CHAPTER-II
SOME PROPERTIES OF ABELIAN FUZZY NUMBER GROUP

2.1. Introduction:

In this chapter, definition of abelian fuzzy number group, properties of abelian fuzzy number group are discussed. Properties of cyclic fuzzy number group are studied.

2.1.1. Definition:

Fuzzy number group \( F'(R),^* \) is said to be an abelian or a commutative fuzzy number group if the composition \( ^* \) is commutative, that is, \( \bar{a}^* \bar{b} = \bar{b}^* \bar{a} \) for all \( \bar{a}, \bar{b} \in F'(R) \).

2.1.2. Proposition:

If any two fuzzy numbers \( \bar{a} = [\bar{a}^-, \bar{a}^+] \) and \( \bar{b} = [\bar{b}^-, \bar{b}^+] \) of a fuzzy number group \( F'(R) \) commute, then the following properties hold

(i) \( \bar{a}^{-1} \cdot \bar{b}^{-1} = \bar{b}^{-1} \bar{a}^{-1} \)

(ii) \( \bar{a}^{-1} \cdot \bar{b} = \bar{b} \cdot \bar{a}^{-1} \)

(iii) \( \bar{a} \cdot \bar{b}^{-1} = \bar{b}^{-1} \bar{a} \).

Proof

i) \( \bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a} \) implies that \( \{ \bar{a} \cdot \bar{b} \}^{-1} = \{ \bar{b} \cdot \bar{a} \}^{-1} \)
implies that $\bar{b}^{-1} \cdot \bar{a}^{-1} = \bar{a}^{-1} \cdot \bar{b}^{-1}$ [by reversal law of inverse]

ii) $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$ implies that $\bar{b} = \bar{a}^{-1} [\bar{b} \cdot \bar{a}] = [\bar{a}^{-1} \cdot \bar{b}] \cdot \bar{a}$

implies that $\bar{b} \cdot \bar{a}^{-1} = [\bar{a}^{-1} \cdot \bar{b}] \cdot \bar{a} \cdot \bar{a}^{-1}$

implies that $\bar{b} \cdot \bar{a}^{-1} = [\bar{a}^{-1} \cdot \bar{b}]$ \[ \therefore \bar{a} \cdot \bar{a}^{-1} = \bar{e} \]

iii) $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$ implies that $\bar{a} = [\bar{b} \cdot \bar{a}] \cdot \bar{b}^{-1} = \bar{b}[\bar{a} \cdot \bar{b}^{-1}]$

implies that $\bar{a} = \bar{b}[\bar{a} \cdot \bar{b}^{-1}]$

implies that $\bar{b}^{-1} \cdot \bar{a} = \bar{b}^{-1} \cdot \bar{b}[\bar{a} \cdot \bar{b}^{-1}]$

implies that $\bar{b}^{-1} \cdot \bar{a} = \bar{e}[\bar{a} \cdot \bar{b}^{-1}]$

implies that $\bar{b}^{-1} \cdot \bar{a} = \bar{a} \cdot \bar{b}^{-1}$.

2.1.3. Proposition:

For any two fuzzy numbers $\bar{a} = [\bar{a}', \bar{a}]$ and $\bar{b} = [\bar{b}', \bar{b}]$ of a fuzzy number group $F'(R)$, $[\bar{a} \cdot \bar{b}]^2 = \bar{a}^2 \cdot \bar{b}^2$ if and only if $F'(R)$ is abelian.

Proof

Let $[\bar{a} \cdot \bar{b}]^2 = \bar{a}^2 \cdot \bar{b}^2$ for all $\bar{a}, \bar{b} \in F'(R)$

Then $[\bar{a} \cdot \bar{b}]^2 = [\bar{a}^2 \bar{b}^2]$. 

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implies that \( [\tilde{a} \cdot \tilde{b}] [\tilde{a} \cdot \tilde{b}] = [\tilde{a} \cdot \tilde{a}] [\tilde{b} \cdot \tilde{b}] \)

implies that \([\tilde{a}] [\tilde{b} \cdot \tilde{a}] \tilde{b} = \tilde{a} [\tilde{a} \cdot \tilde{b}] \tilde{b} \) [by associative]

implies that \(\tilde{b} \cdot \tilde{a} = \tilde{a} \cdot \tilde{b} \) [by cancellation law]

Thus \(\tilde{a} \cdot \tilde{b} = \tilde{b} \cdot \tilde{a} \) for all \(\tilde{a}, \tilde{b} \in F'(R)\).

Hence \(F^*(R)\) is abelian.

Conversely, let \(F^*(R)\) be abelian so that \(\tilde{a} \cdot \tilde{b} = \tilde{b} \cdot \tilde{a} \) for all \(\tilde{a}, \tilde{b} \in F^*(R)\).

Now \(\tilde{a} \cdot \tilde{b})^2 = [\tilde{a} \cdot \tilde{b}] [\tilde{a} \cdot \tilde{b}] = \tilde{a} [\tilde{b} \cdot \tilde{a}] \tilde{b} \) [by associative]

Thus \(\tilde{a} \cdot \tilde{b})^2 = \tilde{a}^2 \cdot \tilde{b}^2 \) for all \(\tilde{a}, \tilde{b} \in F^*(R)\).

2.1.4. Proposition:

If each number of a fuzzy number group except the identity element is of order 2, then the fuzzy number group is abelian.

Proof

Let \(F^*(R)\) be a fuzzy number group.
Let \( \tilde{a} \in F^*(R) \) and \( \tilde{b} \in F^*(R) \), then by closure property, \( \tilde{a} \cdot \tilde{b} \in F^*(R) \).

By hypothesis,

\[ o[\tilde{a}] = 2, \ o[\tilde{b}] = 2 \text{ and } o[\tilde{a} \cdot \tilde{b}] = 2 \]

That is, \( \tilde{a}^2 = \tilde{e}, \ \tilde{b}^2 = \tilde{e} \) and \( [\tilde{a} \cdot \tilde{b}]^2 = \tilde{e} \), where \( \tilde{e} \) is the identity.

Now, \( \tilde{a}^2 = \tilde{e} \) implies that \( \tilde{a} \cdot \tilde{a} = \tilde{e} \) implies that \( \tilde{a}^{-1} = \tilde{a} \)

Similarly \( \tilde{b}^2 = \tilde{e} \) implies that \( \tilde{b}^{-1} = \tilde{b} \)

and \( [\tilde{a} \cdot \tilde{b}]^2 = \tilde{e} \)

implies that \( [\tilde{a} \cdot \tilde{b}]^{-1} = \tilde{a} \cdot \tilde{b} \)

implies that \( \tilde{b}^{-1} \cdot \tilde{a}^{-1} = \tilde{a} \cdot \tilde{b} \) implies that \( \tilde{b} \cdot \tilde{a} = \tilde{a} \cdot \tilde{b} \)

Thus \( \tilde{a} \cdot \tilde{b} = \tilde{b} \cdot \tilde{a} \) for all \( \tilde{a}, \tilde{b} \in F^*(R) \).

Hence \( F^*(R) \) is abelian.

2.1.5 Proposition:

If \( F^*(R) \) be a fuzzy number group of even order, then there exists an element \( \tilde{a} \), other than the identity element, such that \( \tilde{a}^2 = \tilde{e} \).

Proof

Let \( F^*(R) \) be a fuzzy number group of even order, say 2n.

Let \( \tilde{a} = [\tilde{a}^-, \tilde{a}^+] \in F^*(R) \). Now \( \tilde{a}^2 = \tilde{e} \) implies that \( \tilde{a} \cdot \tilde{a} = \tilde{e} \) implies that \( \tilde{a}^{-1} = \tilde{a} \).
Thus, we must show that there exists an element
\[ \bar{a} \neq \bar{e} \] such that \[ \bar{a}^{-1} = \bar{a} \]. Assume that no such element exists. In a fuzzy number group every element possesses a unique inverse.

The identity element is the only element such that \[ \bar{e}^{-1} = \bar{e} \].

The remaining \((2n-1)\) elements must therefore, be divided into pairs in such a way that each pair consists of two distinct elements which are inverses of each other.

But, this is impossible, since \((2n-1)\) is odd.

So there exists \[ \bar{a} \in F'(R) \] such that \[ \bar{a} \neq \bar{e} \] and \[ \bar{a}^{-1} = \bar{a} \].

Hence, there exists \[ \bar{a} \in F'(R) \] such that \[ \bar{a} \neq \bar{e} \] and \[ \bar{a}^2 = \bar{e} \].

2.1.6. Theorem:

If \[ \bar{a} = [ \bar{a}^-, \bar{a}^+ ] \] and \[ \bar{b} = [ \bar{b}^-, \bar{b}^+ ] \] are two arbitrary fuzzy numbers of an abelian fuzzy number group \( F'(R) \) then for all integers \( n \),
\[ [ \bar{a} \bar{b} ]^n = \bar{a}^n \cdot \bar{b}^n. \]
Proof

Case i.

When \( n=0 \)

In this case, \( \bar{a}^0 = \bar{e}, \bar{b}^0 = \bar{e} \) and \( [\bar{a} \cdot \bar{b}]^0 = \bar{e} \).

Therefore \( \bar{a}^0 \cdot \bar{b}^0 = \bar{e} \cdot \bar{e} = \bar{e} = [\bar{a} \cdot \bar{b}]^0 \).

Hence \( [\bar{a} \cdot \bar{b}]^0 = \bar{a}^0 \cdot \bar{b}^0 \)

Case ii.

When \( n>0 \), that is, when \( n \) is a positive integer.

In this case, we shall prove the result by induction.

Clearly \( [\bar{a} \cdot \bar{b}]^l = \bar{a} \cdot \bar{b} = \bar{a}^l \cdot \bar{b}^l \)

So the result is true for \( n = m \). So that \( [\bar{a} \cdot \bar{b}]^m = \bar{a}^m \cdot \bar{b}^m \)

then \( [\bar{a} \cdot \bar{b}]^{m+1} = [\bar{a} \cdot \bar{b}]^m [\bar{a} \cdot \bar{b}] \)

\( = [\bar{a}^m \cdot \bar{b}^m] [\bar{a} \cdot \bar{b}] \)

\( = \bar{a}^m [\bar{b}^m \cdot \bar{a}] \cdot \bar{b} \)

\( = \bar{a}^m [\bar{a} \cdot \bar{b}^m] \cdot \bar{b} \)

\( = [\bar{a}^m \cdot \bar{a}] \cdot [\bar{b}^m \cdot \bar{b}] \)

\( = \bar{a}^{m+1} \cdot \bar{b}^{m+1} \)
Thus, it follows that, if the result is true for \( n = m \) then it is also true for \( n = m+1 \).

So, by the principle of mathematical induction, the required result is true for all positive integers.

**Case iii**

When \( n < 0 \) that is when \( n \) is a negative integer.

Let \( n = -m \), where \( m \) is a positive integer. Then

\[
[\overline{a} \cdot \overline{b}]^n = [\overline{a} \cdot \overline{b}]^{-m} = [(\overline{a} \cdot \overline{b})^m]^{-1} \\
= \frac{1}{[\overline{a}^m \cdot \overline{b}^m]^1} \\
= \frac{1}{[\overline{b}^m \cdot \overline{a}^m]^1} \\
= \frac{1}{[\overline{a}^{-m}] \cdot [\overline{b}^{-m}]^{-1}} \\
= \overline{a}^{-m} \cdot \overline{b}^{-m} \\
= \overline{a}^n \cdot \overline{b}^n
\]

Hence \( [\overline{a} \cdot \overline{b}]^n = \overline{a}^n \cdot \overline{b}^n \) for every integer \( n \).

2.1.7. **Theorem:**

If \( F'(\mathbb{R}) \) is a fuzzy number group in which \( [\overline{a} \cdot \overline{b}]^i = [\overline{a}^i \cdot \overline{b}^i] \) for three consecutive integers \( i \) and for \( \overline{a} = [\overline{a}^-, \overline{a}^+] \), \( \overline{b} = [\overline{b}^-, \overline{b}^+] \in F'(\mathbb{R}) \) then \( F'(\mathbb{R}) \) is abelian.
Proof

Let \([ \tilde{a} \tilde{b} ] = \tilde{a} \tilde{b} \), \([ \tilde{a} \cdot \tilde{b} ] = \tilde{a} \cdot \tilde{b} \)

and \([ \tilde{a} \cdot \tilde{b} ]^{i+2} = \tilde{a}^{i+2} \cdot \tilde{b}^{i+2} \)

Then \([ \tilde{a} \cdot \tilde{b} ]^{i+2} = \tilde{a}^{i+2} \cdot \tilde{b}^{i+2} \)

implies that
\[
[ \tilde{a} \cdot \tilde{b} ]^{i+1} = [ a \cdot \tilde{b} ] = [ a^{i+1} \cdot \tilde{a} ] \cdot [ \tilde{b}^{i+1} \cdot \tilde{b} ]
\]

implies that
\[
[ a^{i+1} \cdot \tilde{b}^{i+1} ] \cdot [ \tilde{a} \cdot \tilde{b} ] = [ a^{i+1} \cdot \tilde{a} ] \cdot [ \tilde{b}^{i+1} \cdot \tilde{b} ]
\]

[since \([ \tilde{a} \cdot \tilde{b} ]^{i+1} = \tilde{a}^{i+1} \cdot \tilde{b}^{i+1} \)]

implies that
\[
[ a^{i+1} \cdot \tilde{b}^{i+1} ] \cdot \tilde{a} = \tilde{a} \cdot \tilde{b}^{i+1}
\]

(by associative)

implies that
\[
\tilde{b}^{i+1} \cdot \tilde{a} = \tilde{a} \cdot \tilde{b}^{i+1}
\]

(by cancellation law)

implies that
\[
[ \tilde{a}^{i+1} \cdot \tilde{b}^{i+1} ] \cdot \tilde{a} = \tilde{a} \cdot \tilde{b}^{i+1}
\]

implies that
\[
[ a^{i+1} \cdot \tilde{b}^{i+1} ] \cdot \tilde{a} = a^{i+1} \cdot \tilde{b}^{i+1}
\]

implies that
\[
[ a \cdot \tilde{b} ]^{i+1} = [ \tilde{a} \cdot \tilde{b} ]^{i+1} = [ \tilde{a} \cdot \tilde{b} ]^{i+1} = [ \tilde{a} \cdot \tilde{b} ]^{i+1}
\]

implies that
\[
\tilde{b} \cdot \tilde{a} = \tilde{a} \cdot \tilde{b}
\]

(by cancellation law)

Thus \( \tilde{a} \cdot \tilde{b} = \tilde{b} \cdot \tilde{a} \) for all \( \tilde{a}, \tilde{b} \in F^*(R) \)

Hence \( F^*(R) \) is abelian.

2.1.8. Proposition:

Let \( F^*(R) \) be a fuzzy number group. Let \( \tilde{a} \in F^*(R) \) such that \( o[ \tilde{a} ] = n \).

Then for any integer \( m \), \( o[ \tilde{a}^m ] = n / (n-m) \), where \( (n, m) \) denotes the H.C.F of \( n \) and \( m \).
Proof

Let \((n,m) = k\), then \(n = pk\) and \(m = qk\) for some integer \(p\) and \(q\) such that \((p,q) = 1\).

Let \(o[\bar{a}^m] = r\). Then \([\bar{a}^m]^r = e\) or \(\bar{a}^{mr} = e\).

Now \(o[\bar{a}] = n\) and \(\bar{a}^{mr} = e\) implies that \(n/mr\).

But \(n/mr\) implies that \(pk/qkr\) \([n = pk\ and \ m = qk]\)

implies that \(p/qr\)

implies that \(p/r\) [since \(p\) and \(q\) are relatively prime]

Again \([\bar{a}^m]^p = [\bar{a}^q]^p = \bar{a}^{q(pk)} = \bar{a}^{qn} = [\bar{a}^n]^q = e^q = e\)

Now, \(o[\bar{a}^m] = r\) and \([\bar{a}^m]^p = e\) implies that \(r/p\)

Thus, \(p/r\) and \(r/p\) implies that \(p = r\)

Therefore \(o[\bar{a}^m] = r = p = n/k = n/(n,m)\)

2.1.9 Proposition:

If the elements \(\bar{a}, \bar{b}\) of a fuzzy number group \(F^*(R)\) commute and
\(o[\bar{a}] = m, o[\bar{b}] = n\) where \(m, n\) are relatively prime then \(o[\bar{a} \cdot \bar{b}] = mn\).
Proof

Let \( o[\bar{a} \cdot \bar{b}] = k \)

Now \( [\bar{a} \cdot \bar{b}]^{mn} = \bar{a}^{mn} \cdot \bar{b}^{mn} \)

\[
= [\bar{a}^{mn}] \cdot [\bar{b}^{mn}]
\]

\[
= e^n \cdot e^m
\]

\[
= e \cdot e
\]

\( [\bar{a} \cdot \bar{b}]^{mn} = e\)

Therefore \( k/mn \)

Again \( [\bar{a} \cdot \bar{b}]^k = e \) implies that \( \bar{a}^k \cdot \bar{b}^k = e \)

implies that \( \bar{a}^k = [\bar{b}^k]^{-1} \)

implies that \( o[\bar{a}^k] = o[[\bar{b}^k]^{-1}] = o[\bar{b}^k] \)

Now, \( o[\bar{a}^k] / o[\bar{a}] \) and \( o[\bar{b}^k] / o[\bar{b}] \)

So, \( o[\bar{a}^k] / m \) and \( o[\bar{a}^k] / n \)

But, \( o[\bar{a}^k] = o[\bar{b}^k] \)

So, \( o[\bar{a}^k] / (H.C.F \text{ of } m \text{ and } n) \)

That is \( o[\bar{a}^k] = 1 \)

Thus, \( o[\bar{a}^k] = o[\bar{b}^k] = 1 \)

\( \bar{a}^k = \bar{e} \) and \( \bar{b}^k = \bar{e} \)
But, \( o\langle a \rangle = m, \quad a^k = e \) implies that \( m/k \)

and \( o\langle b \rangle = n, \quad b^k = e \) implies that \( n/k \) implies that \( mn/k \).

Now, \( mn/k \) and \( k/mn \) implies that \( k=mn \)

Hence, \( o\langle a . b \rangle = mn \).

2.1.10 Proposition:

The composition table for a finite fuzzy number group contains each element once and only once in each of its rows and columns if \( a = [ a^- , a^+ ] \)

Proof

Let \( F^*(R) \) be a finite fuzzy number group and let if possible, an element \( x \in F^*(R) \) be repeated twice in a single row.

Then an element \( a = [ a^- , a^+ ] \in F^*(R) \) operated with distinct elements \( b \) and \( c \) of \( F^*(R) \), gives \( x \) in each case. That is \( a . b = x \) and \( a . c = x \).

Consequently \( a . b = a . c \) and therefore, by cancellation law \( b = c \).

This is a contradiction. So, each row of the table contains each element once and only once. Similarly, each column of the table contain each element once and only once.
2.1.11 Proposition:

Every fuzzy number group consisting of three or less than three elements is always abelian, if \( \bar{a} = [\bar{a}, \bar{a}] \) and \( \bar{b} = [1/\bar{a}, 1/\bar{a}] \).

Proof

Let, \( F^*(R) \) be a fuzzy number group consisting of 3 or less than 3 elements.

Then,

Case (i) : Let \( F^*(R) = \{ \bar{e} \} \), where \( \bar{e} \) is the identity element. In this case

\( F^*(R) \) is clearly abelian.

Case (ii) : Let \( F^*(R) = \{ \bar{e}, \bar{a} \} \) where \( \bar{a} \neq \bar{e} \).

In this case \( \bar{a} \cdot \bar{e} = \bar{e} \cdot \bar{a} = \bar{a} \) and so \( F^*(R) \) is abelian.

Case (iii) : Let \( F^*(R) = \{ \bar{e}, \bar{a}, \bar{b} \} \), where \( \bar{a} \neq \bar{b} \neq \bar{e} \).

In this case we prepare the composition table given below. Since in any row of the composition table of a fuzzy number group each element occurs only once.
So, $\overline{a}^2 = \overline{e}$ or $\overline{a} \cdot \overline{b} = \overline{e}$ and $\overline{b} \cdot \overline{a} = \overline{e}$ or $\overline{b}^2 = \overline{e}$.

Now, $\overline{a}^2 = \overline{e}$ implies that $\overline{a} \cdot \overline{b} = \overline{b}$

implies that $\overline{a} \cdot \overline{b} = \overline{e} \cdot \overline{b}$

implies that $\overline{a} = \overline{e}$ (by cancellation)

This is not possible, since $\overline{a} \neq \overline{e}$

So $\overline{a}^2 \neq \overline{e}$ and therefore $\overline{a} \cdot \overline{b} = \overline{e}$

Similarly $\overline{b}^2 \neq \overline{e}$ and therefore $\overline{b} \cdot \overline{a} = \overline{e}$

Therefore $\overline{a} \cdot \overline{b} = \overline{b} \cdot \overline{a} = \overline{e}$

This shows that $F'(R)$ is abelian.
2.2. Cyclic fuzzy number group:

2.2.1 Definition:

A fuzzy number group $F^*(R)$ is said to be a cyclic fuzzy number group, if there exists an element $\bar{a} \in F^*(R)$ such that every element of $F^*(R)$ is expressible as some integral powers of $\bar{a}$.

In this case, the element $\bar{a}$ is called the generator of the fuzzy number group, and we write

$$F^*(R) = \{ \bar{a} \}$$

A cyclic fuzzy number group is also known as monogenic fuzzy number group.

2.2.2. Theorem:

Every cyclic fuzzy number group is necessarily abelian.

Proof

Let $F^*(R) = \{ \bar{a} \}$ be a cyclic fuzzy number group generated by an element $\bar{a}$. Let $\bar{x}$ and $\bar{y}$ be any two arbitrary elements of $F^*(R)$, then

$$\bar{x} = \bar{a}^m$$

and

$$\bar{y} = \bar{a}^n$$

for some integers $m$ and $n$. 
Since \( x \cdot y = \bar{a}^m \cdot \bar{a}^n = \bar{a}^{m+n} = \bar{a}^{n+m} = \bar{a}^n \cdot \bar{a}^m = y \cdot x \).

Thus \( x \cdot y = y \cdot x \) for all \( x, y \in F^*(R) \)

Hence, \( F^*(R) \) is an abelian group.

2.2.3. Theorem:

If an element \( \bar{a} \) is a generator of a cyclic fuzzy number group \( F^*(R) \), then \( \bar{a}^{-1} \) is also a generator of \( F^*(R) \).

Proof

Let \( F^*(R) = \{ \bar{a} \} \) be a cyclic fuzzy number group generated by an element \( \bar{a} \). Then each element of \( F^*(R) \) is of the form \( \bar{a}^n \) for some integer \( n \).

Now we may write,

\[
\bar{a}^n = \bar{a}^{-1(-n)} = (\bar{a}^{-1})^{-n}
\]

Thus every element of \( F^*(R) \) can be expressed as some integral powers of \( \bar{a}^{-1} \).

2.2.4. Theorem:

If \( \bar{a} = [\bar{a}^~ \bar{a}^-] \) then the order of a cyclic fuzzy number group is the same as the order of its generator.
Proof

Let $\mathcal{F}(R) = \{\tilde{a}\}$ be a cyclic fuzzy number group generated by $\tilde{a}$. Then,

Case (i):

When $o(\tilde{a})$ is finite. Let $o(\tilde{a}) = n$, be a finite positive integer. Then, $\tilde{a}^n = e = \tilde{a}^0$. Now, by closure property. $\tilde{a}$, $\tilde{a}^2$, $\tilde{a}^3$, ..., $\tilde{a}^n = e$ are elements of $\mathcal{F}(R)$. We show that these elements are distinct and these are the only elements of $\mathcal{F}(R)$.

Let $i$ and $j$ be two integers such that $1 \leq j \leq i \leq n$.

We have $\tilde{a}^i = \tilde{a}^j$, Then

$\tilde{a}^i = \tilde{a}^j$ implies that $\tilde{a}^i (\tilde{a}^j)^{-1} = e$

implies that $\tilde{a}^{i-j} = e$

implies that $\tilde{a}^{i-j} = e$

Clearly $0 < i-j < n$.

Hence it shows that $o(\tilde{a}) < n$ which is a contradiction.

Thus $\tilde{a}$, $\tilde{a}^2$, $\tilde{a}^3$, ..., $\tilde{a}^n = e$ are all distinct.
Now, we show that every integral power of \( a \) equals some one of these elements.

Consider, \( a^m \in F^*(R) \), where \( m \) is any integer, by division algorithm, we have \( m = nq + r \), where \( 0 \leq r < n \).

\[
\overline{a}^m = \overline{a}^{nq+r} = (\overline{a}^n)^q \cdot \overline{a}^r = \overline{e} \cdot \overline{a}^r = \overline{a}^r, \text{ where } 0 \leq r < n.
\]

But \( \overline{a}^r \) is one of \( \overline{a}^0, \overline{a}^1, \overline{a}^2, \ldots, \overline{a}^{n-1}, \overline{a}^n = \overline{e} = \overline{a}^0 \).

Thus every integral power of \( \overline{a} \) is one of \( \overline{a}^0, \overline{a}^1, \overline{a}^2, \ldots, \overline{a}^{n-1} \).

Hence \( F^*(R) \) contains \( n \) distinct elements.

\( \overline{a}^0, \overline{a}^1, \overline{a}^2, \ldots, \overline{a}^{n-1} \)

That is \( o[F^*(R)] = n = o[\overline{a}] \).

Case (ii):

When \( o[\overline{a}] \) is infinite

In this case, distinct integral powers of \( \overline{a} \) shall give distinct elements of \( F^*(R) \). For if \( \overline{a}^m = \overline{a}^n \) and \( m > n \), then we get \( \overline{a}^{m-n} = \overline{e} \), which implies that \( o[\overline{a}] \) is finite, a contradiction.

Thus, \( F^*(R) \) will be infinite in this case.
Hence the order of the cyclic fuzzy number group is the same as the order of its generator.

2.2.5. Theorem:

A finite fuzzy number group of order n containing an element of order n must be cyclic.

Proof

Let $F^*(R)$ be a finite number group of order n.

Let $\bar{a} \in F^*(R)$ such that $o[\bar{a}] = n$.

Let $H = \{\bar{a}\}$ be a cyclic fuzzy number group generated by $\bar{a}$.

Then, clearly $H = \{\bar{a}, \bar{a}^2, \bar{a}^3, \ldots, \bar{a}^n = e\}$

Moreover, $\bar{a} \in F^*(R)$ and therefore, by closure property, every integral power of $\bar{a}$ is in $F^*(R)$.

Consequently, $H \subseteq F^*(R)$ and $o[H] = o(F^*(R))$, So $H = F^*(R)$

Therefore $F^*(R)$ is cyclic. Hence $H$ is cyclic.

2.2.6. Theorem:

If $F^*(R) = \{\bar{a}\}$ be a cyclic fuzzy number group generated by $\bar{a}$ and $o[\bar{a}] = n$, then for some integer $m < n$.

$\bar{a}^m$ is a generator if and only if $m$ is relatively prime to $n$. 
Proof

Let $m$ be an integer less than $n$ and relatively prime to $n$. Then the H.C.F of $m$ and $n$ is 1. So, there exist integers $p$ and $q$ such that $mp + nq = 1$.

Let $H = \{ \tilde{a}^m \}$ be a cyclic fuzzy number group generated by $\tilde{a}^m$.

Then each element of $H$ is some integral power of $\tilde{a}^m$ and therefore of $\tilde{a}$. So each element of $H$ is in $F^*(R)$.

That is $H \subseteq F^*(R)$.

Now, $\tilde{a} = \tilde{a}^1 = \tilde{a}^{mp+nq} = [\tilde{a}^m]_p [\tilde{a}^n]_q$

$= [\tilde{a}^m]_p [\cdots \tilde{a}^n = e]$

Thus each integral power of $\tilde{a}$ is some integral power of $\tilde{a}^m$. So $F^*(R) \subseteq H$.

Thus, $F^*(R) = H$.

But, $H$ being the cyclic fuzzy number group, generated by $\tilde{a}^m$, so $F^*(R)$ is also a cyclic fuzzy number group, generated by $\tilde{a}^m$.

Conversely, let $\tilde{a}^m$ be also a generator of the cyclic fuzzy number group $F^*(R)$ generated by $\tilde{a}$, where $\text{ord}[\tilde{a}] = n$ and $m > n$.

Now each element of $F^*(R)$ is some integral power of $\tilde{a}^m$. 
In particular $\bar{a} = [\bar{a}^m]^r$, for some integer $r$.

But $[\bar{a}^m]^r = \bar{a}$ implies that $\bar{a}^m = \bar{a}$ implies that $\bar{a}^{mr+ns} = \bar{a} = \bar{a}^1$.

[since $\bar{a}^{ns} = (\bar{a}^n)^s = \bar{e}$]

This gives $mr + ns = 1$, showing that $m$ and $n$ are relatively prime.