CHAPTER – III

PERMUTATION OF FUZZY NUMBERS:

3.1. Introduction :

In this chapter, different types of permutation of fuzzy numbers are established. Some properties of cyclic fuzzy number permutation and even and odd permutation of fuzzy numbers are discussed.

3.1.1. Definition :

A one-one mapping of a fuzzy number set $\bar{S}^*$ on to its elements is called a fuzzy number transformation.

3.1.2. Definition :

A one-one mapping of a given finite non-empty fuzzy number set $\bar{S}^*$ of n distinct elements on to its elements is called a fuzzy number permutation of degree n.

We denote a fuzzy number permutation $f$ on $\bar{S}^*$ in a two rowed notation.

$$f = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ b_1 & b_2 & b_3 & \ldots & b_n \end{pmatrix}$$
In the first row all the elements of $\tilde{S}^*$ are written in a certain order and $f(\tilde{a}_i) = \tilde{b}_i$ is put under $\tilde{a}_i$ for each $i$. Clearly each $\tilde{b}_i$ is also a member of $\tilde{S}^*$

3.1.3. Example:

Let $\tilde{S}^* = [\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}]$ and let $f$ be a permutation on $\tilde{S}^*$, defined by $f(\tilde{a}) = \tilde{b}$, $f(\tilde{b}) = \tilde{d}$, $f(\tilde{c}) = \tilde{a}$ and $f(\tilde{d}) = \tilde{c}$.

Then we write $f = \begin{pmatrix} \tilde{a} & \tilde{b} & \tilde{c} & \tilde{d} \\ \tilde{b} & \tilde{d} & \tilde{a} & \tilde{c} \end{pmatrix}$

3.1.4. Definition:

Let $f$ and $g$ be two permutations defined on a finite set $\tilde{S}^*$. Then by definition $f$ as well as $g$ is a one-one mapping of $\tilde{S}^*$ on to itself.

Consequently, the composite permutation of fuzzy numbers $g \circ f$ and $f \circ g$ are defined by

$$(g \circ f)(\tilde{x}) = g(f(\tilde{x})) \text{ for all } \tilde{x} \in \tilde{S}^*$$

and $$(f \circ g)(\tilde{x}) = f(g(\tilde{x})) \text{ for all } \tilde{x} \in \tilde{S}^*,$$

both are one-one mappings of $\tilde{S}^*$ on to itself.
So, whenever \( f \) and \( g \) are permutations of degree \( n \) then so are \( g \cdot f \) and \( f \cdot g \).

### 3.1.5. Example:

Let \( f = \begin{pmatrix}
  a & b & c & d \\
  c & a & d & b
\end{pmatrix} \) and

let \( g = \begin{pmatrix}
  a & b & c & d \\
  b & c & d & a
\end{pmatrix} \)

Then

\[
\begin{align*}
(g \cdot f)(a) &= g \cdot [f(a)] = g.(c) = d; \\
(g \cdot f)(b) &= g \cdot [f(b)] = g.(a) = b; \\
(g \cdot f)(c) &= g \cdot [f(c)] = g.(d) = a; \\
\end{align*}
\]

and

\[
\begin{align*}
(g \cdot f)(d) &= g \cdot [f(d)] = g.(b) = c; \\
\end{align*}
\]

Therefore \( g \cdot f = \begin{pmatrix}
  a & b & c & d \\
  d & b & a & c
\end{pmatrix} \)

Similarly \( f \cdot g = \begin{pmatrix}
  a & b & c & d \\
  a & d & b & c
\end{pmatrix} \).

### 3.1.6. Definition:

If \( \tilde{S}^* \) is a finite set containing \( \text{'}n\text{'} \) elements, then the set \( \tilde{S}n^* \) of all permutations on \( \tilde{S}^* \) is called a fuzzy number symmetric set.
Clearly $Sn^*$ contains $n!$ elements.

### 3.1.7. Example:

Let $S^* = \{ \overline{a}, \overline{b} \}$. Then the symmetric set $S_{2^*}$ of all permutations of degree two on $S^*$ contains $2!$ elements given below.

$$f_1 = \left( \begin{array}{c} \overline{a} \\ \overline{b} \end{array} \right), \quad f_2 = \left( \begin{array}{c} \overline{a} \\ \overline{b} \end{array} \right)$$

### 3.1.8. Theorem:

Let $S^*$ be a finite fuzzy number set containing $n$ distinct elements. Then the symmetric set of all fuzzy number permutations of degree on $S^*$ forms a finite fuzzy number group of order $n!$ with respect to composite of fuzzy number permutations as the composition.

**Proof**

Let $S^* = \{ \overline{a_1}, \overline{a_2}, \ldots, \overline{a_n} \}$. Then the composite composition on $S_{n^*}$ satisfies the following axioms.

1. **Closure Property**

   Let $f \in S_{n^*}$ and $g \in S_{n^*}$. Then $f$ as well as $g$ is a one-one mapping of $S^*$ onto its elements and therefore so is $g \circ f$.  

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Consequently, $g \cdot f$ is a permutation of degree $n$ on $\tilde{S}^*$. Thus $f \in \tilde{S}^*$ and $g \in \tilde{S}^*$ implies that $g \cdot f \in \tilde{S}^*$ for all $f, g \in \tilde{S}^*$

(ii) **Associative law.**

Since the composite composition of mappings is associative, so it is associative in the case of fuzzy number permutations also.

(iii) **Existence of identity.**

The identity permutation

$$\tilde{I} = \begin{pmatrix} \tilde{a_1} & \tilde{a_2} & \ldots & \tilde{a_n} \\ \tilde{b_1} & \tilde{b_2} & \ldots & \tilde{b_n} \end{pmatrix}$$

in $\tilde{S}^*$ is the identity for composite composition on $\tilde{S}^*$. For if $f = \begin{pmatrix} \tilde{a_1}, \tilde{a_2}, \ldots, \tilde{a_n} \\ \tilde{b_1}, \tilde{b_2}, \ldots, \tilde{b_n} \end{pmatrix}$ then

$$f \cdot \tilde{I} = \begin{pmatrix} \tilde{a_1} & \tilde{a_2} & \ldots & \tilde{a_n} \\ \tilde{b_1} & \tilde{b_2} & \ldots & \tilde{b_n} \end{pmatrix} \begin{pmatrix} \tilde{a_1} & \tilde{a_2} & \ldots & \tilde{a_n} \\ \tilde{b_1} & \tilde{b_2} & \ldots & \tilde{b_n} \end{pmatrix} = \begin{pmatrix} \tilde{a_1} & \tilde{a_2} & \ldots & \tilde{a_n} \\ \tilde{b_1} & \tilde{b_2} & \ldots & \tilde{b_n} \end{pmatrix} = f$$

Thus $f \cdot \tilde{I} = f$

Similarly $\tilde{I} \cdot f = f$

(iv) **Existence of inverse**

Corresponding to each fuzzy number permutation
f = \begin{pmatrix} a_1, a_2, \ldots, a_n \\ b_1, b_2, \ldots, b_n \end{pmatrix} \in \mathcal{S}_n^*, \text{ the fuzzy number permutation}

f^{-1} = \begin{pmatrix} b_1, b_2, \ldots, b_n \\ a_1, a_2, \ldots, a_n \end{pmatrix} \in \mathcal{S}_n^* \text{ is the inverse of } f

since

f \cdot f^{-1} = \begin{pmatrix} a_1, a_2, \ldots, a_n \\ b_1, b_2, \ldots, b_n \end{pmatrix} \begin{pmatrix} b_1, b_2, \ldots, b_n \\ a_1, a_2, \ldots, a_n \end{pmatrix} = \begin{pmatrix} b_1 b_2, \ldots, b_n \\ b_1 b_2, \ldots, b_n \end{pmatrix}

and similarly \quad f^{-1} \cdot f = I

Hence (\mathcal{S}_n^*, \cdot) \text{ is a finite fuzzy number group of order } n!.

3.1.9. Definition:

Let f be a fuzzy number permutation on a set \( \mathcal{S}^* \). On \( \mathcal{S}^* \), we define a relation \( \sim \) by (\( \bar{a} \sim \bar{b} \)) if and only if \( f^n(\bar{a}) = \bar{b} \) for some integer n.

Each one of these equivalence classes is called an orbit of \( f \).

3.1.10. Theorem:

Let f be a fuzzy number permutation on a finite set \( \mathcal{S}^* \) and let \( \bar{x} \in \mathcal{S}^* \). Then there exists a positive integer k such that the only orbit of \( \mathcal{S}^* \) containing \( \bar{x} \) is \{ (\bar{x}), f(\bar{x}), f^2(\bar{x}), \ldots, f^{k-1}(\bar{x}) \}. \)
Proof

Let $f \in \mathbb{S}^n$. Then the order of each element of a finite fuzzy number group being finite it follows that $o(f)$ is finite.

Let $o[f] = m$.

Then, $f^m = I$ and so $f^m(x) = I(x) = x$

Let $k$ be the smallest positive integer such that $f^k(x) = x$.

Now, we claim that $x = f^0(x), f^1(x), f^2(x), f^3(x), \ldots, f^{k-1}(x)$ are all distinct, for if $0 < p < q < (k-1)$ such that $f^q(x)$ implies that $f^p(x)$, then $f^{q-p}(x) = x$

where $0 < (q-p) < k$.

This contradicts the fact that $k$ is the least positive integer such that $f^k(x) = x$.

Now, if $y$ is in the orbit containing $x$, then

$y = f^s(x)$, for some integer $s$. But $s = qk + r$ where $0 \leq r < k$

therefore $y = f^s(x) = f^{qk+r}(x) = f^r[f^k(q)(x)] = f^r(x)$.
Now, since $0 \leq r < k$, it follows that $f^r(x)$ is one of the elements of $\{x, f(x), f^2(x), \ldots, f^{k-1}(x)\}$. Hence the only orbit of $f$ containing $x$ is \{ $x, f(x), f^2(x), \ldots, f^{k-1}(x)$\}.

3.1.11. Exercise:

Let $S\{[1,2], [2,3], [3,4], [4,5], [5,6]\}$


Determine the orbit of $f$.

Solution:

Since $f[1,2]=[2,3]$ and $f[2,3]=[1,2]$

So orbit of $f$ is \{ $[1,2], f[1,2]$\}

that is \{ $[1,2], [2,3]$\}

Again $f[3,4]=[5,6]$, and $f[5,6]=[3,4]$ so the orbit of $f$ is

\{ $[3,4], f[3,4]$\} that is \{ $[3,4], [5,6]$\}

Also $f[4,5]=[4,5]$, so $[4,5]$ is itself an orbit.

Hence the orbits of $f$ are \{ $[1,2], [2,3]$\}, \{ $[3,4], [5,6]$\}, \{ $[4,5]$\}. 

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3.2. Cyclic permutations of fuzzy numbers:

3.2.1. Definition:

Let $\bar{S}^*$ be a fuzzy number finite set containing $n$ elements. Then a permutation $f$ on $\bar{S}^*$ is said to be a cyclic fuzzy number permutation of length $k$, or simply a $k$ - cycle if $(nk)$ elements of $\bar{S}^*$ remain invariant and the variant elements are capable of being expressed in one row, in such a way that the image of each element in this row is the element following it and the image of the last element in the row is the first element with which we started.

3.2.2. Example:

The permutation $f = \left( [1,2] [2,3] [3,4] [5,6] [6,7] \right)$

\[ [2,3] [3,4] [5,6] [1,2] [6,7] \]

is a cyclic fuzzy number permutation of length 4.

It is denoted by $f = \left( [1,2] [2,3] [3,4] [5,6] \right)$

and $g = \left( [1,2] [2,3] [3,4] [4,5] [5,6] \right)

\[ [2,3] [3,4] [1,2] [5,6] [4,5] \]

is not cyclic. Since the elements of $g$ cannot be expressed in the requisite form of one row notation.
3.2.3. Properties of cyclic permutation of fuzzy number

(1) A cycle of fuzzy number of length 1 is the identity fuzzy number permutation.

(2) A cycle of fuzzy number of length 2 is called a fuzzy number transposition.

\[
\begin{bmatrix}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
[2,3], [3,4]
\end{bmatrix}
\text{is a fuzzy number transposition.}
\]

(3) The inverse of a cyclic fuzzy number permutation is obtained by writing the elements of the row of the cycle in a reverse order.

Thus if \( f = \begin{bmatrix}
\end{bmatrix} \) then

\[
f^{-1} = \begin{bmatrix}
\end{bmatrix}
\]

(4) Two cycles of fuzzy number permutation are said to be disjoint fuzzy number permutation if when expressed in one row notation, they have no elements in common.
(5) If f is cyclic fuzzy number permutation then f has almost one orbit having more than one elements.

3.2.4. Theorem:

The product of disjoint fuzzy number cycles is commutative.

Proof

Let f and g be any two disjoint fuzzy number cycles defined on a given finite set. Then f and g when expressed in one row notation have no common element.

So, the elements permuted by f are left unchanged by g and those permuted by g remain invariant under f.

Consequently \( f \cdot g = g \cdot f \).

3.2.5. Example:

Let \( f = \begin{bmatrix} 1,2 \end{bmatrix} \begin{bmatrix} 2,3 \end{bmatrix} \begin{bmatrix} 5,6 \end{bmatrix} \) and

\( g = \begin{bmatrix} 3,4 \end{bmatrix} \begin{bmatrix} 4,5 \end{bmatrix} \begin{bmatrix} 6,7 \end{bmatrix} \) be two disjoint fuzzy number cycle defined on six symbols \( \begin{bmatrix} 1,2 \end{bmatrix} \begin{bmatrix} 2,3 \end{bmatrix} \begin{bmatrix} 3,4 \end{bmatrix} \begin{bmatrix} 4,5 \end{bmatrix} \begin{bmatrix} 5,6 \end{bmatrix} \begin{bmatrix} 6,7 \end{bmatrix} \)
Then \( f = \left[ \begin{array}{cccccccc} 1,2 & 2,3 & 3,4 & 4,5 & 5,6 & 6,7 \\ 2,3 & 5,6 & 3,4 & 4,5 & 1,2 & 6,7 \end{array} \right] \)

\( g = \left[ \begin{array}{cccccccc} 1,2 & 2,3 & 3,4 & 4,5 & 5,6 & 6,7 \\ 1,2 & 2,3 & 4,5 & 6,7 & 5,6 & 3,4 \end{array} \right] \)

\( g.f = \left[ \begin{array}{cccccccc} 1,2 & 2,3 & 3,4 & 4,5 & 5,6 & 6,7 \\ 1,2 & 2,3 & 4,5 & 6,7 & 5,6 & 3,4 \end{array} \right] \cdot \left[ \begin{array}{cccccccc} 1,2 & 2,3 & 3,4 & 4,5 & 5,6 & 6,7 \\ 2,3 & 5,6 & 3,4 & 4,5 & 1,2 & 6,7 \end{array} \right] \)

\( g.f = \left[ \begin{array}{cccccccc} 1,2 & 2,3 & 3,4 & 4,5 & 5,6 & 6,7 \\ 2,3 & 5,6 & 4,5 & 6,7 & 1,2 & 3,4 \end{array} \right] \)

and \( f.g = \left[ \begin{array}{cccccccc} 1,2 & 2,3 & 3,4 & 4,5 & 5,6 & 6,7 \\ 2,3 & 5,6 & 3,4 & 4,5 & 1,2 & 6,7 \end{array} \right] \cdot \left[ \begin{array}{cccccccc} 1,2 & 2,3 & 3,4 & 4,5 & 5,6 & 6,7 \\ 1,2 & 2,3 & 4,5 & 6,7 & 5,6 & 3,4 \end{array} \right] \)

\( f.g = \left[ \begin{array}{cccccccc} 1,2 & 2,3 & 3,4 & 4,5 & 5,6 & 6,7 \\ 2,3 & 5,6 & 4,5 & 6,7 & 1,2 & 3,4 \end{array} \right] \)

\( g.f = f.g \)

3.2.6. Theorem:

Every fuzzy number permutation can be expressed as a composite of disjoint fuzzy number cycles.
Proof

Let $f$ be a given fuzzy number permutation of degree $n$ defined on $\tilde{S}^*$. First we take up all the cycles of length one, each determined by the invariant element.

Then we take an element which is non-invariant and construct a row starting with the same and putting after each elements image under $f$.

Since, the number of elements in $\tilde{S}^*$ is finite after a finite number of steps, we arrive at an element whose image under $f$ is the one with which we started.

This row represents a cycle.

Now, if all the element of $\tilde{S}^*$ have not been contained in the cycles, so far obtained, then we start with one such elements and obtain another cycle in the above manner. Continuing in this way each and every element of $\tilde{S}^*$ will be included in one or the other cycle.

Clearly these cycles have no element in common and so they are disjoint.
Thus, $f$ can be expressed as a composite permutation of disjoint fuzzy number cycles.

### 3.2.7. Example:

The fuzzy number permutation


as a composite of disjoint fuzzy number cycles.

**Solution:** We may write


### 3.2.8. Theorem:

Every fuzzy number permutation can be expressed as a composite of fuzzy number transpositions.

**Proof**

By theorem (3.2.6) every fuzzy number permutation can be expressed as a composite of disjoint fuzzy number cycles.
Now it is easy to verify that

\[ [\overline{a_1}, \overline{a_2}, \ldots, \overline{a_k}] = [\overline{a_1}, \overline{a_k}] \cdot [\overline{a_1}, \overline{a_{k-1}}] \ldots [\overline{a_1}, \overline{a_2}] \text{ when } k > 1 \]

That is every cycle other than the identity fuzzy number permutation can be expressed as a composite of fuzzy number transpositions.

Also,

\[
\begin{pmatrix}
\bar{a}_1 & \bar{a}_2 & \ldots & \bar{a}_n \\
\bar{a}_1 & \bar{a}_2 & \ldots & \bar{a}_n
\end{pmatrix}
= [\bar{a}_1 \bar{a}_2] \cdot [\bar{a}_2 \bar{a}_1]
\]

That is the identity fuzzy number permutation can also be expressed as a composite of fuzzy number transpositions.

Thus, every cycle and therefore, every fuzzy number permutation can be expressed as the product of fuzzy number transpositions.

3.2.9. Example:

The fuzzy number permutation

\[
f = \begin{pmatrix}
\end{pmatrix}
\]

as a product of fuzzy number transpositions.

Solution:

\[
f = \begin{pmatrix}
\end{pmatrix}
\]
\[= [1,2] \cdot [2,3] \cdot [6,7] \cdot [3,4] \cdot [5,6] \cdot [4,5]\]

\[= [1,2] \cdot [2,3] \cdot [2,3] \cdot [1,2] \cdot [2,3] \cdot [6,7] \cdot [3,4] \cdot [4,5] \cdot [3,4] \cdot [5,6]\]

**3.2.10 Theorem:**

For any manner of expressing a given fuzzy number permutation of the form \([ a^{-1}, \ a^- ]\) as a composite of fuzzy number transpositions, the number of fuzzy number transpositions is either necessarily odd or even.

**Proof**

Let a given fuzzy number permutation \( f \) of degree \( n \) be expressed in two ways as the product of \( r \) and \( s \) transpositions respectively.

Consider a polynomial in \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \) defined by

\[P = (\bar{x}_1 - \bar{x}_2)(\bar{x}_1 - \bar{x}_3) \ldots (\bar{x}_1 - \bar{x}_n)(\bar{x}_2 - \bar{x}_3) \ldots (\bar{x}_2 - \bar{x}_n) \ldots (\bar{x}_{n-1} - \bar{x}_n)\]

Now if \( \sigma = (ij) \) be any transposition, then by \( P\sigma \), we mean the polynomial obtained by permuting the suffixes \( 1,2,3,\ldots, n \) of the \( \bar{x}_i \)'s are prescribed by \( \sigma \). The effect of this transposition \( \sigma \) on \( P \) is to change its sign, that is \( P\sigma = - P \).

For example if \( n = 3 \) then
\[
\begin{aligned}
P &= (x_1 - x_2) (x_1 - x_3) (x_2 - x_3) \\
\text{Now if } \sigma = (1,2) \text{ be any transposition, then} \\
P \sigma &= (x_2 - x_1) (x_2 - x_3) (x_1 - x_3) = -P.
\end{aligned}
\]

Now, when \( f \) is expressible as the product of \( r \) transpositions, then, since each operation of a transposition on \( p \) gives \(-P\),

So \( Pf = (-1)^r P \).

Similarly, \( Pf = (-1)^s P \). Consequently \((-1)^r P = (-1)^s P \).

But this will hold only when \( r \) and \( s \) are either both even or both odd.

3.3 Even and odd permutation of fuzzy numbers:

3.3.1. Definition:

A fuzzy number permutation is said to be even or odd according as it is expressible as the product of an even or an odd number of fuzzy number transpositions.

3.3.2. Proposition:

A fuzzy number cycle of length \( n \) is an even or an odd fuzzy number permutation according as \( n \) is odd or even.
Proof

Let \( f = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \) be a cycle of length \( n \).

Then \( f = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = (\bar{a}_1, \bar{a}_n) \cdot (\bar{a}_1, \bar{a}_{n-1}) \cdot \ldots \cdot (\bar{a}_1, \bar{a}_2) \)

That is \( f \) is expressible as the product of \((n-1)\) transpositions clearly when \( n \) is odd, \((n-1)\) is even and so \( f \) is even. And when \( n \) is even \((n-1)\) is odd and \( f \) is odd.

3.3.3. Proposition:

Identity fuzzy number permutation is always an even fuzzy number permutation.

Proof

We have \( I = \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \ldots & \bar{a}_n \\ \bar{a}_1 & \bar{a}_2 & \ldots & \bar{a}_n \end{bmatrix} = [\bar{a}_1, \bar{a}_2] \cdot [\bar{a}_2, \bar{a}_1] \)

That is the identity permutation is expressible as the product of two fuzzy number transpositions. Hence it is an even fuzzy number permutation.

3.3.4. Proposition:

(i) The product of two even fuzzy number permutation is an even fuzzy number permutation.
(ii) The product of two fuzzy number odd permutations is an even permutation.

(iii) The product of an even fuzzy number permutations and an odd fuzzy number permutation is an odd fuzzy number permutation.

Proof

Let \( f \) and \( g \) be two even fuzzy number permutations so that each one is expressible as a product of an even fuzzy number of transpositions say \( m \) and \( n \) respectively.

Then \( g.f \) is the product of \( (m+n) \) transpositions.

But, \( (m+n) \) being the sum of two even numbers it is even. Hence \( g.f \) is an even fuzzy number permutation.

Similarly (ii) and (iii) can be proved.

3.3.5 Proposition:

(i) The inverse of an even fuzzy number permutation is even

(ii) The inverse of an odd fuzzy number permutation is odd.
Proof

(i) Let $f$ be an even fuzzy number permutation and if possible $f^{-1}$ be odd.

Then $f \cdot f^{-1} = I$ is odd, which contradicts the fact that identity fuzzy number permutation is always even. Hence whenever $f$ is even, then $f^{-1}$ is even.

Similarly (ii) follows.

3.3.6. Proposition:

Of the $n!$ fuzzy number permutations on $n$ symbols, $n!/2$ are even and $n!/2$ are odd.

Proof

Let the symmetric fuzzy number group $\overline{S}_n^*$ of all $n!$ permutations of degree $n$ consist of even fuzzy number permutation $f_1, f_2, \ldots, f_k$ and odd fuzzy number permutation $g_1, g_2, \ldots, g_m$ so that $(k+m) = n!$ Now, let $f$ be any transposition in $\overline{S}_n^*$. Then $f$ is an odd fuzzy number permutation.

By the closure property, $f \cdot f_i \in \overline{S}_n^*$ for $1 \leq i \leq k$
and \( f \cdot g_j \in \overline{S}_n^* \) for \( 1 \leq j \leq m \)

Since, \( f \) is odd and \( f_i \) is even, so \( f \cdot f_i \) is odd, for each \( i \).

Again, \( f \) is odd and \( g_j \) is odd, So \( f \cdot g_j \) is even for each \( j \).

Also \( f . f_i = f . f_p \) implies that \( f_i = f_p \) (by cancellation law) and \( f . g_j = f . g_i \) implies that \( g_j = g_i \) (by cancellation law)

Thus \( f . f_1, f . f_2, \ldots f . f_k \) are \( k \) distinct odd permutations and \( f . g_1, f . g_2, \ldots f . g_m \) are \( m \) distinct even permutations.

This shows that \( \overline{S}_n^* \) contain \( k \) odd fuzzy number permutation and \( m \) even fuzzy number permutation.

But, a permutation can not be both even and odd.

\[
\frac{n!}{2}
\]

Therefore \( m = k = \frac{n!}{2} \)

Hence \( \overline{S}_n^* \) contains \((n!/2)\) even fuzzy number permutations and \((n!/2)\) odd fuzzy number permutation.

3.3.7. Proposition:

The set \( A_n \) of all even fuzzy number permutations of degree \( n \), defined and a set \( \overline{S}_n^* \) forms a finite fuzzy number group of order \((n!/2)\) with respect to composite composition.
Proof

It has already been shown that the composite of two even fuzzy number permutations is even the identity fuzzy number permutation is even and the inverse of an even fuzzy number permutation is even.

We also know that the composite composition of fuzzy number permutations is associative. So it is associative an $\bar{\mathcal{A}} n$. More over out of $n!$ fuzzy number permutations in $\bar{\mathcal{S}}_n^*$, $n!/2$ are even. Hence $[ \bar{\mathcal{A}} n, . ]$ is a fuzzy number group of order $n!/2$

3.3.8. Example :

The four fuzzy number permutations $\bar{1}, [ \bar{a}, \bar{b} ] [ \bar{c}, \bar{d} ]$

$[ \bar{a}, \bar{b} ] . [ \bar{c}, \bar{d} ]$

on four symbols $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ form a finite abelian fuzzy number group with respect to the composite composition

Solution : Let $\bar{f} = \bar{1} = \left( \begin{array}{cccc} \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \end{array} \right)$

$\bar{f}2 = [ \bar{a} \bar{b} ] = \left( \begin{array}{cccc} \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{b} & \bar{a} & \bar{c} & \bar{d} \end{array} \right)$

$\bar{f}3 = [ \bar{c} \bar{d} ] = \left( \begin{array}{cccc} \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{a} & \bar{b} & \bar{d} & \bar{c} \end{array} \right)$ and
\[
\vec{f}_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b & c & d \\ b & a & c & d \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ a & b & d & c \end{bmatrix}
\]

Let \( F^*(R) = \{ \vec{f}_1, \vec{f}_2, \vec{f}_3, \vec{f}_4 \} \)

By computing the various products, we get the table given below

<table>
<thead>
<tr>
<th>.</th>
<th>\vec{f}_1</th>
<th>\vec{f}_2</th>
<th>\vec{f}_3</th>
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<td>\vec{f}_3</td>
<td>\vec{f}_2</td>
<td>\vec{f}_1</td>
</tr>
</tbody>
</table>

Clearly all the entries in the table are members of \( F^*(R) \)

\( \vec{f}_1 \) is the identity and \( \vec{f}_1^{-1} = \vec{f}_1 \), \( \vec{f}_2^{-1} = \vec{f}_2 \), \( \vec{f}_3^{-1} = \vec{f}_3 \) and \( \vec{f}_4^{-1} = \vec{f}_4 \)

Also composite composition on \( F^*(R) \) is associative. Moreover, each row of the table coincides with the corresponding column. Hence \([F^*(R),.]\) is an abelian fuzzy number group.