CHAPTER IV

INEQUALITIES ON k-EP MATRICES
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In this chapter, it is shown that all standard partial orderings are preserved under 'k'. Also, conditions for all those matrices that lie below (or) above a given k-EP matrix relative to a particular matrix partial ordering to be k-EP are determined. Inequalities $M \geq B^*MB$ where $M$ is a k-EP matrix and $B$ is a complex matrix are derived.
4.1 k-INVARIANT PARTIAL ORDERINGS ON MATRICES:

In this section, first it is shown that all standard partial orderings are preserved under 'k'. Next, it is shown that all the standard partial orderings are preserved for 'k'-unitary similarity.

The characterizations of the ordering can be done in different ways. The Lowener partial order, the star order and the minus order (or) rank subactivity order denoted by \( \geq_L \), \( \geq^* \) and \( \geq_{rs} \) respectively are defined as follows:

**Definition 4.1.1:**

For \( A, B \in \mathbb{C}^{n \times n} \),

(i) \( A \geq^*_L B \) if \( A-B \geq 0 \).

(ii) \( A \geq^* B \) if \( B^*B = B^*A \) and \( BB^* = AB^* \).

(iii) \( A \geq_{rs} B \) if \( \text{rk}(A-B) = \text{rk}(A) - \text{rk}(B) \).

The relationship between the star and minus orderings are studied by Baksalary [1], Mitra [39], Mitra and Puri [40] and Hartwig et al. [12, 13].

In the sequel, the following known results will be used.
Result 4.1.1: [14]

For $A, B \in \mathbb{C}^{n \times n}$,

$$ A \geq B \iff \rho(A+B) \leq 1 \text{ and } R(B) \subseteq R(A) $$

where $\rho(A) = \max \{ |\lambda| : \lambda \text{ an eigenvalue of } A \}$ is the spectral radius.

Result 4.1.2: [12]

For $A, B \in \mathbb{C}^{n \times n}$,

$$ A \geq B \iff A \geq B \text{ and } (A-B)^+ = A^+ - B^+. $$

For other conditions to be added to rank subreductivity to be equivalent to star order, one may refer [1].

Result 4.1.3: [13]

For $A, B \in \mathbb{C}^{n \times n}$,

$$ A \geq B \iff B = BA^*B = BA^*A = AA^*B. $$

Theorem 4.1.1:

For $A, B \in \mathbb{C}^{n \times n}$, $K$ is the permutation matrix associated with 'k'.

(i) $A \geq B \iff KA \geq KB \iff AK \geq BK.$
(ii) \( A \geq B \iff KA \geq KB \iff AK \geq BK \).

(iii) \( A \geq B \iff KA \geq KB \iff AK \geq BK \).

Proof:

(i) \( A \geq B \iff \rho(A^+B) \leq 1 \text{ and } R(B) \subseteq R(A) \) (By Result (4.1.1))

\[ \iff \rho(A^+KB) \leq 1 \text{ and } B = AA^+B \] (By Theorem (1.2.7))

\[ \iff \rho(A^+KB) \leq 1 \text{ and } (KB) = (KA)(A^+K)(KB) \]

\[ \iff \rho((KA)^+(KB)) \leq 1 \text{ and } R(KB) \subseteq R(KA) \] (By (2.1.2) and Theorem (1.2.7))

\[ \iff KA \geq KB \] (By Result (4.1.1)).

Also,

\( A \geq B \iff \rho(A^+B) \leq 1 \text{ and } R(B) \subseteq R(A) \) (By Result (4.1.1))

\[ \iff \rho(KA^+BK) \leq 1 \text{ and } B = AA^+B \] (By Theorem (1.2.7))

\[ \iff \rho((AK)^+(BK)) \leq 1 \text{ and } (BK) = (AK)(AK^+(BK)) \] (By (2.1.2))

\[ \iff \rho((AK)^+(BK)) \leq 1 \text{ and } R(BK) \subseteq R(KA) \] (By Theorem (1.2.7))

\[ \iff AK \geq BK \] (By Result (4.1.1)).

(ii) \( A \geq B \iff \rho(B^*B = B^*A) \text{ and } BB^* = AB^* \) (By Definition of star ordering)

\[ \iff \rho(B^*KKB = B^*KKA \text{ and } KBB^*K = KAB^*K) \]
\[
(\text{KB})^*(\text{KB}) = (\text{KB})^*(\text{KA}) \text{ and } (\text{KB})(\text{KB})^* = (\text{KA})(\text{KB})^*
\]
\[
\text{< = > } \text{KA} \geq_{*} \text{KB} \text{ (By Definition of star ordering ).}
\]

Similarly, it can be proved that \( A \geq_{*} B \text{ < = > } AK \geq_{*} BK. \)

(iii) \( A \geq_{rs} B \text{ < = > } \text{rk}(A-B) = \text{rk}(A) - \text{rk}(B) \)

\[
\text{< = > } \text{rk}(K(A-B)) = \text{rk}(KA) - \text{rk}(KB)
\]

\[
\text{< = > } \text{rk}(KA-KB) = \text{rk}(KA) - \text{rk}(KB)
\]

\[
\text{< = > } KA \geq_{rs} KB.
\]

Similarly, it can be proved that \( A \geq_{rs} B \text{ < = > } AK \geq_{rs} BK. \)

Thus, all the three standard partial orderings are preserved under 'k'.

The following results can be easily verified by using the Definition (1.2.6).

**Result 4.1.4:**

Lowener ordering is preserved under unitary similarity, that is,

\[
A \geq_{L} B \text{ < = > } P^*AP \geq_{L} P^*BP.
\]

**Result 4.1.5:**

Star ordering is preserved under unitary similarity, that is,

\[
A \geq_{*} B \text{ < = > } P^*AP \geq_{*} P^*BP.
\]
Result 4.1.6:

Rank subactivity order is preserved under unitary similarity, that is,

\[ A \geq B \iff P^*AP \geq P^*BP. \]

Theorem 4.1.2:

Lowener order, star order and rank subactivity order are all preserved for k-unitary similarity.

Proof:

(i) Lowener ordering is preserved for k-unitary similarity. We have to prove that

\[ A \geq B \iff KP^{-1}KAP \geq KP^{-1}KBP, \]

for some unitary matrix \( P \).

For,

\[ A \geq B \iff KA \geq KB \quad \text{(By Theorem (4.1.1))} \]

\[ \iff P^*KAP \geq P^*KBP \quad \text{(By Result (4.1.4))} \]

\[ \iff KP^*KAP \geq KP^*KBP \quad \text{(By Theorem (4.1.1))} \]

\[ \iff (KP^{-1}K)AP \geq (KP^{-1}K)BP \]

\[ \iff C \succeq D \quad \text{(By Theorem (4.1.1))} \]
Where
\[ C = K P^{-1} K A P \text{ is unitarily } k\text{-similar to } A, \]
\[ D = K P^{-1} K B P \text{ is unitarily } k\text{-similar to } B. \]

Thus, Löwner ordering is preserved for \( k \)-unitary similarity.

(ii) Star ordering is preserved for \( k \)-unitary similarity. We have to prove that

\[ A \geq B \iff (K P^{-1} K) A P \geq (K P^{-1} K) B P \text{ for some unitary matrix } P. \]

For,
\[
A \geq B \iff KA \geq KB \quad \text{(By Theorem (4.1.1))}
\]
\[ \iff P^* K A P \geq P^* K B P \quad \text{(By Result (4.1.5))} \]
\[ \iff KP^* K A P \geq KP^* K B P \quad \text{(By Theorem (4.1.1))} \]
\[ \iff (K P^{-1} K) A P \geq (K P^{-1} K) B P \]

Thus, star ordering is preserved for \( k \)-unitary similarity.

(iii) Rank subtractivity ordering is preserved for \( k \)-unitary similarity. We have to prove that

\[ A \succeq B \iff (K P^{-1} K) A P \succeq (K P^{-1} K) B P \text{ for some unitary matrix } P. \]
\[ A \geq B \iff KA \geq KB \quad \text{(By Theorem (4.1.1))} \]
\[ \iff P^r KAP \geq P^r KBP \quad \text{(By Result (4.1.6))} \]
\[ \iff KP^r KAP \geq KP^r KBP \quad \text{(By Theorem (4.1.1))} \]
\[ \iff (KP^{-1}K)AP \geq (KP^{-1}K)BP. \quad \text{(By Result (4.1.6))} \]

Thus, rank subtractivity ordering is preserved for k-unitary similarity.

Thus, all the three standard partial orderings are preserved for k-unitary similarity. For \( K = I \) it reduce to the known Results (4.1.4), (4.1.5) and (4.1.6).
§4.2 PARTIAL ORDERINGS ON k-EP MATRICES:

In this section, conditions for all those matrices that lie below (or) above a given k-EP matrix relative to a particular matrix partial ordering to be k-EP are determined.

Theorem 4.2.1:

If $A \succeq B$, $B$ is k-EP and $N(A) \subseteq N(B)$, then $A$ is k-EP.

Proof:

Since $A \succeq B$, by Theorem (4.1.1), $K_A \succeq K_B$, $K_{A-K_B} \succeq 0$, $K(A-B) \succeq 0 \Rightarrow (A-B)$ is k-Hermitian positive semi definite.

By Theorem (1.2.22), $(A-B)^* = K(A-B)K$. Since $B$ is k-EP, by Theorem (2.1.1), $B^* = HKBK$.

Now,

$$A^*B^* = KAK-KBK$$

$$A^* = KAK-KBK + B^*$$

$$A^* = KAK-KBK + HKBK$$

$$A^* = KAK+(H-I)KBK$$

$$A^* = KAK+LKBK \quad \text{where} \ L = H-I$$

$$A^*K = KA+LKB.$$
Since \( N(A) \subseteq N(B) \) it follows that \( N(KA) \subseteq N(LKB) \). Then for any \( x \in N(A) \),
\[
Ax = 0 \implies A^*Kx = KA + LKBx = 0.
\]
Therefore, \( N(A) \subseteq N(A^*K) \) and \( \text{rk}(A) = \text{rk}(A^*K) \).

\[
\implies N(A) = N(A^*K)
\]

\[
\implies A \text{ is k-EP.}
\]

Hence the Theorem.

**Theorem 4.2.2:**

If \( A \succeq B \), \( B \) is k-EP and \( (A-B) \) is k-hermitian, then \( A \) is k-EP.

**Proof:**

Since \( (A-B) \) is k-hermitian and \( B \) is k-EP, as in the above proof we get \( A^*K = KA + LKB \).

\[
A \succeq B \implies B = BA^*A
\]

\[
\implies N(A) \subseteq N(B) = N(LKB)
\]

Therefore, for any \( x \in N(A) \),

\[
Ax = 0 \implies A^*Kx = 0. \text{ Hence } N(A) \subseteq N(A^*K) \text{ and } \text{rk}(A) = \text{rk}(A^*K)
\]

\[
\implies N(A) = N(A^*K). \text{ Thus, } A \text{ is k-EP. Hence the Theorem.}
\]

**Corollary 4.2.1:**

If \( A \succeq B \), \( A \succeq B \) and \( B \) is k-EP, then \( A \) is k-EP.
Proof:

\[ A \geq B \Rightarrow KA \geq KB \Rightarrow KA-KB \geq O = \Rightarrow K(A-B) \geq O \]

\[ = \Rightarrow (A-B) \text{ is } k\text{-hermitian and then } A \text{ is } k\text{-EP directly follows from Theorem (4.2.2).} \]

Corollary 4.2.2:

If \( A \geq B \), \( B \) is \( k\)-EP and \( (A-B) \) is \( k\)-hermitian then, \( A \) is \( k\)-EP.

Proof:

This follows from Theorem (4.2.2) and Result (4.1.2).

Next the concept of p.s.d, a.d and a.p.d [22] are generalized respectively to p.k-s.d, h.p.k-s.d, a.k-d and a.p.k-d as follows:

Definition 4.2.1:

\( A \in \mathbb{C}_{\text{nn}} \) is said to be positive \( k\)-semi definite (p.k-s.d) if

\[ \text{Re}(x^*KAx) \geq 0 \text{ for } x \in \mathbb{C}^n. \]

Definition 4.2.2:

If \( A \) is \( k\)-hermitian and p.k-s.d then \( A \) is h.p.k-s.d denoted as

\[ A \geq_k O. \]
Definition 4.2.3:

\( A \in \mathcal{C}_{n \times n} \) is said to be almost k-definite (a.k-d) if for \( x \in \mathcal{C}_n \),
\[ x^*KAx = 0 \implies KAx = 0 \implies Ax = 0. \]

Definition 4.2.4:

\( A \in \mathcal{C}_{n \times n} \) is said to be almost positive k-definite (a.p.k-d) if it is both a.k-d and p.k-s.d.

Theorem 4.2.3:

Let \( A \) be k-EP and p.k-s.d and \( B \) be any matrix, so that \((A-B)\) is k-hermitian. Then
\[ A \succeq B \implies A \succeq B \implies A \succeq B. \]

Proof:

By Result (4.1.2), \( A \succeq B \implies A \succeq B \). Now, we need only to prove the second implication. \( A \succeq B \implies B = BA^+B = BA^+A = AA^+B \)
\[ \text{(By Result (4.1.3))} \]
\[ A-B = A-2B+B \]
\[ = A-B+B \]
\[ = A-BA^+A-AA^+B+BA^+B \]
\[ = AA^+A-AA^+B-BA^+A+BA^+B \]
\[ \begin{align*}
&= AA^+(A-B)-BA^+(A-B) \\
&= (AA^+-BA^+)(A-B) \\
&= (A-B)A^+(A-B) \\
&= K(A-B)^*KA^+(A-B). \quad (\text{Since } (A-B) \text{ is } k\text{-hermitian})
\end{align*} \]

\[ K(A-B) = (A-B)^*KA^+(A-B) \]

\[ = (K(A-B))^*A^+KK(A-B) \]

\[ = (K(A-B))^*(KA)^+(K(A-B)). \]

Since \( A \) is \( k\)-EP and p.k-s.d, \( KA \) is EP and p.s.d. and \( (KA)^+B \) is p.s.d, \( K(A-B) = (K(A-B))^*(KA)^+(K(A-B)) \) is p.s.d. \( (A-B) \) is p.k-s.d. Also \( (A-B) \) is k-hermitian. Therefore \( (A-B) \geq O \Rightarrow K(A-B) \geq O \Rightarrow \]

\[ A-B \geq O \Rightarrow A \geq B. \]

Remark 4.2.1:

The condition \( (A-B) \) to be \( k\)-hermitian is essential can be seen by the following example.

Example 4.2.1:

Let \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \).

\[ KA-KB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \neq 0. \]
(A-B) is not k-hermitian. KA is p.s.d => A is p.k.s.d and
\[ \text{rk}(A-B) = 1 = 2 - 1 = \text{rk}(A) - \text{rk}(B) => A \geq B. \]

Theorem 4.2.4:

Let A $\geq$ B with B k-EP. Then A is a.p.k-d => B is a.p.k-d.

Proof:

A is a.p.k-d => KA is a.p.d
  => (KA)$^+$ is a.p.d
  => $P^*(KA)^+P$ is a.p.d for any P.

Now A $\geq$ B $\Rightarrow$ A $\geq$ B $\Rightarrow$ B = BA$^+$B (By Result (4.1.2))
  => B$^+$ = HKBK = HK(BA$^+$B)K (By Theorem (2.1.1))
  => B$^+$ = HKBKK$^+$BK
  => B$^+$ = (HKBK)(AK)$^+$BK
  => B$^+$ = B$^*(AK)^+(BK)$
  => KB$^+$ = KB$^*(AK)^+(BK)$
  => (BK)$^+$ = (BK)$^*(AK)^+(BK)$
  => (BK)$^+$ is a.p.d
  => KB$^+$ is a.p.d
  => B$^+$ is a.p.k-d
  => B is a.p.k-d.

Hence the Theorem.
**Remark 4.2.2:**

The converse of the Theorem (4.2.4) need not be true. For example,

\[
A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Let \( A > B \). However, \( A \) is not k-EP. Hence, \( A \) is not a.p.k-d.

**Remark 4.2.3:**

The condition on \( B \) to be k-EP in Theorem (4.2.4) cannot be relaxed. For example, consider the matrices,

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{with} \quad K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

\[
KA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad KB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.
\]

Clearly, \( KA \geq KB \Rightarrow A \geq B \). \( A \) is h.p.k-s.d.\(^{rs}\)

Hence, \( A \) is a.p.k-d. But, \( B \) is not k-EP. Hence, \( B \) is not a.p.k-d.\(^{rs}\)
Theorem 4.2.5:

If \( A \geq B \), \( A \) is \( k \)-EP and \( L(N(B)) \subseteq L(N(A)) \) then \( B \) is \( k \)-EP.

Proof:

Similar to that of Theorem (4.1.1).

The following corollary can be deduced from Theorem (4.2.1) and Theorem (4.2.5).

Corollary 4.2.3:

If \( A \geq B \) and \( L(N(A)) \subseteq L(N(B)) \) then \( A \) is \( k \)-EP \( \iff \) \( B \) is \( k \)-EP.
§4.3 INEQUALITIES $M \geq B^*MB$:

Inequalities $M \geq B^*MB$ where $M$ is h.p.s.d and '$\geq$' is one of the three standard partial orders have been discussed in [2]. In this section, we have extended these results for k-EP matrices. Inequalities $M \geq B^*MB$ where $M$ is a k-EP matrix and $B$ is a complex matrix are derived.

**Theorem 4.3.1:**

Let $M$ be k-EP and p.k-s.d, $B \in \mathbb{C}^{n \times n}$. Then the pair of conditions

\begin{align*}
&\text{(4.3.1)} \quad M \geq (KB^*K)^{LM}B \\
&\text{(4.3.2)} \quad MB = MB^2
\end{align*}

is equivalent to the pair of conditions

\begin{align*}
&\text{(4.3.3)} \quad KB^*KM = MB = KB^*KMB \\
&\text{(4.3.4)} \quad KM-B^*KMB \text{ is hermitian.}
\end{align*}

**Proof:**

Since $M$ is k-EP and p.k-s.d, $KM$ is EP and p.s.d. By Theorem (4.1.1), $M \geq (KB^*K)^{LM}B \Rightarrow KM \geq B^*KMB$. By definition,

$$N = KM-B^*KMB$$

is hermitian non-negative definite. Hence (4.3.4) is trivial. Let us prove (4.3.3) by using (4.3.1) and (4.3.2). Pre and post multiplication of $N$ by $B^*$ and $B$ give,
\[ B^*NB = B^*KMB - (B^*)^2KMB^2 \]

\[ = B^*KMB - (B^*)^2KMB \quad \text{(By (4.3.2))} \]

\[ = B^*KMB - (B^*)^2M^*H^{-1}KB \quad \text{(By Theorem (2.1.1))} \]

\[ = B^*KMB - (MB^2)^*H^{-1}KB \quad \text{(By (4.3.2))} \]

\[ = B^*KMB - (MB)^*H^{-1}KB \quad \text{(By (4.3.2))} \]

\[ = B^*KMB - B^*M^*H^{-1}KB \]

\[ = B^*KMB - B^*KMB \quad \text{(By Theorem (2.1.1))} \]

\[ = 0. \]

Now,

\[ B^*NB = 0 = \Rightarrow NB = 0 \]

\[ \Rightarrow KMB = B^*KMB^2 \]

\[ \Rightarrow KMB = B^*KMB^2 = B^*KMB \quad \text{(By (4.3.2))} \]

\[ \Rightarrow KMB = B^*KMB = B^*KMB \]

\[ \Rightarrow MB = KB^*KM = KB^*KMB. \]

Thus (4.3.3) holds.

Conversely,

\[ KB^*KM = MB = KB^*KMB \]

\[ \Rightarrow MB^2 = (MB)B \]

\[ = (KB^*KM)B \]

\[ = MB. \]

Thus \( MB^2 = MB. \) Thus (4.3.2) holds.
Since $KB'KM = MB$ we have $B'KM = KMB$.

$KM-B'KMB$ can be written as

$$KM-B'KMB = (I-B')KM(I-B).$$

Since $M$ is $p.k$-$s.d$, $KM$ is $p.s.d.$ and $KM-B'KMB$ is hermitian, $LM > B'KMB => L > (KB'K) MB$.

Hence the Theorem.

**Theorem 4.3.2:**

Let $M$ be $k$-$EP$ and $p.k$-$s.d$, $B \in C_{nn}$. Then the pair of conditions,

(4.3.5) $M \geq B'M(KBK)$ and

(4.3.6) $MKB = MKB^2$

is equivalent to the pair of conditions

(4.3.7) $B'M = MKBK = B'MKBK$

(4.3.8) $MK-B'MKB$ is hermitian.

**Proof:**

Since $M$ is $k$-$EP$ and $p.k$-$s.d$, we have $MK$ is $EP$ and $p.s.d.$

Proof is similar to that of Theorem (4.3.1) using $MK$ as $EP$.

**Corollary 4.3.1:**

Let $M$ be $k$-$EP$ and $p.k$-$s.d$, $B \in C_{nn}$ such that $MB = MB^2$ then $M \geq (KB'K) MB \Leftrightarrow KB'KM = MB$ and $KM-B'KMB$ is hermitian.
Remark 4.3.1:

We note that condition (4.3.4) is essential. For example, consider

\[ M = \begin{bmatrix} 1+i & 1-i \\ 1+i & 1+i \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

\[ KM = \begin{bmatrix} 1+i & 1+i \\ 1+i & 1-i \end{bmatrix} \]

is EP and p.s.d.

Therefore, M is k-EP and p.k-s.d.

\[ B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = B^2. \quad \text{Hence} \]

\[ MB = \begin{bmatrix} 1+i & 1-i \\ 1+i & 1+i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1+i & 1+i \\ 1+i & 1+i \end{bmatrix}. \]

Also \( MB^2 = \begin{bmatrix} 1+i & 1+i \\ 1+i & 1+i \end{bmatrix} \). Thus \( MB = MB^2 \).

\[ KB^*KM = \begin{bmatrix} 1+i & 1+i \\ 1+i & 1+i \end{bmatrix} \]
\[ KB^*KMB = \begin{bmatrix} 1+i & 1+i \\ 1+i & 1+i \end{bmatrix} \]

Thus \( MB = MB^2 = KB^*KMB. \)

\[ KM - B^*KMB = \begin{bmatrix} 0 & 0 \\ 0 & -2i \end{bmatrix} \] is not hermitian.

Thus \( KM \neq B^*KMB \Rightarrow M \neq (KB^*K)MB. \)

Theorem (4.3.1) fails.

\[ M \text{ is } k\text{-EP and } p.k\text{-s.d, } B = B^2. \text{ Hence } MB = MB^2 = KB^*KMB. \]

\( KB^*KM = MB \) but \( KM - B^*KMB \) is not hermitian. Thus (4.3.1) fails.

Theorem 4.3.3:

Let \( M \) be \( k\text{-EP and } p.k\text{-s.d, } B \in \mathbb{C}_{nn} \) such that \( KM - B^*KMB \) is hermitian. Then \( M \geq (KB^*K)MB \) and \( MB = MB^2 \iff (KB^*K)M = MB = (KB^*K)MB. \)

Proof:

Since \( M \geq (KB^*K)MB, \) \( KM \geq B^*KMB, \)

By Result (4.1.3), \( KM - B^*KMB \) can be written as

\[ KM - B^*KMB = (KM - B^*KMB)(KM) + (KM - B^*KMB). \]

Since \( M \) is \( p.k\text{-s.d}, \)

\( KM \) is \( p.s.d. \Rightarrow (KM)^+ \) is \( p.s.d \Rightarrow M^+ \) is \( p.k\text{-s.d.} \) \( KM - B^*KMB \) is
hermitian by hypothesis, hence $KM - B^*KMB$ is hermitian positive semi
definite which implies, $KM \succeq B^*KMB \Rightarrow M \succeq (KB^*)MB$. By applying

Theorem (4.3.1), we get $(KB^*)M = MB = (KB^*)MB$.

Conversely, suppose $(KB^*)M = MB = (KB^*)MB$. Then, together with $KM - B^*KMB$ is hermitian, by Theorem (4.3.1), it follows that $MB = MB^2$. Since $M$ is k-EP by Theorem (2.1.1), $M^* = HKMK$. Now we will prove $B^*KMB$ is EP.

$$(B^*KMB)^* = B^*M^*KB
= (MB)^*KB
= (KB^*KM)^*KB
= M^*KB^2KB
= M^*KB^2
= (HKMK)KB^2
= HKMB^2
= HKMB
= HKKB^2KB
= HKKB^2KB
= H(B^*KMB).$$

By Theorem (1.2.12), $B^*KMB$ is EP. Thus, both $KM$ and $B^*KMB$ are EP and p.s.d. To prove $M \succeq (KB^*)MB$, we have to prove $KM \succeq B^*KMB$.

For that it is enough to show that, $R(B^*KMB) \subseteq R(KM)$ and

$$(B^*KMB)(KM)^+(B^*KMB) = (B^*KMB).$$
For, since \( KB'KMB = MB \), \( B'KMB = KMB \).

\[
R(B'KMB) = R(KMB) \subseteq R(KM).
\]

Also,

\[
(B'KMB)(KM)'(B'KMB) = (B'KM)M'^{+}K(KMB)
= B'KMM'^{+}MB
= B'KMB.
\]

Hence the Theorem.

**Corollary 4.3.2:**

Let \( M \) be k-EP and p.k-s.d, \( B \in C_{\text{max}} \) such that \( MB = MB^2 \).

Then \( M \geq (KB'K)MB \) if and only if \( M \geq (KB'K)MB \) and \( KM-B'KMB \) is hermitian.

**Proof:**

Since \( M \geq (KB'K)MB \), \( KM \geq B'KMB \). Also \( MB = MB^2 \). By Theorem (4.3.1), \( KM-B'KMB \) is hermitian and \( MB = KB'KM = KB'KMB \). \( M \geq (KB'K)MB \) follows from Theorem (4.3.3).

Conversely, if \( KM-B'KMB \) is hermitian and \( M \geq (KB'K)MB \), then using \( MB = MB^2 \) by Theorem (4.3.3), it follows that \( MB = KB'KM = KB'KMB \).

Again by Theorem (4.3.1), \( M \geq (KB'K)MB \).
Remark 4.3.2:

We note that the condition $K_{M} - B^{*}K_{M}B$ is hermitian cannot be relaxed. For example,

Let $M = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $K_{M} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ is EP and p.s.d.

Therefore, $M$ is k-EP and is p.k-s.d. Let $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = B^{2}$. $MB = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$.

$MB^{2} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$, $KB^{*}KMB = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$.

Thus, $MB = MB^{2} = KB^{*}KMB$.

$K_{M} - B^{*}K_{M}B = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ is not hermitian.

$\text{rk}(K_{M} - B^{*}K_{M}B) = 1 = 2 - 1 = \text{rk}(K_{M}) - \text{rk}(B^{*}K_{M}B)$.

Hence, $K_{M} \succeq B^{*}K_{M}B = \succ_{rs} M \succeq (K_{B}^{*}K_{M}B)_{rs}$ and $M \not\succeq_{L} (K_{B}^{*}K_{M}B)_{rs}$. Thus, corollary (4.3.2) fails.

Theorem 4.3.4:

Let $M$ be a k-EP matrix and p.k-s.d, $B \in \mathbb{C}_{n\times n}$ such that $K_{M} - B^{*}K_{M}B$ is hermitian. Then, the following are equivalent:

(i) $M \succeq_{*} (K_{B}^{*}K_{M}B)$ and $MB = MB^{2}$.

(ii) $K_{B}^{*}K_{M} = MB = K_{B}^{*}K_{M}B$ and $MBKM = MB^{*}K_{M}$. 
Proof:

(i) $\Rightarrow$ (ii):

$$M \geq (KB^*K)MB \Rightarrow KM \geq B^*KMB \Rightarrow KM \geq B^*KMB.$$  

Since $KM - B^*KMB$ is hermitian, by Corollary (4.3.2), $M \geq (KB^*K)MB$.

Now by Theorem (4.3.1) and using $MB = MB^2$, $KB^*KM = MB = KB^*KMB$.

By definition of star ordering of $KM$ and $B^*KMB$, we can see that,

$$(B^*KMB)(KM) = (KM)(B^*KMB) = (B^*KMB)(KM) = (KM)(B^*KMB).$$  

Then

$$KMBKM = (KMB)(KM) = (B^*KMB)(KM) = (KM)(B^*KMB) = (KM)(B^*KM) = MBKM = MB^*KM.$$  

Thus (ii) holds.

(ii) $\Rightarrow$ (i):

Suppose (ii) holds. Then the condition for star ordering of $KM$ and $B^*KMB$ can be directly verified by using (ii). Since $KM - B^*KMB$ is hermitian and $KB^*KM = MB = KB^*KMB$, by Theorem (4.3.1), it follows that $MB = MB^2$. Hence the Theorem.
Corollary 4.3.3:

Let $M$ be k-EP and p.k-s.d. Let $B \in \mathbb{C}^{n \times n}$ such that $MB = MB^2$. $KM - B^*KMB$ is hermitian and $MBKM = MB^*KM$.

Then $M \geq (KB^*K)MB \iff \mathbb{L} M \geq (KB^*K)MB \iff \mathbb{L} M \geq (KB^*K)MB$.

Remark 4.3.3:

In particular, if $KM$ is hermitian non-negative definite and $B$ is a projection, then the three orderings on $KM$ and $B^*KMB$ are equivalent.