CHAPTER I

INTRODUCTION
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§1.1 REVIEW OF LITERATURE:

In matrix theory, we come across many special types of matrices and one among them is the normal matrix, which plays an important role in the spectral theory of rectangular matrices and in the theory of generalized inverses.

In 1918, the concept of a normal matrix with entries from the complex field was introduced by O. Toeplitz [50] who gave a necessary and sufficient condition for a complex matrix to be normal. Since then, many researchers have developed the concept and many generalizations of normality were studied [3, 15, 20].

As a generalization of normality the concept of EP matrices over the complex field was introduced by Schwerdtfeger [49]. The class of complex EP matrices includes the class of all nonsingular matrices, hermitian matrices and normal matrices.

A complex matrix \( A \) of order \( n \) is called EP if the range spaces of \( A \) and \( A^* \) are equal (\( A^* \) denotes the conjugate transpose of \( A \)). Greville [10] termed an EP matrix as a range hermitian matrix. Pearl [43] has proved that \( A \) is an EP matrix if and only if \( A \) commutes with \( A^+ \), that is, \( AA^+ = A^+A \) (\( A^+ \) is the unique solution of the four Moore-Penrose
equations \[45\] \(AXA = A, XAX = X, (AX)^* = AX\) and \((XA)^* = (XA)\).
Since \(AA^+\) and \(A^+A\) are the orthogonal projectors onto the spaces \(R(A)\) and \(R(A^*)\), the matrix \(A\) is named as an equiprojector matrix (or) in short 'EP' matrix.

EP matrices have wide applications in the theory of generalized inverses, since the class of EP matrices is the larger class for which spectral property holds and for EP matrices with same range space, reverse order law holds. These properties are not valid in general for generalized inverses of arbitrary matrices [9]. Pearl [43] has proved that \(A\) is EP if and only if \(A^+\) can be expressed as a polynomial in \(A\) with scalar co-efficients and deduced that \(A\) is EP if and only if \(A^+ = A^\#\) (\(A^\#\) is the group inverse of \(A\) which satisfy the equations \(AXA = A, XAX = X\) and \(AX = XA\)). The class of all matrices for which group inverse exists is denoted as G.P. It is well known that \(A \in G.P\) if and only if \(\text{rank}(A) = \text{rank}(A^2)\) [5]. In [15], Hearon has studied about the construction of EP\(_r\) (EP matrix with rank \(r\)) generalized inverse by inversion of nonsingular matrices.

In general, the product of EP matrices need not be EP and sums of EP matrices need not be EP. Many attempts have been made on these aspects. Ballantine [3], Baskett and Katz [4] and Meenakshi [26] have studied about the products of complex EP\(_r\) matrices. Integral EP\(_r\) matrices were investigated by Meenakshi [28]. Necessary and sufficient conditions for sums of EP matrices to be EP were obtained by Meenakshi [27]. A pair of matrices \(A\) and \(B\) are parallel summable if the product \(A(A+B)^*B\) remains
invariant for all choices of generalized inverse \((A+B)^-\) of \((A+B)\) and parallel sum of \(A\) and \(B\) is denoted as \(A \mp B\) (p.188, [47]). The class of parallel summable matrices includes the class of hermitian semi definite matrices. In [27], Meenakshi has proved that the parallel sum of a pair of parallel summable EP matrices is EP.

For a complex matrix \(M\) partitioned in the form
\[
M = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
a Schur complement of \(A\) in \(M\) denoted by \((M/A)\) is defined as \(D - CA^-B\) [6] where \(A^-\) is a generalized inverse of \(A\) satisfying \(AXA = A\). For properties of Schur complement one may refer [6, 8, 24, 29, 33]. Relationships among matrices \(A, B, C, D\) and \((M/A)\) in terms of null spaces are \(N(A) \subseteq N(C)\), \(N(A^*) \subseteq N(B^*)\), \(N((M/A)^*) \subseteq N(C^*)\) and \(N(M/A) \subseteq N(B)\). It is well known (p. 21, [47]), that \((M/A)\) is invariant for all choices of \(A^-\) of \(A\) if and only if \(A\) satisfy \(C = CA^-A\) and \(B = AA^-B\), for every \(A^-\) of \(A\).

In [26], Meenakshi has studied on the product of EP, partitioned matrix of the form
\[
M = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
with \(\text{rank } (M) = \text{rank } (A)\) and in [29], she has established equivalent conditions for \((M/A)\) to be EP matrix in an EP matrix
\[
M = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
with \(\text{rank}(M) = \pm \text{rank}(A)\). In [30], necessary and sufficient conditions are determined for all the principal submatrices and their Schur complement to be EP in a partitioned matrix. Similar study for integral matrices were also made by her in [33]. In [31],
Meenakshi has developed necessary and sufficient condition for partitioned matrix of the form \( M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) to be almost definite (a.d). In particular when \( \text{rank}(M) = \text{rank}(A) \), she has shown that under certain conditions \( M \) is a.d \( \iff \) \( M \) is EP.

The concept of a complex EP matrix was extended to matrices over an arbitrary field by Pearl [41, 42, 44], for elements in a ring with involution by Hartwig [11], for matrices over a skew field with involution by Zhuang [51], for polynomial matrices over the field \( F(\lambda) \) whose elements are rational functions of the form \( f(\lambda)/g(\lambda) \) where \( f(\lambda), g(\lambda) \neq 0 \) are polynomials in \( \lambda \) by Meenakshi and Anandam [35], for morphism of a category with involution by Robinson and Puystjens [48]. Katz and Pearl [21] had pointed out that for a matrix over an arbitrary field, the concepts of normal and EP are independent and obtained conditions for their equivalence. Algebraic structure of EP matrices over an arbitrary field was determined by Meenakshi [25] and that of EP morphisms having the same kernel was studied in [38].

In recent years, generalization of EP matrices have found place in the literature. Campbell and Meyer [7] introduced the concept of bi-EP. A complex matrix \( A \) of order \( n \) is called bi-EP if \( [AA^+, A^+A] = 0 \) (For a pair of matrices \( A \) and \( B \), \( [A,B] = AB-BA \) is the commutator of \( A \) and \( B \)). An EP matrix is automatically a bi-EP matrix. The interrelationship between EP, bi-EP and G.P are discussed in [7]. Hartwig [11] has generalized the
concept of EP elements of a \(^*-\)regular ring to \(E^q\) elements for \(q \geq 1\). Meenakshi \([32]\) has studied the relation between EP elements and \(E^q\) elements in a \(*\)-regular ring. The relations between EP-\(\lambda\)-matrices and \(E^q\)-\(\lambda\)-matrices over an arbitrary field \(F\) were exhibited by Meenakshi and Anandam \([36]\). A is said to be an \(E^q\)-\(\lambda\)-matrix for some \(q \geq 1\) if 
\[ A^q A^+ = A^+ A^q \quad \text{and} \quad (A^+)^q A = A (A^+)^q. \]
In particular for \(q = 1\), this definition coincides with the definition of EP-\(\lambda\)-matrices \([35]\) over the polynomial domain \(F[\lambda]\). The concept of conjugate EP (Con-EP) or range symmetric matrix is introduced and developed by Meenakshi and Indira \([37]\) which coincides with range hermitian matrix, in particular for real matrices.

In \([16]\), Hestenes has developed a spectral theory for a rectangular matrix \(A\), through elementary matrices (A matrix \(T\) is elementary if \(T = TT^*T\) and plays the role of identity matrix) analogous to that given in the hermitian case and which reduces to the usual spectral theory when \(A\) is hermitian. Hestenes has treated \(TA^*T\) as the conjugate transpose of \(A\) relative to \(T\) and defined that \(A\) is hermitian matrix relative to \(T\) if \(A = TA^*T\). A matrix \(A\) is said to be normal relative to an elementary matrix \(T\) if \(A = AT^*T = TT^*A\) and \(AA^*T = TA^*A\).

Recently, Richard D.Hill and Stevan R.Waters \([19]\) have developed a theory for \(k\)-real and \(k\)-hermitian matrices, where '\(k\)' is a fixed product of disjoint transpositions in \(S_n\), the set of all permutations on \(\{1, 2, \ldots, n\}\). A complex matrix \(A = (a_{ij})\) of order \(n\) is said to be \(k\)-hermitian if and only if \(a_{ij} = \overline{a}_{k(0), k(i)}\) for all \(1 \leq i, j \leq n\). In \([19]\), some basic results
for k-hermitian matrices, their spectral properties and characterizations of linear transformations which preserve them are discussed. The authors have provided examples to illustrate that in general k-hermitian and k-real matrices are not normal.

Hermitian matrices and perhermitian matrices [18] are special cases of k-hermitian matrices with \( k(i) = i \) and \( k(i) = n-i+1 \) respectively. Further, real matrices and centrohermitian matrices [17] are special cases of k-real matrices with \( k(i) = i \) and \( k(i) = n-i+1 \) respectively. In [19], it is shown that, for a k-hermitian matrix \( A \), \( A = KA^*K \), where \( K \) is the permutation matrix associated with the fixed transposition \( 'k' \). Since \( K \) is also an elementary matrix, a matrix being k-hermitian is equivalent to that, it is hermitian relative to \( K \). Thus, k-hermitian matrix is a particular case of relative hermitian matrix studied by Hestenes [16].

In the present work, the concept of k-EP (or) range k-hermitian matrix is introduced as a generalization of k-hermitian matrices where \( 'k' \) is the fixed product of disjoint transpositions in \( S_n \), the set of all permutations on \( \{1, 2, \ldots, n\} \). A theory for k-EP matrices is developed which reduces to that of EP matrices as a special case when \( 'k' \) is the identity transposition. Characterizations of a k-EP matrix analogous to that of an EP matrix are determined. Relations between k-EP and EP matrices are discussed. Necessary and sufficient conditions are established for a matrix to be k-EP. The conditions for the sums and products of k-EP matrices to be k-EP are investigated. Necessary and sufficient conditions for
the product of \( k\)-EP\(_r\) partitioned matrices to be \( k\)-EP\(_r\) and for Schur complement \( (M/A) \) of \( A \) in a partitioned matrix \( M \) of the form

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

to be \( k\)-EP are obtained. Necessary sufficient conditions for a \( k\)-EP matrix to have its principal sub-matrices and their Schur complement to be \( k\)-EP are determined. As an application it is shown that the property of a matrix being \( k\)-EP\(_r\) is preserved under the principal pivot transformation. Partial orderings on matrices which are invariant under 'k' are established. Some well known inequalities on hermitian positive semi definite matrices are extended for \( k\)-EP matrices. Various generalized inverses and in particular the group inverse of a \( k\)-EP matrix to be \( k\)-EP are analysed.
§1.2 NOTATIONS AND PRELIMINARIES:

In this section, the notations, definitions and theorems used in the thesis are given. Throughout, it is concerned with complex square matrices.

\( C_{nxn} \) : The space of nxn complex matrices of order n.
\( C_n \) : The space of complex n-tuples.
\( I_n \) : Identity matrix of order n.
\( O \) : Zero matrix of appropriate size.
\( S_n \) : The set of all permutations on \( \{1, 2, \ldots, n\} \).

For \( A \in C_{nxn} \),

\( \dim (A) \) : The dimension of A.
\( \det (A) \) : Determinant of A.
\( \text{rk}(A) \) : Rank of A is the maximum number of linearly independent rows or columns of A.
\( R(A) \) : Range space of A = \( \{y \in C_n / y = Ax \text{ for some } x \in C_n\} \).
\( N(A) \) : Null space of A = \( \{x \in C_n / Ax = 0\} \).
\( A^T \) : The transpose of A.
\( \overline{A} \) : The conjugate of A.
\( A^* \) : The conjugate transpose of A.
\( A^- \) : 1-inverse of A, is a solution of the equation \( AXA = A \).
\( A\{1\} \) : Set of all 1 - inverses of A.
A" : (1, 2)-inverse of A, is a solution of the equations AXA = A and XAX = X.

A{1,2} : Set of all (1,2) - inverses of A.

A{1,2,3} : Set of all (1,2,3) - inverses of A, is a solution of the equations
AXA = A, XAX = X and AX = (AX)*.

A{1,2,4} : Set of all (1,2,4)-inverses of A, satisfying the equations
AXA = A, XAX = A and XA=(XA)*.

A+: Moore-Penrose inverse of A is the unique solution of the
following equations: AXA = A, XAX = X, (AX)* = AX and
(XA)* = XA.

A⁺ exists and is unique for AeC_{nn}.

A‡ : Group inverse of A, satisfying the equations AXA = A,
XAX = X and AX = XA. If A‡ exists, then it is unique.

Definitions of some special matrices are given in Table I,
where A is taken to be a square matrix.
### TABLE - I

<table>
<thead>
<tr>
<th>Type of matrix A</th>
<th>Definitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hermitian</td>
<td>$A = A^*$</td>
</tr>
<tr>
<td>Idempotent</td>
<td>$A^2 = A$</td>
</tr>
<tr>
<td>Normal</td>
<td>$AA^* = A^*A$</td>
</tr>
<tr>
<td>Unitary</td>
<td>$AA^* = A^*A = I$</td>
</tr>
<tr>
<td>EP (or) range hermitian</td>
<td>$N(A) = N(A^<em>)$ (or) $R(A) = R(A^</em>)$</td>
</tr>
<tr>
<td>EP_r</td>
<td>$N(A) = N(A^*)$ and $rk(A) = r$</td>
</tr>
<tr>
<td>Hermitian positive definite (h.p.d)</td>
<td>$x^*Ax &gt; 0$ for all non-zero $x \in \mathbb{C}^n$</td>
</tr>
<tr>
<td>Positive semi definite</td>
<td>$\text{Re}(x^*Ax) \geq 0$ for all non-zero $x \in \mathbb{C}^n$</td>
</tr>
<tr>
<td>Hermitian positive semi definite (h.p.s.d)</td>
<td>$x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$</td>
</tr>
</tbody>
</table>

$A \bigoplus B$: The block diagonal matrix

\[
\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}
\]

$V^\perp$: Orthogonal complement of subspace $V$

Throughout 'k' refers to a fixed product of disjoint transposition in $S_n$. $K$ be the permutation matrix associated with 'k' unless otherwise specified.
Definition 1.2.1:

A permutation matrix is one which is obtained from the identity matrix by performing elementary row or column operations.

Theorem 1.2.1: [5]

Let \( A \in \mathbb{C}^{n \times n} \) be of rank \( r \). Then there exist nonsingular matrices \( P \) and \( Q \) such that

\[
A = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q
\]

Theorem 1.2.2: (Full rank factorization) (p. 22, [5])

Let \( A \in \mathbb{C}^{n \times n} \) be of rank \( r \). Then there exist matrices \( F \in \mathbb{C}^{n \times r} \) and \( G \in \mathbb{C}^{r \times n} \) both of rank \( r \) such that \( A = FG \).

Theorem 1.2.3: [5]

For \( A, B \in \mathbb{C}^{n \times n} \), the following hold:

(i) \( \text{rk}(AA^*) = \text{rk}(A^*A) = \text{rk}(A) = \text{rk}(A^T) = \text{rk}(A^+) = \text{rk}(\overline{A}) = \text{rk}(A^+) \).

(ii) \( \text{rk}(AB) = \text{rk}(B) - \dim(N(A) \cap N(B^*)) \).
Theorem 1.2.4: [23]

For $A, B \in \mathbb{C}_{n \times n}$, the following statements are equivalent:

(i) $\text{rk}(A + B) = \text{rk}(A) + \text{rk}(B)$.
(ii) $\dim(\text{R}(A) \cap \text{R}(B)) = 0$.

Theorem 1.2.5: [4]

Let $A, B \in \mathbb{C}_{n \times n}$ and $U \in \mathbb{C}_{n \times n}$ be any nonsingular matrix. Then

(i) $\text{R}(A) = \text{R}(B) \iff \text{R}(UAU^*) = \text{R}(UBU^*)$.
(ii) $\text{N}(A) = \text{N}(B) \iff \text{N}(UAU^*) = \text{N}(UBU^*)$.

Theorem 1.2.6: [5]

Let $A, B \in \mathbb{C}_{n \times n}$. Then we have the following:

(i) $\text{R}(AB) \subseteq \text{R}(A), \text{N}(B) \subseteq \text{N}(AB)$.
(ii) $\text{R}(AB) = \text{R}(A) \iff \text{rk}(AB) = \text{rk}(A)$ and $\text{N}(AB) = \text{N}(B) \iff \text{rk}(AB) = \text{rk}(B)$.
(iii) $\text{N}(A) = \text{N}(A^*A)$ and $\text{R}(A) = \text{R}(AA^*)$.

Theorem 1.2.7: (p.21, [47])

Let $A, B \in \mathbb{C}_{n \times n}$. Then

(i) $\text{N}(A) \subseteq \text{N}(B) \iff \text{R}(B^*) \subseteq \text{R}(A^*)$

$\iff B = BA^*A$ for all $A \in \mathbb{C}_{n \times n} \setminus \{0\}$. 
(ii) \( N(A^*) \subseteq N(B^*) \iff R(B) \subseteq R(A) \iff B = AA^*B \) for every choice of \( A \in A \{1\} \).

**Theorem 1.2.8:** [5]

For \( A, B \in \mathbb{C}_{nn} \), the following statements hold:

(i) \( R(A^+) = R(A^*) \) and \( N(A^+) = N(A^*) \).

(ii) \( R(A) = R(B) \iff AA^+ = BB^+ \).

**Theorem 1.2.9:** (p. 24, [5])

For any \( A \in \mathbb{C}_{nn} \), the following hold:

(i) \( (A^*)^+ = A \).

(ii) \( A^+ = A^{-1} \iff A \) is nonsingular.

(iii) \( (A^*)^+ = (A^+)^* \).

**Theorem 1.2.10:** (p. 68 [5])

For \( A, B \in \mathbb{C}_{nn} \), \( N(A^*)^\perp = R(A) \).

**Theorem 1.2.11:** (p. 25, [5])

If \( U \) and \( V \) are unitary matrices of order \( n \), then for any matrix \( A \in \mathbb{C}_{nn} \), \( (UAV)^* = V^*A^+U^* \).
Theorem 1.2.12: (p. 91, [4])

For $A \in \mathbb{C}^{n \times n}$, the following statements are equivalent:

(i) $A$ is an EP$_r$ matrix.

(ii) There exists a unitary $U \in \mathbb{C}^{n \times n}$ such that $A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$ where $D \in \mathbb{C}^{r \times r}$ is nonsingular.

Theorem 1.2.13: (p. 90, [4])

Let $A$ and $B \in \mathbb{C}^{n \times n}$ be EP$_r$ matrices. Then $AB$ is an EP$_r$ matrix $\iff R(A) = R(B)$.

Theorem 1.2.14: (p. 162, [5])

Let $A \in \mathbb{C}^{n \times n}$. Then group inverse $A^g$ exists $\iff \text{rk}(A) = \text{rk}(A^2)$.

Theorem 1.2.15: (p. 164, [5])

Let $A \in \mathbb{C}^{n \times n}$. Then $A$ is EP $\iff A^g = A^+$ when $A^g$ exists.

Theorem 1.2.16: (p. 163, [5])

Let $A = FG$ be a full rank factorization of $A \in \mathbb{C}^{n \times n}$. Then $A$ has group inverse $A^g$ $\iff GF$ is nonsingular in which case $A^g = F(GF)^2 G$. 
Theorem 1.2.17: (p. 163, [5])

Let $A \in C_{\text{nxn}}$. $A$ is EP $\iff$ $A^+$ is EP.

Theorem 1.2.18: (Theorem 2.1, [25])

Let $E = E^* = E^2C_{\text{nxn}}$. Then $H(E) = \{A: A$ is EP and $R(A) = R(E)\}$ forms a maximal subgroup containing $E$ as identity.

Definition 1.2.2: [6]

Let $M$ be an nxn matrix of the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. A Schur complement of $A$ in $M$ is $(M/A) = D - CA^{-1}B$.

Theorem 1.2.19: [8]

For an nxn matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ the following are equivalent:

(i) $\text{rk}(M) = \text{rk}(A)$.

(ii) $N(A) \subseteq N(C)$, $N(A^*) \subseteq N(B^*)$ and $(M/A) = 0$.

Theorem 1.2.20: (Theorem 1, [8])

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then,

$\text{rk}(M) \geq \text{rk}(A) + \text{rk}(M/A)$,
with equality if and only if
\[ N(M/A) \subseteq N(I-AA^+)B, \]
\[ N((M/A)^*) \subseteq N((I-A^+A)C^*), \]
and \((I-AA^+)B(M/A)^+ \subseteq (I-A^+A) = 0.\)

In particular, we have equality if \(M\) satisfies \(N(A) \subseteq N(C)\)
and \(N(A') \subseteq N(B').\)

**Theorem 1.2.21:** (Theorem 1[6] and [46])

Let \(M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\). Then,
\[
M^+ = \begin{bmatrix}
A^+ + A^+ B(M/A)^+ CA & -A^+ B(M/A)^+ \\
-(M/A)^+ CA^+ & (M/A)^+
\end{bmatrix}
\]
\[
\iff N(A) \subseteq N(C), N(A^*) \subseteq N(B^*), N((M/A)^*) \subseteq N(C^*) \text{ and } N((M/A)) \subseteq N(B).
\]
Also,
\[
M^+ = \begin{bmatrix}
(M/D)^+ & -A^+ B(M/A)^+ \\
-D^+ C(M/D)^+ & (M/A)^+
\end{bmatrix}
\]
\[
\iff N(A) \subseteq N(C), N(A^*) \subseteq N(B^*), N((M/A)^*) \subseteq N(C^*) \text{ and } N((M/A)) \subseteq N(B)
\]
and \(N(D) \subseteq N(B), N(D^*) \subseteq N(C^*), N((M/D)^*) \subseteq N(B^*), N((M/D)) \subseteq N(C).\)

When \(rk(M) = rk(A)\), then
\[ M = \begin{bmatrix} A & B \\ C & CA^{-B} \end{bmatrix} \quad \text{and} \quad M^+ = \begin{bmatrix} A^*PA^* & A^*PC^* \\ B^*PA^* & B^*PC^* \end{bmatrix} \]

where \( P = (AA^*+BB^*)^{-1}A(A^*A+C^*C)^{-1} \).

**Definition 1.2.3:**

For a square matrix \( A \) of order \( n \), the eigenvalues of \( A \) are the roots of the equation \( \det (\lambda I_n - A) = 0 \).

**Definition 1.2.4:** (p. 17, [19])

\( A \in \mathbb{C}^{n \times n} \) is said to be \( k \)-hermitian if and only if \( a_{ij} = \overline{a_{kj}} \), \( i, j = 1, \ldots, n \), where \( k \) is a fixed product of disjoint transpositions in \( S_n \), the set of permutations on \( \{1, 2, \ldots, n\} \).

**Theorem 1.2.22:** (Result 2.1, [19])

For \( A \in \mathbb{C}^{n \times n} \) and \( K \) is the permutation matrix associated with the fixed transposition \( 'k' \) in \( S_n \), the following are equivalent:

(i) \( A \) is \( k \)-hermitian.

(ii) \( A = KA^*K \).

(iii) \( KA \) is hermitian.

(iv) \( AK \) is hermitian.

(v) \( A \) is hermitian relative to \( K \).
Definition 1.2.5: (p. 21, [19])

For $A \in \mathbb{C}_{n \times n}$, the roots of the $k$-characteristic equation $\det(\lambda K - A) = 0$ are called $k$-eigenvalues of a matrix $A$. The corresponding $k$-eigenvectors is a non-zero $x \in \mathbb{C}^n$ such that $(\lambda K - A)x = 0$ or equivalently $Ax = \lambda Kx$.

Theorem 1.2.23:

If $A \in \mathbb{C}_{n \times n}$ is EP, $\lambda$ is an eigenvalue of $A$ $\iff$ $1/\lambda$ is an eigenvalue of $A^+$. 

Definition 1.2.6: (p. 22, [19])

$A, B \in \mathbb{C}_{n \times n}$ are said to be $k$-similar if and only if there exists a nonsingular $P \in \mathbb{C}_{n \times n}$ such that $B = KP^{-1} KAP$. Note that $A$ is $k$-similar to $B$ via $P$ if and only if $KA$ is similar to $KB$ via $P$. 
§1.3 SUMMARY OF RESULTS:

In this section, a short account of the results obtained in this thesis are given. Throughout 'k' is a fixed product of disjoint transpositions in $S_n$ and $K$ is the permutation matrix associate with 'k'.

**k-EP MATRICES:**

The concept of k-EP matrix is introduced for complex matrices and exhibited as a generalization of k-hermitian and EP matrices. Many of the basic results on k-hermitian matrices were extended for the larger class of k-EP matrices. Equivalent conditions for a matrix to be k-EP are determined. Relations between EP matrices and k-EP matrices are studied.

**SUMS OF k-EP MATRICES:**

Necessary and sufficient conditions are determined for a sum of k-EP matrices to be k-EP, analogous to the results of Meenakshi [27] on EP matrices. In particular, for a pair of parallel summable k-EP matrices, the conditions established automatically hold and it is deduced that their sum and parallel sum are k-EP.

**PRODUCT OF k-EP MATRICES:**

Equivalent conditions for a product of k-EP matrices to be k-EP are obtained. As a special case it includes the results of Baskett and
Katz [4]. It is shown that, if A, B and AB are k-EP matrices then BA, A+B+, B+A+ are all k-EP matrices.

SCHUR COMPLEMENT IN A k-EP MATRIX:

Necessary and sufficient conditions are determined for a Schur complement (M/A) of a matrix A in the partitioned matrix M = [A B; C D] to be k-EP, for 'k' whose associated permutation matrix is of the form K = [K₁ 0; 0 K₂] for the cases rk(M) = rk(A) and for M and K of the above form, with rk(M) = rk(A) = r, it is proved that M is k-EP if any only if A is k₁-EP and CA+K₁ = (A+BK₂)*.

FACTORIZATION OF k-EP MATRICES:

It is established that a matrix of the form M = [A B; C D] with rk(M) = rk(A) = r can be expressed as a product of k-EP matrices. Necessary and sufficient conditions are determined for the product of two k-EP partitioned matrices to be k-EP. The results are analogous to that of EP matrices studied by Baskett and Katz [4] and Meenakshi [26].

PIVOTAL TRANSFORM ON k-EP MATRICES:

Necessary and sufficient conditions for a k-EP matrix to have its principal sub-matrices and their Schur complement to be k-EP are
determined. This is a generalization of the result found in [30]. As an application, it is shown that the property of a matrix being $k$-EP, is preserved under the principal pivot transformation.

**INEQUALITIES ON $k$-EP MATRICES:**

It is shown that all standard partial orderings are preserved under 'k'. Conditions for all those matrices that lie below (or) above a given $k$-EP matrix relative to a particular matrix partial ordering to be $k$-EP are determined. Some matrix inequalities involving almost positive $k$-definite matrices are derived.

**GENERALIZED INVERSE OF A $k$-EP MATRIX:**

The existence of the group inverse of a $k$-EP matrix are investigated. Various generalized inverses of a $k$-EP matrix to be $k$-EP are discussed. If $A$ and $B$ are $k$-EP, matrices, necessary and sufficient conditions for the reverse order law $(AB)^+ = B^+A^+$ to hold are determined.