CHAPTER II

ON k-EP MATRICES
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The concept of k-EP matrix is introduced. Relations between k-EP and EP matrices are discussed. Necessary and sufficient conditions are determined for a matrix to be k-EP. Necessary and sufficient conditions for sums of k-EP matrices to be k-EP are obtained. As an application, it is shown that sum and parallel sum of parallel summable k-EP matrices are k-EP. Set of conditions for the product of k-EP matrices to be k-EP are determined.
In this section, the concept of k-EP matrix as a generalization of k-hermitian and EP matrix is defined and a theory is developed for k-EP matrices. Relations between k-EP and EP matrices are discussed. Necessary and sufficient conditions are determined for a matrix to be k-EP. Equivalent characterizations of a k-EP matrix are discussed. As an application, it is shown that the class of all k-EP matrices having the same range space form a group under multiplication.

For \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \), let us define the function
\[ k(x) = (x_{k(1)}, x_{k(2)}, \ldots, x_{k(n)})^T \in \mathbb{C}^n \] where 'k' is the fixed product of disjoint transpositions in \( S_n \). Since 'k' is involutory, it can be verified that the associated permutation matrix \( K \) satisfy the following:

\[ K = K^T = K^{-1} \text{ and } k(x) = Kx. \]

\[ (KA)^+ = A^+K \text{ and } (AK)^+ = KA^+ \text{ for } A \in \mathbb{C}_{n \times n} \] (By Theorem (1.2.11))

**Definition 2.1.1:**

An \( A \in \mathbb{C}_{n \times n} \) is said to be k-EP if it satisfies the condition \( Ax = 0 \) \( \iff \) \( A^*k(x) = 0 \) (or) equivalently \( N(A) = N(A^*K) \). \( A \) is said to be k-EP, if \( A \) is k-EP and of rank \( r \).

In particular, when \( k(i) = i \) for each \( i=1 \) to \( n \), then the associated permutation matrix \( K \) reduces to the identity matrix and
Definition (2.1.1), reduces to $N(A) = N(A^*)$ which implies that $A$ is EP matrix [49]. If $A$ is nonsingular, then $A$ is $k$-EP for all transpositions 'k' in $S_n$.

**Remark 2.1.1:**

We note that a $k$-hermitian matrix $A$ is $k$-EP. For, if $A$ is $k$-hermitian, then by Theorem (1.2.22), $A = KA^*K$. Hence, $N(A) = N(KA^*K) = N(A^*K)$ which implies $A$ is $k$-EP. However the converse need not be true.

**Example 2.1.1:**

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. For a transposition $k = (1 \ 2)$ the associated permutation matrix $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$KA^*K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq A$.

Therefore $A$ is not $k$-hermitian. $A$ being a nonsingular matrix, is a $k$-EP matrix. Thus, the class of $k$-EP matrices is a wider class containing the class of $k$-hermitian matrices.
Theorem 2.1.1:

For $A \in \mathbb{C}_{n \times n}$ the following are equivalent:

1. $A$ is $k$-EP.
2. $KA$ is EP.
3. $AK$ is EP.
4. $A^+$ is $k$-EP.
5. $N(A) = N(A^+K)$.
6. $N(A^+) = N(AK)$.
7. $R(A) = R(KA^*)$.
8. $R(A^*) = R(KA)$.
9. $KA^+A = AA^+K$.
10. $A^+AK = KAA^+$.
11. $A = KA^*KH$ for a nonsingular $n \times n$ matrix $H$.
12. $A = HKA^*K$ for a nonsingular $n \times n$ matrix $H$.
13. $A^* = HKAK$ for a nonsingular $n \times n$ matrix $H$.
15. $C_n^+ = R(A) \oplus N(AK)$.
16. $C_n^+ = R(KA) \oplus N(A)$.

Proof:

The proof for the equivalence of (1), (2) and (3) runs as follows:
A is k-EP \iff N(A) = N(A^*K) \quad \text{(By Definition (2.1.1))}
< = > N(KA) = N(KA)^* \quad \text{(By (2.1.1))}
< = > KA is EP \quad \text{(By Definition of EP matrix in Table 1)}
< = > K(KA)K^* is EP \quad \text{(By Theorem (1.2.5))}
< = > AK is EP \quad \text{(By (2.1.1))}

Thus (1) \iff (2) \iff (3) hold.

(2) \iff (4):

KA is EP \iff (KA)^+ is EP \quad \text{(By Theorem (1.2.17))}
< = > A^+K is EP \quad \text{(By (2.1.2))}
< = > A^+ is k-EP \quad \text{(By equivalence of (1) and (3) applied to A^+)}

Thus equivalence of (2) and (4) is proved.

(1) \iff (5):

A is k-EP \iff N(A) = N(A^*K) \quad \text{(By Definition (2.1.1))}
< = > N(A) = N(KA)^* \quad \text{(By (2.1.1))}
< = > N(A) = N(KA)^+ \quad \text{(By Theorem (1.2.8))}
< = > N(A) = N(A^+K) \quad \text{(By (2.1.2))}

Thus, equivalence of (1) and (5) is proved. The other equivalences with (2) can be proved along the same lines and hence the proof is omitted.
Remark 2.1.2:

In particular, when $A$ is $k$-hermitian, then Theorem (2.1.1) reduces to result 2.1 of [19]. In result 2.8 of [19], it is stated that $A^+$ is $k$-hermitian for a $k$-hermitian matrix $A$ only when $A$ is normal. We note that, without any condition on $A$ to be normal the result follows:

$$A \text{ is } k\text{-hermitian} \iff A = KA^*K \iff A^+ = (KA^*K)^+$$

$$= K(A^+)^*K \iff A^+ \text{ is } k\text{-hermitian}.$$  

When $k(i) = i$, for each $i = 1$ to $n$, then Theorem (2.1.1) reduces to Theorem 1 of [4], Theorem 1 of [41], Theorem 1 of [21].

Further, the class of complex normal matrices is a subclass of EP matrices. However this is not the case with $k$-EP matrices (refer example (2.1.2)(iii)).

Example 2.1.2:

Let us consider $k = (1 \ 2)$.

Then $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; with reference to this $k$,

(i) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is EP as well as $k$-EP.
(ii) $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is k-EP but not EP.

(iii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is hermitian, normal and EP but not k-EP and hence not k-hermitian.

This motivates the following result.

**Theorem 2.1.2:**

Let $A \in \mathbb{C}_{nn}$, then any two of the following conditions imply the other one:

1. $A$ is EP.
2. $A$ is k-EP.
3. $R(A) = R(KA)$.

**Proof:**

First, we prove that, whenever (1) holds, then (2) and (3) are equivalent. Suppose (1) holds; then by definition of EP matrix, $R(A) = R(A^*)$, now by Theorem (2.1.1), $A$ is k-EP $\iff$ $R(A^*) = R(KA)$. Therefore, $A$ is k-EP $\iff$ $R(A) = R(KA)$. This completes the proof of (1) and (2) $\iff$ (3); (1) and (3) $\iff$ (2).
(2) and (3) => (1): Since $A$ is $k$-EP by Theorem (2.1.1), $KA$ is EP. Hence, $R(KA) = R(KA)^*$. By using (3), we have $R(A) = R(KA) = R(KA)^* = R(A^*K) = R(A^*)$. Again by definition, $A$ is EP. Thus (1) holds.

**Corollary 2.1.1:**

If $A \in C_{\text{ad}}$ is normal and $AA^*$ is $k$-EP, then $A$ is $k$-EP.

**Proof:**

Since $A$ is normal, $A$ is EP.

$AA^*$ is k-EP $\Rightarrow$ $R(AA^*) = R(KAA^*) = R(A) = R(KA)$. Then $A$ is k-EP follows from Theorem (2.1.2).

**Corollary 2.1.2:**

Let $E = E^* = E^2 \in C_{\text{ad}}$ be a hermitian idempotent that commutes with $K$, the permutation matrix associated with a fixed product of disjoint transpositions 'k' in $S_n$. Then, $H_k(E) = \{A: A$ is k-EP and $R(A) = R(E)\}$ forms a maximal subgroup containing $E$ as identity.

**Proof:**

Since $E = KE$, by (2.1.1) and (2.1.2), we have $E = KEK$ and $EE^* = E^2 = E = (KE)(EK) = (KE)(KE)^*$, then by Theorem (1.2.8), $R(E) = R(KE)$. $E$ being hermitian, is automatically EP and by Theorem (2.1.2), $E$ is k-EP. Thus $E \in H_k(E)$. For $AEH_k(E)$, $A$ is k-EP and
R(A) = R(E) = R(KE) = > AA^+ = EE^+ = E and AA^+ = E = (KE)(KE)^+ = KEE^+K^+ = KAA^+K^+ = (KA)(KA)^+ . Therefore by Theorem (1.2.8), R(A) = R(KA). Hence by Theorem (2.1.2), A is EP. Thus H_k(E) = H(E) = \{A : A is EP and R(A) = R(E)\}. By Theorem (1.2.18), H_k(E) forms a maximal subgroup containing E as identity.

**Remark 2.1.3:**

For 0 \not= E \not= I_n, by corollary 2.3 of [25], H_k(E) is a non-abelian group \( n > 2 \).

For \( A \in C_{nn} \), there exist unique k-hermitian matrices P and Q such that \( A = P + iQ \) where \( P = (1/2)(A + KA^*K) \) and \( Q = (1/2i)(A - KA^*K) \) (Refer 2.11 of [19]).

In the following Theorem, an equivalent condition for a matrix \( A \) to be k-EP is obtained in terms of P, the k-hermitian part of A.

**Theorem 2.1.3:**

For \( A \in C_{nn} \), \( A \) is k-EP \( \iff \) \( N(A) \subseteq N(P) \) where P is the k-hermitian part of A.

**Proof:**

If \( A \) is k-EP, then by Theorem (2.1.1), KA is EP. Since K is nonsingular, we have \( N(A) = N(KA) = N(KA^*) = N(A^*K) = N(KA^*K) \). Then, for \( x \in N(A) \), both KA\( x \) = 0 and KA^*K\( x \) = 0 which implies that
P(x) = (\frac{1}{2})(A + KA^*K)x = 0. Thus N(A) \subseteq N(P). Conversely, let N(A) \subseteq N(P) then Ax = 0, implies Px = 0, hence Qx = 0. Therefore, N(A) \subseteq N(Q). Thus, N(A) \subseteq N(P) \cap N(Q). Since both P and Q are k-hermitian, by Theorem (1.2.22), P = KP^*K and Q = KQ^*K. Hence N(P) = N(KP^*K) = N(P^*K) and N(Q) = N(KQ^*K) = N(Q^*K). Now N(A) \subseteq N(P) \cap N(Q) = N(P^*K) \cap N(Q^*K) \subseteq N(P^* - iQ^*)K. Therefore N(A) \subseteq N(A^*K) and \text{rk}(A) = \text{rk}(A^*K). Hence N(A) = N(A^*K). Therefore A is k-EP. Hence the Theorem.

Next, we derive equivalent conditions for a matrix to be k-EP.

To make the proof simpler, first let us prove certain lemmas.

**Lemma 2.1.1:**

Let $B \in \mathbb{C}^{n \times n}$ be of the form $B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ where $D$ is $r \times r$ nonsingular matrix.

Then the following are equivalent:

1. $B$ is k-EP.
2. $R(KB) = R(B)$.
3. $BB^*$ is k-EP.
4. $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$ where $K_1$ and $K_2$ are permutation matrices of order $r$ and $(n-r)$ respectively.
(5) $k = k_1k_2$ where $k_1$ is the product of disjoint transpositions in $S_n$ leaving $(r+1, r+2, \ldots, n)$ fixed and $k_2$ is the product of disjoint transpositions leaving $(1,2,\ldots,r)$ fixed.

**Proof:**

(1) $\iff$ (2): Since $B$ is $EP_r$, the equivalence (1) and (2) follows from Theorem (2.1.2).

(2) $\iff$ (3) follows from Theorem (2.1.1).

(2) $\iff$ (4): $R(KB) = R(B) \iff (KB)(KB)^+ = BB^+ \iff KBB^+K = BB^+ \iff KBB^+ = BB^+K$.

Let us partition $K = \begin{bmatrix} K_1 & K_3 \\ K_3^T & K_2 \end{bmatrix}$ where $K_1$ is $r \times r$.

$KBB^+ = BB^+K \iff K \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} K$

$\iff \begin{bmatrix} K_1 & 0 \\ K_3^T & 0 \end{bmatrix} = \begin{bmatrix} K_1 & K_3 \\ 0 & 0 \end{bmatrix}$

$\iff \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} = K$.

Thus, equivalence of (2) and (4) holds. The equivalence of (4) and (5) is clear from the definition of $'k'$. 
Remark 2.1.4:

In Lemma (2.1.1), the form of \( B \) is essential can be seen by the following example:

\[
B = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
\end{bmatrix}
\]
is not EP.

For \( K = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix} \), \( KB = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix} \) is EP.

Hence \( B \) is \( k \)-EP.

But \( K = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} \).

Lemma 2.1.2:

\( A \in \mathbb{C}^{n \times n} \) is \( k \)-EP, \( \iff \) There exist unitary matrix \( U \) and \( r \times r \) nonsingular matrix \( F \) such that

\[
A = KU \begin{bmatrix}
F & 0 \\
0 & 0 \\
\end{bmatrix} U^*.
\]

Proof:

Let us assume that \( A \) is \( k \)-EP. Then by Theorem (2.1.1), \( A^* = HKAK \), where \( H \) is a nonsingular matrix. Now \( A^*A = H(KA)(KA) \)
\( \iff \) \( A^*A = H(KA)^2 \iff \) \( \text{rk}(A^*A) = \text{rk}(H(KA)^2) \)
\( \iff \) \( \text{rk}(A^*A) = \text{rk}(KA)^2. \)
Over the complex field $A^*A$ and $A$ have the same rank. Therefore, 
$\text{rk}((KA)^2) = \text{rk}(A^*A) = \text{rk}(A) = \text{rk}(KA) \implies R(KA) \cap N(KA) = \{0\} \implies R(KA) \cap N(A) = \{0\}$. Thus $C_n = R(KA) \oplus N(A)$.

Choose an orthonormal basis $\{x_1, x_2, \ldots, x_n\}$ of $R(KA) = R(A^*)$ and extend it to a basis $\{x_1, x_2, \ldots, x_r, x_{r+1}, \ldots, x_n\}$ of $C_n$ where $\{x_{r+1}, \ldots, x_n\}$ is an orthonormal basis of $N(A)$.

If $(u, v)$ denotes the usual inner product on $C_n$ and $1 \leq i \leq r < j \leq n$ it follows that $x_i \in R(KA) = R(A^*) \implies x_i = A^*y$. Therefore, 
$(x_i, x_j) = (A^*y, x_j) = (y, Ax_j) = 0$ (since $x_j \in N(A)$). Hence $\{x_1, x_2, \ldots, x_n\}$ is an orthonormal basis of $C_n$. If we consider $KA$ as the matrix of a linear transformation relative to any orthonormal basis of $C_n$ then 

$$U^*KAU = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$$

where $F$ is $r \times r$ nonsingular matrix.

$\implies A = KU \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} U^*$.

Conversely, if $A = KU \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} U^*$, $U^* KAU = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$.

$N(KA) = N(KA)^*$ which implies $KA$ is EP and by Theorem (2.1.1), $A$ is $k$-EP.
Theorem 2.1.4:

$A \in C_{n \times n}$ is k-EP, where $k = k_1k_2 \iff A$ is unitarily k-similar to a diagonal block k-EP matrix $B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ where $D$ is $r \times r$ nonsingular matrix.

Proof:

Since $A$ is k-EP, by Lemma (2.1.2), there exist unitary matrix $U$ and a $r \times r$ nonsingular matrix $F$ such that $A = (KUK)K'F$. Since $k = k_1k_2$, the associated permutation matrix is $K = \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix}$. Hence

$A = KUK\begin{bmatrix} K_1F & 0 \\ 0 & 0 \end{bmatrix}U' = KUK\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}U'$ where $D = K_1F$.

Thus $A$ is unitarily k-similar to a diagonal block matrix $B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ where $D$ is $r \times r$ nonsingular. $B$ is k-EP, follows from Lemma (2.1.1).

Conversely, if $B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ with $D$ $r \times r$ nonsingular is k-EP, then again by using Lemma (2.1.1), $k = k_1k_2$ and $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$. Since $A$ is unitarily
k-similar to $B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ there exists unitary matrix $U$ such that $A = KUK \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$. Since $B$ is k-EP, then by Theorem (2.1.1),

$KB = K \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = U^*KAU$ is EP. By Theorem (1.2.5), $KA$ is EP. Now $A$ is k-EP follows from Theorem (2.1.1) and $\text{rk}(A) = r$. Hence $A$ is k-EP.

The proof is complete.

Remark 2.1.5:

As a particular case, for $k(i) = i$ for each $i=1$ to $n$,

Theorem (2.1.4) and Lemma (2.1.2) reduce to Theorem (1.2.12).

To conclude, we note that the k-spectral property (p. 21, [19]) holds for k-EP matrices.

Theorem 2.1.5:

If $A$ is k-EP, then $(\lambda, x)$ is a k-eigenvalue, k-eigenvector pair for $A$ $\iff$ $(1/\lambda, k(x))$ is a k-eigenvalue, k-eigenvector pair for $A^+$.

Proof:

$(\lambda, x)$ is a k-eigenvector, k-eigenvector pair for $A$

$\iff Ax = \lambda Kx$ \hspace{1cm} (By Definition (1.2.5))
\( \Rightarrow \) \( KAx = \lambda x \) \hspace{1cm} \text{(By (2.1.1))}

\( \Rightarrow \) \( (KA)^+x = (1/\lambda)x \) \hspace{1cm} \text{(By Theorem (1.2.23))}

\( \Rightarrow \) \( A^+Kx = (1/\lambda)x \) \hspace{1cm} \text{(By (2.1.2))}

\( \Rightarrow \) \( A^+k(x) = (1/\lambda)K(k(x)) \)

\( \Rightarrow \) \( (1/\lambda, k(x)) \) is a \( k \)-eigenvalue, \( k \)-eigenvector pair for \( A^+ \).
§2.2 SUMS OF k-EP MATRICES:

In this section, we give necessary and sufficient conditions for sums of k-EP matrices to be k-EP. As an application it is shown that sum and parallel sum of parallel summable k-EP matrices are k-EP.

Theorem 2.2.1:

Let $A_i$ (i = 1 to m) be k-EP matrices. Then $A = \sum_{i=1}^{m} A_i$ is k-EP if any one of the following equivalent conditions hold:

(i) $N(A) \subseteq N(A_i)$ for each i.

(ii) $\text{rk} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \text{rk}(A).$

Proof:

(i) $\iff$ (ii):

$N(A) \subseteq N(A_i)$ for each i implies $N(A) \subseteq \cap N(A_i)$. Since $N(A) = N(\sum A_i) \subseteq N(A_1) \cap N(A_2) \ldots \cap N(A_m)$, it follows that $N(A) \subseteq \cap N(A_i)$. Hence, $N(A) = \cap N(A_i) = N \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}$.
Therefore, \( \text{rk}(A) = \text{rk} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \) and (ii) holds. Conversely, since

\[
N \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \bigcap \text{N}(A_i) \subseteq \text{N}(A), \text{rk} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \text{rk}(A).
\]

\( \Rightarrow \) \( N(A) = \bigcap \text{N}(A_i) \). Hence, \( N(A) \subseteq N(A_i) \) for each i and (i) holds. Since each \( A_i \) is k-EP, \( N(A_i) = N(A_i^*K) \), for each i. Now \( N(A) \subseteq N(A_i) \) for each i.

\[
N(A) \subseteq \bigcap \text{N}(A_i) = \bigcap \text{N} (A_i^*K) \subseteq N(A^*K)
\]

and \( \text{rk}(A) = \text{rk}(A^*K) \).

Hence \( N(A) = N(A^*K) \). Thus \( A \) is k-EP. Hence the Theorem.

**Remark 2.2.1:**

In particular, if \( A \) is nonsingular the conditions automatically hold and \( A \) is k-EP. Theorem (2.2.1) fails if we relax the conditions on the \( A_i \)'s.

**Example 2.2.1:**

Consider \( A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Let \( k = (1 \ 2) \) the associated permutation matrix \( K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). \( KA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) is EP \( \Rightarrow A \) is k-EP.
KB = \[
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
\end{bmatrix}
\] is not EP. Therefore B is not k-EP.

A + B = \[
\begin{bmatrix}
1 & 0 \\
1 & 0 \\
\end{bmatrix}
\] and K(A + B) = \[
\begin{bmatrix}
1 & 0 \\
1 & 0 \\
\end{bmatrix}
\] which is not EP. Therefore (A + B) is not k-EP. However N(A + B) \subseteq N(A^*K) \subseteq N(A) and

N(A + B) \subseteq N(B^*K) \subseteq N(B). Moreover \( \text{rk} \begin{bmatrix} A \\ B \end{bmatrix} = \text{rk} (A + B). \)

**Remark 2.2.2:**

If rank is additive, that is \( \text{rk}(A) = \sum \text{rk}(A_i) \) then by Theorem (1.2.4), \( R(A_i) \cap R(A_j) = \{0\}, i \neq j \) which implies that \( N(A) \subseteq N(A_i) \) for each \( i \geq 1 \), \( N(A) \subseteq N(A_i^*K) \) for each \( i \).

Hence A is k-EP.

The conditions given in Theorem (2.2.1) are weaker than the condition of rank additivity can be seen by the following example.

**Example 2.2.2:**

Let \( A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \). Let the associated permutation matrix be \( K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). \( A + B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \). Hence A, B and (A + B) are k-EP matrices. Conditions (i) and (ii) in Theorem (2.2.1) hold, but \( \text{rk}(A + B) \neq \text{rk}(A) + \text{rk}(B) \).
Theorem 2.2.2:

Let \( A_i (i = 1 \text{ to } m) \) be \( k \)-EP matrices such that
\[
\sum_{i \neq j} A_i^* A_j = 0 \quad \text{then } A = \Sigma A_i \text{ is } k\text{-EP.}
\]

Proof:

Since \( \sum_{i \neq j} A_i^* A_j = 0 \)
\[
A^* A = (\Sigma A_i)^* (\Sigma A_i)
= (\Sigma A_i^*) (\Sigma A_i)
= \Sigma A_i^* A_i.
\]

\( N(A) = N(A^* A) = N (\Sigma A_i^* A_i) \)
\[
= N \left( \begin{bmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_m^* \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \right)
= N \left( \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \right)
= N(A_1) \cap N(A_2) \cap \ldots \cap N(A_m)
= N(A_1^* K) \cap N(A_2^* K) \cap \ldots \cap N(A_m^* K).
\]

Hence \( N(A) \subseteq N(A_i^* K) \) for each \( i \)
\[
= N(A_i) \text{ for each } i.
\]

Now, \( A \) is \( k \)-EP follows from Theorem (2.2.1).
Remark 2.2.3:

Theorem (2.2.2) fails if we relax the condition that $A_i$'s are k-EP. For, let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and let the associated permutation matrix be

$$K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Therefore $A$ is not k-EP. $KB = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is not EP. Therefore $B$ is not k-EP.

$A + B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$. $K(A + B) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ is not EP. Therefore, $(A + B)$ is not k-EP. But $A^*B + B^*A = 0$.

Remark 2.2.4:

The condition given in Theorem (2.2.2) implies those in Theorem (2.2.1) but not conversely. This can be seen by the following example.

Example 2.2.3:

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. For $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, both $A$ and $B$ are k-EP matrices.
\[ N(A + B) = N(K(A + B)) = N(KA + KB) \subseteq N(KA) = N(A). \]

Therefore \( N(A + B) \subseteq N(A) \). Also \( N(A + B) \subseteq N(B) \). But \( A^*B + B^*A = 0 \).

**Remark 2.2.5:**

The conditions given in Theorem (2.2.1) and Theorem (2.2.2) are only sufficient for the sum of k-EP matrices to be k-EP, but not necessary is illustrated by the following example.

**Example 2.2.4:**

Let \[ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \]. For \( K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \),

A and B are k-EP. Neither the conditions in Theorem (2.2.1) nor in Theorem (2.2.2) hold. However \( (A + B) \) is k-EP.

If A and B are k-EP matrices, then by Theorem (2.1.1), \( A^* = H_1KAK \) and \( B^* = H_2KAK \) where \( H_1 \) and \( H_2 \) are nonsingular \( n \times n \) matrices. If \( H_1 = H_2 \) then \( A^* + B^* = H_1K(A+B)K => (A+B)^* = H_1K(A+B)K = > (A+B) \) is k-EP by Theorem (2.1.1).

If \( (H_1-H_2) \) is nonsingular, then the above conditions are also necessary for the sum of k-EP matrices to be k-EP is given in the following Theorem.
Theorem 2.2.3:

Let $A^* = H_1KAK$ and $B^* = H_2KBK$ such that $(H_1-H_2)$ is nonsingular and $K$ be a permutation matrix, then $(A+B)$ is $k$-EP $\iff N(A+B) \subseteq N(B)$, where 'k' be the fixed transposition whose associative permutation matrix is $K$.

Proof:

Since $A^* = H_1KAK$ and $B^* = H_2KBK$, by Theorem (2.1.1), $A$ and $B$ are $k$-EP matrices. Since $N(A+B) \subseteq N(B)$, by Theorem (2.2.1), $(A+B)$ is $k$-EP. Conversely, let us assume that $(A+B)$ is $k$-EP. By Theorem (2.1.1), there exists a nonsingular matrix $G$ such that

$$(A+B)^* = GK(A+B)K$$

$=> A^*+B^* = GK(A+B)K$

$=> H_1KAK + H_2KBK = GK(A+B)K$

$=> (H_1KA + H_2KB)K = GK(A+B)K$

$=> (H_1KA + H_2KB) = GKA + GKB$

$=> (H_1K-GK)A = (GK-H_2K)B$

$=> (H_1G)KA = (G-H_2)KB$

$=> LKA = MKB$ where $L = H_1-G$, $M = G-H_2$.

Now $(L+M)(KA) = LKA + MKA = MKB + MKA = MK(A+B)$ and $(L+M)(KB) = LK(A+B)$.

By hypothesis, $L+M = H_1-G + G-H_2 = H_1-H_2$ is nonsingular.
Therefore,
\[ N(A+B) \subseteq N(MK(A+B)) \]
\[ = N((L+M)KA) \]
\[ = N(KA) \]
\[ = N(A). \]

Therefore, \( N(A+B) \subseteq N(A). \)

Also, \( N(A+B) \subseteq N(LK(A+B)) \)
\[ = N((L+M)KB) \]
\[ = N(KB). \]
\[ = N(B). \]

Therefore, \( N(A+B) \subseteq N(B). \)

Thus \( (A+B) \) is k-EP \( \Rightarrow \) \( N(A+B) \subseteq N(A) \) and \( N(A+B) \subseteq N(B). \)

Hence the Theorem.

Remark 2.2.6:

The condition \( (H_1,H_2) \) to be nonsingular is essential in Theorem (2.2.3) is illustrated in the following example.

Example 2.2.5:

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \] and \( B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \) are both k-EP matrices for \( K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

Further \( A^* = A = KAK \) and \( B^* = B = KBK \) \( \Rightarrow \) \( H_1 = H_2 = I. \)
Thus, Theorem (2.2.3) fails.

We note that when $k(i) = i$, for $i = 1$ to $n$ the permutation matrix $K$ reduces to $I$ and Theorem (2.2.1), Theorem (2.2.2) and Theorem (2.2.3) reduce to results found in [27].

PARALLEL SUMMABLE $k$-EP MATRICES:

Here, it is shown that, sum and parallel sum of parallel summable $k$-EP matrices are $k$-EP.

First, we shall quote the definition and some properties of parallel summable matrices [47] which are used in this section.

**Definition 2.2.1:**

$A$ and $B$ are said to be parallel summable (p.s) if $N(A+B) \subseteq N(B)$ and $N(A+B)^* \subseteq N(B^*)$ (or) equivalently $N(A+B) \subseteq N(A)$ and $N(A+B)^* \subseteq N(A^*)$.

**Definition 2.2.2:**

If $A$ and $B$ are parallel summable then parallel sum of $A$ and $B$ denoted by $A \boxplus B$ is defined as $A \boxplus B = A(A+B)^*B$. The product $A(A+B)^*B$ is invariant for all choices of generalized inverse $(A+B)^*$ of $(A+B)$ under the conditions that $A$ and $B$ are parallel summable (p.188,[47]).
Properties 2.2.1:

Let $A$ and $B$ be a pair of parallel summable (p.s) matrices. Then the following hold:

P.1 $A \oplus B = B \oplus A$.

P.2 $A^*$ and $B^*$ are p.s and $(A \oplus B)^* = A^* \oplus B^*$.

P.3 If $U$ is nonsingular then $UA$ and $UB$ are p.s and $UA \oplus UB = U(A \oplus B)$.

P.4 $R(A \oplus B) = R(A) \cap R(B)$;

R(A \oplus B) = N(A) \oplus N(B).

P.5 $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ if all the parallel sum operations involved are defined.

Lemma 2.2.1:

Let $A$ and $B$ be $k$-EP matrices. Then $A$ and $B$ are p.s $\iff$ $N(A+B) \subseteq N(A)$.

Proof:

$A$ and $B$ are p.s $\implies$ $N(A+B) \subseteq N(A)$ follows from Definition (2.2.1). Conversely, if $N(A+B) \subseteq N(A)$, then $N(KA+KB) \subseteq N(KA)$. Also $N(KA+KB) \subseteq N(KB)$. Since $KA$ and $KB$ are EP matrices and $N(KA+KB) \subseteq N(KA)$ and $N(KA+KB) \subseteq N(KB)$, by Theorem (2.2.1), $(KA+KB)$ is EP.
Hence $N(KA+KB)^* = N(KA+KB) = N(KA) \cap N(KB) = N(KA)^* \cap N(KB)^*$

Therefore, $N(KA+KB)^* \subseteq N(KA)^*$ and $N(KA+KB)^* \subseteq N(KB)^*$.

Also, $N(KA+KB) \subseteq N(KA)$ by hypothesis. Hence, by Definition (2.2.1), $KA$ and $KB$ are p.s.

$N(KA+KB) \subseteq N(KA)$

$N(K(A+B)) \subseteq N(KA)$

$N(A+B) \subseteq N(A)$.

Also,

$N(KA+KB)^* \subseteq N(KA)^*$

$N(K(A+B))^* \subseteq N(KA)^*$

$N(A+B)^* \subseteq N(A^*)$.

Therefore, $A$ and $B$ are p.s. Hence the Theorem.

**Remark 2.2.7:**

Lemma (2.2.1) fails if we relax the condition that $A$ and $B$ are $k$-EP. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Let the associated permutation matrix be $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $A$ is $k$-EP. $B$ is not $k$-EP. $N(A+B) \subseteq N(A)$ and $N(A+B) \subseteq N(B)$, but $N(A+B)^* \nsubseteq N(A^*)$; $N(A+B)^* \nsubseteq N(B^*)$. Hence, $A$ and $B$ are not parallel summable.
Theorem 2.2.4:

Let $A$ and $B$ be p.s k-EP matrices. Then $(A \oplus B)$ and $(A+B)$ are k-EP.

Proof:

Since $A$ and $B$ are p.s k-EP matrices, by Lemma (2.2.1), $N(A+B) \subseteq N(A)$ and $N(A+B) \subseteq N(B)$.

$$N(K(A+B)) \subseteq N(KA) \text{ and } N(K(A+B)) \subseteq N(KB).$$

$$N(KA+KB) \subseteq N(KA) \text{ and } N(KA+KB) \subseteq N(KB).$$

Therefore, $K(A+B) = (KA+KB)$ is EP. Then, $(A+B)$ is k-EP follows from Theorem (2.2.1).

Since $A$ and $B$ are p.s k-EP matrices $KA$ and $KB$ are p.s EP matrices. Therefore, $R(KA)^* = R(KA)$ and $R(KB)^* = R(KB)$.

$$R(KA \oplus KB)^* = R((KA)^* \oplus (KB)^*) \quad \text{(By P.2)}$$

$$= R((KA)^*) \cap R((KB)^*) \quad \text{(By P.4)}$$

$$= R(KA) \cap R(KB) \quad \text{(since KA and KB are EP)}$$

$$= R(KA \oplus KB).$$

Thus $(KA \oplus KB)$ is EP.

$=> K(A \oplus B)$ is EP $=> (A \oplus B)$ is k-EP.

Thus $(A \oplus B)$ is k-EP whenever $A$ and $B$ are k-EP.

Hence the Theorem.
Remark 2.2.8:

The sum and parallel sum of p.s k-EP matrices are k-EP.

Corollary 2.2.1:

Let A and B be k-EP matrices such that $N(A+B) \subseteq N(B)$. If C is k-EP commuting with both A and B, then C(A+B) and C(A±B) = (CA±CB) are k-EP.

Proof:

A and B are k-EP with $N(A+B) \subseteq N(B)$. By Theorem (2.2.1), (A+B) is k-EP. Now KA, KB and K(A+B) are EP. Since C commutes with A, B and (A+B), KC commutes with KA, KB and K(A+B) and by Theorem (1.3) of [21], K(CA), K(CB) and K(C(A+B)) are EP. Therefore, CA, CB and C(A+B) are k-EP. Now by Theorem (2.2.4), CA±CB is k-EP. By P.3 (properties (2.2.1)), $K(C(A\pm B)) = K(CA\pm CB)$. Since CA±CB is k-EP, K (CA ± CB) is EP => K(C(A±B)) is EP => C(A±B) is k-EP. Hence the Corollary.

Remark 2.2.9:

In particular for $k(i) = i$, Theorem (2.2.4) reduces to the following:

Corollary 2.2.2: (Theorem 4, [27])

Let A and B be p.s EP matrices. Then (A±B) and (A+B) are EP.
§2.3 PRODUCTS OF k-EP MATRICES:

It is well known that the product of nonsingular matrices is nonsingular. In general, the product of symmetric, hermitian, normal and EP matrices need not be respectively symmetric, hermitian, normal and EP matrices. Similarly, the product of k-EP matrices need not be k-EP. For instance, let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $A$ is k-EP, $B$ is k-EP. $AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not k-EP. In this section, we explore the conditions for the product of k-EP matrices to be k-EP. Also, we study the question of when $BA$ is k-EP, for k-EP matrices $A$, $B$ and $AB$.

**Theorem 2.3.1:**

Let $A_1$ and $A_n$ ($n > 1$) be k-EP matrices and let $A = A_1 A_2 A_3 A_4 \ldots A_n$. Then the following statements are equivalent:

(i) $A$ is k-EP.

(ii) $R(A_1) = R(A_n)$ and $rk(A) = r$.

(iii) $R(A_1^*) = R(A_n^*)$ and $rk(A) = r$.

**Proof:**

(i) $\leq \Rightarrow$ (ii):

Since $A_1$ and $A_n$ are k-EP, $R(A_1) = R(KA_1^*)$, $R(A_n) = R(KA_n^*)$. Since $A = A_1 A_2 \ldots A_n$, $R(A) \subseteq R(A_1)$ and
\[
\text{rk}(A) = \text{rk}(A_i) \implies R(A) = R(A_i). \text{ Also } A^* = A_1^* \ldots A_i^* \implies R(A^*) \subseteq R(A_{n}^*)
\]
and \(\text{rk}(A) = \text{rk}(A_{n}^*) = r \implies \text{rk}(A^*) = \text{rk}(A_{n}^*) = r.\) Therefore,
\[
R(A^*) = R(\text{Diag}(A^*)) \implies R(KA^*) = R(KA_{n}^*). \text{ Now,}
\]
\(\text{rk}(A^*) = \text{rk}(A_{n}^*) \implies \text{rk}(A_{n}^*) = r.\) A is \(k\)-EP \(\implies R(A) = R(KA^*) \text{ and } \text{rk}(A) = r \) (By Theorem (2.1.1))
\[
\implies R(A_i) = R(KA_{n}^*) \text{ and } \text{rk}(A) = r
\]
\[
\implies R(A_i) = R(A_{n}^*) \text{ and } \text{rk}(A) = r \text{ (By Theorem (2.1.1))}
\]

(ii) \(\implies (iii):\)
\[
R(A_i) = R(A_{n}^*) \implies R(KA_i^*) = R(KA_{n}^*)
\]
\[
\implies R(A_i^*) = R(A_{n}^*)
\]
Hence the Theorem.

**Corollary 2.3.1:**

Let \(A\) and \(B\) be \(k\)-EP \(\text{ matrices. Then } AB \text{ is } k\)-EP \(\implies \text{rk}(AB) = r \text{ and } R(A) = R(B).\)

**Proof:**

Proof follows from Theorem (2.3.1) for the product of two \(k\)-EP \(\text{ matrices } A \text{ and } B.\)

**Remark 2.3.1:**

In the above corollary both the conditions that \(\text{rk}(AB) = r \text{ and } R(A) = R(B)\) are essential for the product of two \(k\)-EP \(\text{ matrices to be } k\)-EP. This can be seen in the following example.
Example 2.3.1:

Let \( A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \)

\( KA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, KB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \) \( A \) is k-EP and \( B \) is k-EP.

\( AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) is not k-EP. Here, \( rk(AB) \neq 1. R(A) = R(B). \)

Example 2.3.2:

Let \( A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \)

\( A \) is k-EP and \( B \) is k-EP. \( R(A) \neq R(B). AB = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \) is not k-EP, \( rk(AB) = 1. \)

Remark 2.3.2:

In particular for \( k(i) = i, \) Corollary (2.3.1) reduces to Theorem (1.2.13).

Theorem 2.3.2:

Let \( rk(AB) = rk(B) = r_1 \) and \( rk(BA) = rk(A) = r_2. \) If \( AB, B \) are k-EP, \( A \) is k-EP then \( BA \) is k-EP.
Proof:

Since \( \text{rk}(BA) = \text{rk}(A) = r_2 \), it is enough to show that \( N(BA) = N((BA)^*K) \), to prove \( BA \) is \( k\text{-EP}_{r_2} \). Now, \( N(A) \subseteq N(BA) \) and \( \text{rk}(BA) = \text{rk}(A) = r_2 \Rightarrow N(A) = N(BA) \). Also, \( N(B) \subseteq N(AB) \) and \( \text{rk}(AB) = \text{rk}(B) = r_2 \Rightarrow N(B) = N(AB) \).

Now \( N(BA) = N(A) \)
\[
= N(A^*K) \quad \text{(since } A \text{ is } k\text{-EP})
\subseteq N(B^*A^*K)
= N((AB)^*K)
= N(AB) \quad \text{(since } AB \text{ is } k\text{-EP})
= N(B)
= N(B^*K) \quad \text{(since } B \text{ is } k\text{-EP})
\subseteq N((BA)^*K)
\]
\( N(BA) \subseteq N(BA^*K) \).

Further, \( \text{rk}(BA) = \text{rk}(BA)^* = \text{rk}((BA)^*K) = r_2 \Rightarrow N(BA) = N((BA)^*K) \).

Thus, \( BA \) is \( k\text{-EP}_{r_2} \).

Hence the Theorem.

Lemma 2.3.1:

If \( A, B \) are \( k\text{-EP}_r \) matrices and \( AB \) has rank \( r \), then \( BA \) has rank \( r \).
Proof:

By Theorem (1.2.3), \( \text{rk}(AB) = \text{rk}(B) - \dim (N(A) \cap N(B^*)^\perp) \).

Since \( \text{rk}(AB) = \text{rk}(B) = r, N(A) \cap N(B^*)^\perp = \{0\} \)

\( N(A) \cap N(B^*)^\perp = \{0\} \)
\( \Rightarrow N(A) \cap N(BK)^\perp = \{0\} \) (since \( B \) is \( k\)-EP)
\( \Rightarrow N(A)^\perp \cap N(BK) = \{0\} \)
\( \Rightarrow N(A^*)^\perp \cap N(BK) = \{0\} \) (since \( A \) is \( k\)-EP)

Now,

\( \text{rk}(BA) = \text{rk}((BK)(KA)) \)
\( = \text{rk}(KA) - \dim(N(BK) \cap N(A^*K)^\perp) \)
\( = \text{rk}(KA) - 0 \)
\( = \text{rk}(A) \)
\( = r. \)

Hence the Lemma.

Theorem 2.3.3:

If \( A, B \) and \( AB \) are \( k\)-EP matrices then \( BA \) is \( k\)-EP.

Proof:

Since \( A, B \) are \( k\)-EP matrices and \( \text{rk}(AB) = r, \) by Lemma (2.3.1), \( \text{rk}(BA) = r. \) Now the Theorem follows from Theorem (2.3.2) for \( r_1 = r_2 = r. \)
Corollary 2.3.2:

Let $A$, $B$ be $k$-$EP^r$ matrices. Then the following statements are equivalent:

(i) $AB$ is $k$-$EP^r$.
(ii) $(AB)^+$ is $k$-$EP^r$.
(iii) $A^+B^+$ is $k$-$EP^r$.
(iv) $B^+A^+$ is $k$-$EP^r$.

Remark 2.3.3:

In particular for $k(i) = i$, Theorem (2.3.3) reduces to the following:

Corollary 2.3.3: (Corollary 1, [4])

If $A$, $B$ and $AB$ are $EP^r$ matrices, then $BA$ is an $EP^r$ matrix.