path resolving number for hypercube networks and one-size resolving number and one-factor resolving number are investigated for enhanced hypercube networks. The results pertaining to hypercube have appeared in Applied Mathematics, Vol.3, pages 473-477, 2012. and results about enhanced hypercube have appeared in International Journal of Computer Applications (0975 - 8887) Volume 43- No.23, pages 1-5, 2012.