CHAPTER 4

Asymmetric Type II Compound Laplace Distributions and its Properties

4.1 Introduction

Recently there is a growing trend in the literature on parametric families of asymmetric distributions which are deviated from symmetry as well as from the classical normality assumptions. Various researchers have developed different methods to construct asymmetric distributions with heavy tails. In Chapter 2 we have introduced skew slash distributions generated by Cauchy kernel, skew slash t and asymmetric slash Laplace distributions for modelling microarray data. The form of the density functions of this family is not convenient. This motivated us to introduce a convenient distribution which can account asymmetry, peakedness and heavier tails. In the present chapter we introduce asymmetric type II compound Laplace

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Results included in this chapter form the paper Bindu et al. (2012a).
(ACL) density which is the asymmetric version of the type II compound Laplace distribution and is a generalization of asymmetric Laplace distribution (AL). This four-parameter probability distribution provides an additional degree of freedom to capture the characteristic features of the microarray data. We derive the \( pdf \), \( cdf \), \( qf \) and study various properties of \( ACL \).

### 4.2 Symmetric Type II Compound Laplace Distribution

The (symmetric) type II compound Laplace distribution \( (CL) \) is introduced by Kotz et al. (2001) which results from compounding a Laplace distribution with a gamma distribution. Let \( X \) follow a classical Laplace distribution given \( s \) with density given by

\[
f(x|s) = \frac{s}{2} e^{-s|x-\theta|}, \quad x \in \mathbb{R},
\]

and let \( s \) follow a \( Gamma(\alpha, \beta) \) distribution with density

\[
f(s; \alpha, \beta) = \frac{s^{\alpha-1}e^{-s/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad \alpha > 0, \beta > 0, \ s > 0.
\]

Then the unconditional distribution of \( X \) is the type II compound Laplace distribution with parameters \( (\theta, \alpha, \beta) \), denoted by \( X \sim CL(\theta, \alpha, \beta) \) and the density function

\[
f(x) = \frac{1}{2} \alpha \beta [1 + |x - \theta|\beta^{-(\alpha+1)}], \quad \alpha > 0, \beta > 0, \theta \in \mathbb{R}, \ x \in \mathbb{R},
\]

The \( CL \) can be represented as a mixture of Laplace distribution as, \( Y \overset{d}{=} \theta + \sigma X \), where \( X \) has the standard classical Laplace distribution. The mixture on \( \sigma = 1/s \) of the distribution of \( X \) is the type II compound Laplace distribution with parameter \( \theta, \alpha \) and \( \beta \) if \( 1/s \) or \( \sigma \) has the \( Gamma(\alpha, \beta) \) distribution.
4.2.1 Distribution, Survival and Quantile Functions

The cumulative distribution function (cdf) of the type II compound Laplace distribution is given by

\[
F(x) = \begin{cases} 
1 - \frac{1}{2}[1 + \beta(x - \theta)]^{-\alpha}, & \text{for } x > \theta, \\
\frac{1}{2}[1 - \beta(x - \theta)]^{-\alpha}, & \text{for } x \leq \theta.
\end{cases}
\]

(4.2.4)

The survival function (sf) of the type II compound Laplace distribution is given by

\[
S(x) = \begin{cases} 
\frac{1}{2}[1 + \beta(x - \theta)]^{-\alpha}, & \text{for } x > \theta, \\
1 - \frac{1}{2}[1 - \beta(x - \theta)]^{-\alpha}, & \text{for } x \leq \theta.
\end{cases}
\]

(4.2.5)

The \(q^{th}\) quantile function (qf) of \(CL\) distribution is,

\[
\xi_q = \begin{cases} 
\theta + \frac{1}{\beta} \left[1 - \frac{1}{(2q)^{1/\alpha}}\right], & \text{for } q \in (0, \frac{1}{2}], \\
\theta + \frac{1}{\beta} \left[\frac{1}{(2(1-q))^{1/\alpha}} - 1\right], & \text{for } q \in (\frac{1}{2}, 1).
\end{cases}
\]

(4.2.6)

**Remark 4.2.1.** If \(X \sim CL(\theta, \alpha, \beta)\) then for \(\alpha \to \infty, \beta \to 0\) such that \(\alpha \beta = s\), a constant, the density \(f(x)\) in Eq.(4.2.3) converges to the classical Laplace density, \(\frac{s}{2} e^{-sx-\theta}\), \(s > 0\), \(\theta \in \mathbb{R}\).

**Remark 4.2.2.** If \(X \sim CL(\theta, \alpha, \beta)\) then for \(\alpha = 1\) the density \(f(x)\) in Eq.(4.2.3) reduce to the double Lomax density, which is the ratio of two independent standard Laplace densities. The double Lomax distribution (Bindu (2011c) and Bindu et al. (2013e)) is given by

\[
f(x) = \frac{1}{2}[1 + \beta|x - \theta|]^{-2}, \ \theta \in \mathbb{R}, \ x \in \mathbb{R}.
\]

(4.2.7)

**Remark 4.2.3.** If \(X \sim CL(\theta, \alpha, \beta)\) then the \(r^{th}\) moment of \(X\) around \(\theta\), \(E(X - \theta)^r\) is
Figure 4.1: Type II compound Laplace density functions \((\theta = 0, \kappa = 1)\) for various values of \(\alpha\) and \(\beta\).

given as follows.

\[
m_r = E(x - \theta)^r = \begin{cases} \frac{\alpha}{\beta^r} B(r + 1, \alpha - r), & \text{for } r \text{ even}, \ 0 < r < \alpha, \\ 0, & \text{for } r \text{ odd}, \ 0 < r < \alpha. \end{cases} \quad (4.2.8)
\]

Put \(r = 1\) in Eq.(4.2.8), we get \(E(X - \theta) = 0\). Hence, the mean \(E(X) = \theta\), for \(\alpha > 1\). Therefore \(E(X - \theta)^r\) is the \(r^{th}\) central moment \(\mu_r\) of the CL distribution.

The expression for variance is given by,

\[
V(X) = \frac{2}{\beta^2(\alpha - 1)(\alpha - 2)}, \text{ for } \alpha > 2.
\]

Hence, the type II compound Laplace distribution has finite mean if \(\alpha > 1\) and has finite variance if \(\alpha > 2\).
4.3 Asymmetric Type II Compound Laplace Distribution

Here we introduce asymmetry into the symmetric type II compound Laplace distribution using the method of Fernandez and Steel (1998). The idea is to postulate inverse scale factors in the positive and negative orthants of the symmetric distribution to convert it into an asymmetric distribution. Thus a symmetric density \( f \) generates the following class of skewed distribution, indexed by \( \kappa > 0 \).

If \( g(\cdot) \) is symmetric on \( \mathbb{R} \), then for any \( \kappa > 0 \), a skewed density can be obtained as

\[
    f(x) = \frac{2\kappa}{1 + \kappa^2} \begin{cases} 
        g(x\kappa), & \text{for } x > 0, \\
        g(\frac{x}{\kappa}), & \text{for } x \leq 0.
    \end{cases}
\]

In the above expression, when \( g \) is the symmetric type II compound Laplace distribution with density Eq.(4.2.3), we get a skewed distribution with the density function defined as follows.

**Definition 4.3.1.** A random variable \( X \) is said to have an asymmetric type II compound Laplace distribution (ACL) with parameters \((\theta, \alpha, \beta, \kappa)\), denoted by \( X \sim ACL(\theta, \alpha, \beta, \kappa) \) if its probability density function is given by

\[
    f(x) = \frac{\kappa \alpha \beta}{1 + \kappa^2} \begin{cases} 
        (1 + \kappa\beta(x - \theta))^{-(\alpha+1)}, & \text{for } x > \theta, \\
        (1 - \frac{\beta}{\kappa}(x - \theta))^{-(\alpha+1)}, & \text{for } x \leq \theta.
    \end{cases}
\]

and \( \theta \in \mathbb{R}, \alpha, \beta, \kappa > 0 \).

The parameters \((\theta, \alpha, \beta, \kappa)\) are the location, shape, scale, and skewness parameters, respectively.

4.3.1 Distribution, Survival and Quantile Functions

The cdf of the ACL distribution is given by

\[
    F(x) = \begin{cases} 
        1 - \frac{1}{1+\kappa^2} [1 + \kappa\beta(x - \theta)]^{-\alpha}, & \text{for } x > \theta, \\
        \frac{\kappa^2}{1+\kappa^2} [1 - \frac{\beta}{\kappa}(x - \theta)]^{-\alpha}, & \text{for } x \leq \theta.
    \end{cases}
\]
When $\kappa = 1$ we get the symmetric type II compound Laplace distribution. The survival function ($sf$) of the $ACL$ distribution is given by

$$S(x) = \begin{cases} 
\frac{1}{1 + \kappa^2}[1 + \kappa\beta(x - \theta)]^{-\alpha}, & \text{for } x > \theta, \\
1 - \frac{\kappa^2}{1 + \kappa^2}[1 - \frac{\beta}{\kappa}(x - \theta)]^{-\alpha}, & \text{for } x \leq \theta.
\end{cases} \tag{4.3.3}$$

The $q^{th}$ quantile function ($qf$) of $ACL$ distribution is,

$$\xi_q = \begin{cases} 
\theta + \kappa \frac{1}{\beta} \left[ 1 - \frac{1}{q^{1/\alpha}} \right], & \text{for } q \in \left(0, \frac{\kappa^2}{1 + \kappa^2}\right), \\
\theta + \kappa \frac{1}{\beta} \left[ \frac{1}{(1-q)(1+\kappa^2))^{1/\alpha}} - 1 \right], & \text{for } q \in \left(\frac{\kappa^2}{1 + \kappa^2}, 1\right).
\end{cases} \tag{4.3.4}$$

The cdf and $qf$ can be useful for goodness-of-fit and simulation purposes. For $q = \kappa^2/(1 + \kappa^2)$, the $q^{th}$ quantile is given by $\xi_q = \theta$. Hence, for given $\kappa$ the location parameter is given by $\hat{\theta} = \xi_{\kappa^2/(1+\kappa^2)}$.

**Remark 4.3.1.** If $X \sim ACL(\theta, \alpha, \beta, \kappa)$ then for $\alpha \to \infty$, $\beta \to 0$ such that $\alpha \beta = s$, a constant, the density $f(x)$ in Eq.(4.3.1) converges to the $AL$ density of Kotz et al. (2001) denoted by $\mathcal{AC}^\ast(\theta, \kappa, \sqrt{2}/s)$.

### 4.3.2 Properties

Fig. 4.1 shows density plots of symmetric (for various values of $\alpha$, $\beta$) and Fig. 4.2 shows the asymmetric (for various values of $\kappa$) type II compound Laplace distributions. For asymmetric type II compound Laplace distribution both tails are power tails and the rate of convergence depends on the values of $\kappa$. When $\kappa < 1$ the curve moves to the right of the symmetric curve and the left tail moves towards $\theta$ giving heavier right tail, and vice versa when $\kappa > 1$. Below we list a few important properties of type II compound Laplace distributions. For properties similar to asymmetric Laplace we refer to Kotz et al. (2001).
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Figure 4.2: Asymmetric type II compound Laplace density functions for various values of $\kappa$ and for fixed $(\theta = 0, \alpha = 1.5, \beta = 2)$.

(i) If $X \sim ACL(\theta, \alpha, \beta, \kappa)$, then $Y = aX + b \sim ACL(b + a\theta, \alpha, \beta/a, \kappa)$ where $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $a \neq 0$. Hence, the distribution of a linear combination of a random variable with $ACL(\theta, \alpha, \beta, \kappa)$ distribution is also $ACL$. If $X \sim ACL(\theta, \alpha, \beta, \kappa)$, then $Y = (X - \theta)/\beta \sim ACL(0, \alpha, 1, \kappa)$, which can be called as the standard $ACL$ distribution.

(ii) The mode of the distribution is $\theta$ and the value of the density function at $\theta$ is $(\kappa/(1 + \kappa^2))\alpha\beta$.

(iii) The value of the distribution function at $\theta$ is $\kappa^2/(1 + \kappa^2)$ and hence, $\theta$ is also the $\kappa^2/(1 + \kappa^2)$-quantile of the distribution.

(iv) The $r$th moment of $X$ around $\theta$, $E(X - \theta)^r$, exists for $0 < r < \alpha$ and is given as follows.

$$m_r = E(X - \theta)^r = \frac{1 + (-1)^r\kappa^{2(r+1)}}{\kappa^r(1 + \kappa^2)} \frac{\alpha}{\beta^r} B(r + 1, \alpha - r),$$

where $B(a, b)$ is a beta function. It is clear that the moments of order $\alpha$ or greater do not exist. For the symmetric distribution ($\kappa = 1$), all odd moments
around $\theta$ are zero and even moments are given by $\left(\frac{\alpha}{\beta r}\right) B(r + 1, \alpha - r)$.

(v) The type II compound Laplace distributions have heavier tails than classical Laplace distributions. Note that the tail probability of the type II compound Laplace density is $\bar{F} \sim cx^{-\alpha}$, as $x \to \pm \infty$. The heavy tail characteristic makes this densities appropriate for modeling network delays, signals and noise, financial risk or microarray gene expression or interference which are impulsive in nature.

(vi) The type II compound Laplace distributions are completely monotonic on $(\theta, \infty)$ and absolutely monotonic on $(-\infty, \theta)$. As noted by Dreier(1999), every symmetric density on $(-\infty, \infty)$, which is completely monotonic on $(0, \infty)$, is a scale mixture of Laplace distributions.

### 4.3.3 Stochastic Representation $ACL$

Here we give two stochastic representations for the $ACL$ distribution based on the two stochastic representation of the $AL$ distribution. Let $X$ has the $ACL(\theta, \alpha, \beta, \kappa)$ and $Y$ has $AL(0, 1, \kappa)$

$$X \overset{d}{=} \theta + \sigma Y,$$

where $1/\sigma$ has the $Gamma(\alpha, \beta)$ distribution (Eq. (4.2.2)) or $\sigma$ has the Inverse Gamma distribution with parameters $\alpha$ and $\beta$. Then $ACL(\theta, \alpha, \beta, \kappa)$ can be represented as normal mixture as follows,

$$X \overset{d}{=} \theta + \mu W + \sigma \sqrt{W} Z,$$

where $\mu = \sigma \left(\frac{1}{\kappa} - \kappa\right) / \sqrt{2}$, $W$ is the standard exponential variate, $Z$ follows $N(0, 1)$ independent of $W$ and $\sigma$ has the $Inverse Gamma(\alpha, \beta)$ distribution. Equation (4.3.7) says that $X$ can be viewed a continuous mixture of normal random variables whose scale and mean parameters are dependent and vary according to an exponential distribution. Then $X|W \sim N(\theta + \mu W, \sigma^2 W)$, where $W$ is $\exp(1)$ and $\sigma$ has the $Inverse Gamma(\alpha, \beta)$ distribution.
Another representation as the log-ratio of two independent random-variables with Pareto I distributions is given below.

\[ X \overset{d}{=} \theta + \frac{\sigma}{\sqrt{2}} \log \left( \frac{P_1}{P_2} \right), \]  
(4.3.8)

where \( \sigma \) has the Inverse Gamma(\( \alpha, \beta \)) distribution, \( P_1 \sim \text{Pareto I}(\kappa, 1) \) and \( P_2 \sim \text{pareto}(1/\kappa, 1) \).

### 4.4 Estimation of ACL

In this section we study the problem of estimating four unknown parameters, \( \Theta = (\theta, \alpha, \beta, \kappa) \), of ACL distribution. To estimate the parameter \( \theta \) we use the quantile estimation. The quantile estimate of \( \theta \) is given by \( \hat{\theta} = \xi_{[\kappa^2/(1+\kappa^2)]} \). Given \( \kappa \), the quantile estimate of \( \theta \) is the sample quantile of order \( \kappa^2/(1+\kappa^2) \), which is (for large \( n \)) \( ([n\kappa^2/(1+\kappa^2)] + 1)^{th} \) ordered observation, (\([c] \) denoted the integral part of \( c \)). When the data are approximately symmetric the estimate of \( \theta \) will be close to median. The method of moments or maximum likelihood estimation method can be employed to estimate \( \Theta \) as described below. Let \( X = (X_1, \cdots, X_n) \) be independent and identically distributed samples from an asymmetric type II compound Laplace distribution with parameters \( \Theta \).

#### 4.4.1 Method of Moments

To estimate \( \Theta \) under the method of moments, four first moments, \( E(X^r), r = 1, 2, 3, 4 \), are equated to the corresponding sample moments and the resulted system of equations are solved for the unknown parameters. These moments can be obtained from Eq.(4.3.5) but they exist only when \( \alpha > 4 \). Hence, the method is not applicable to the entire parametric space. An alternative method is a maximum likelihood estimation where the likelihood function is maximized to estimate the unknown parameters. We describe this alternative method briefly in the following subsection.
4.4.2 Maximum Likelihood Estimation

The log-likelihood function of the data $X$ takes the form

$$logL(\Theta; X) = n \log \kappa - n \log(1 + \kappa^2) + n \log \alpha + n \log \beta - (\alpha + 1)S(\theta, \beta, \kappa).$$

Where

$$S(\theta, \beta, \kappa) = \sum_{i=1}^{n} S_i(\theta, \beta, \kappa) = \sum_{i=1}^{n} \log \left[ 1 + (\kappa \beta)(x_i - \theta)^+ + \frac{\beta}{\kappa}(x_i - \theta)^- \right],$$

and $(x - \theta)^+ = (x - \theta)$, if $x > \theta$, and $= 0$ otherwise, and $(x - \theta)^- = (\theta - x)$, if $x \leq \theta$, and $= 0$ otherwise.

Existence, uniqueness and asymptotic normality of maximum likelihood estimators (MLEs) can be derived on the same lines as described in detail for an $AL$ distribution in Kotz et al. (2001).

The MLEs of $(\alpha, \beta)$ for given $\theta = \hat{\theta}$ and $\kappa = \hat{\kappa}$ are obtained by solving the score equations for $\alpha$ and $\beta$. This leads to the following equations which are solved iteratively.

$$\alpha = \frac{n}{S(\hat{\theta}, \beta, \hat{\kappa})},$$

$$\frac{n}{\beta} = (\alpha + 1) \sum_{i=1}^{n} \frac{\hat{\kappa}(x_i - \hat{\theta})^+ + \frac{1}{\hat{\kappa}}(x_i - \hat{\theta})^-}{1 + \hat{\kappa} \beta(x_i - \hat{\theta})^+ + \frac{\beta}{\hat{\kappa}}(x_i - \hat{\theta})^-}.$$

In our illustrations, the maximization of the likelihood is implemented using the `optim` function of the R statistical software, applying the $BFGS$ algorithm ($R$ Development Core Team (2006)). Estimates of the standard errors were obtained by inverting the numerically differentiated information matrix at the maximum likelihood estimates. We discuss the performance of our numerical maximization algorithm (programmed in R) using the simulated data sets in the chapter 5. Now we discuss the stress-strength reliability $Pr(X > Y)$, when $X$ and $Y$ are two indepen-
dent but non-identically distributed random variables belonging to the asymmetric, heavy-tailed and peaked distribution, ACL.

4.5 Stress-strength Reliability of ACL

In the context of reliability, the stress-strength model describes the life of a component which has a random strength $X$ and is subjected to a random stress $Y$. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X > Y$. Thus, $R = Pr(X > Y)$ is a measure of component reliability. The parameter $R$ is referred to as the reliability parameter. This type of functional can be of practical importance in many applications. For instance, if $X$ is the response for a control group, and $Y$ refers to a treatment group, $Pr(X < Y)$ is a measure of the effect of the treatment. $R = Pr(X > Y)$ can also be useful when estimating heritability of a genetic trait. Bamber (1975) gives a geometrical interpretation of $A(X,Y) = Pr(X < Y) + \frac{1}{2}Pr(X = Y)$ and demonstrates that $A(X,Y)$ is a useful measure of the size of the difference between two populations.

Weerahandi and Johnson (1992) proposed inferential procedures for $Pr(X > Y)$ assuming that $X$ and $Y$ are independent normal random variables. Gupta and Brown (2001) illustrated the application of skew normal distribution to stress-strength model. Bindu (2011c) introduced the double Lomax distribution, which is the ratio of two independent and identically distributed Laplace distributions and presented its application to the IQ score data set from Roberts (1988). The Roberts IQ data gives the Otis IQ scores for 87 white males and 52 non-white males hired by a large insurance company in 1971. Where $X$ represent the IQ scores for whites and $Y$ represent the IQ scores for non-whites and estimated the probability that the IQ score for a white employee is greater than the IQ score for a non-white employee.

The functional $R = Pr(X > Y)$ or $\lambda = Pr(X > Y) - Pr(X < Y)$ is of practical importance in many situations, including clinical trials, genetics, and reliability.
We are interested in applying $R = Pr(X > Y)$ as a measure of the difference between two populations, in particular where $X$ and $Y$ refer to the log intensity measurements of red dye (test sample) and the log intensity measurements for the green dye (control) in cDNA microarray gene expression data. In microarray gene expression studies the investigators are interested in “is there any significant difference in expression values for genes”, “what is the estimate of the number of genes which are differentially expressed”, “what proportion of genes are really differentially expressed” and so on.

Here we explored the applications of stress-strength analysis in microarray gene expression studies using the $ACL$ distribution. We used this concept for checking array normalization in microarray gene expression data. First we derived the stress-strength reliability $R = Pr(X > Y)$ for asymmetric type II compound Laplace. Then we calculated $R$ for microarray datasets for before and after normalization by taking $X$ as the log intensity measurements of red dye (test sample) and let $Y$ represent the log intensity measurements of green dye (control sample). We developed R program and Maple program for the computation of $Pr(X > Y)$. We also used the stress strength reliability $Pr(X > Y)$ for comparing test and control intensity measurements for each gene in replicated microarray experiments and computed the proportion of differentially expressed genes.

4.5.1 $Pr(X > Y)$ for the Asymmetric Type II Compound Laplace Distribution

Let $X$ and $Y$ are two continuous and independent random variables. Let $f_2$ denote the pdf of $Y$ and $F_1$ denote the cdf $X$. Then $Pr(X > Y)$ can be given as,

$$Pr(X > Y) = \int_{-\infty}^{\infty} F_2(z)f_1(z)dz.$$  \hfill (4.5.1)

Now we evaluate the $Pr(X > Y)$ for two independent $ACL$ distributions. Let $X$ and $Y$ are continuous and independent variables having $ACL$ distribution with parameters $\theta_i, \alpha_i, \beta_i$ and $\kappa_i$, $i = 1, 2$ respectively. The pdf and cdf of $ACL$ is given
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by

\[ f(x) = \frac{\kappa \alpha \beta}{1 + \kappa^2} \begin{cases} (1 + \kappa \beta(x - \theta))^{-(\alpha + 1)}, & \text{for } x > \theta \\ (1 - \frac{\beta}{\kappa}(x - \theta))^{-(\alpha + 1)}, & \text{for } x \leq \theta, \end{cases} \]  \hspace{1cm} (4.5.2)

and \( \theta \in \mathbb{R}, \alpha, \beta, \kappa > 0. \)

\[ F(x) = \begin{cases} 1 - \frac{1}{1+\kappa^2}[1 + \kappa \beta(x - \theta)]^{-\alpha}, & \text{for } x > \theta \\ \frac{\kappa^2}{1+\kappa^2}[1 - \frac{\beta}{\kappa}(x - \theta)]^{-\alpha}, & \text{for } x \leq \theta. \end{cases} \]  \hspace{1cm} (4.5.3)

From equation (4.3.1) we get the reliability \( R \) for the ACL distribution as follows.

For \( \theta_1 < \theta_2, \) \( R \) can be expressed as

\[
R_{[\theta_1 < \theta_2]} = \frac{\kappa_1 \kappa_2 \alpha_1 \beta_1}{(1 + \kappa_1^2)(1 + \kappa_2^2)} \left\{ \int_{-\infty}^{\theta_1} \left[ 1 - \frac{\beta_1}{\kappa_1} x \right]^{-(\alpha_1+1)} \left[ 1 - \frac{\beta_2}{\kappa_2} y \right]^{-\alpha_2} dz + \int_{\theta_1}^{\theta_2} \left[ 1 + \beta_1 \kappa_1 x \right]^{-(\alpha_1+1)} \left[ 1 - \frac{\beta_2}{\kappa_2} y \right]^{-\alpha_2} dz \right\} + \frac{\kappa_1 \alpha_1 \beta_1}{(1 + \kappa_1^2)} \int_{\theta_2}^{\infty} \left[ 1 + \beta_1 \kappa_1 x \right]^{-(\alpha_1+1)} \left[ 1 - \frac{1}{(1 + \kappa_2^2)} \left( 1 + \beta_2 \kappa_2 y \right)^{-\alpha_2} \right] dz,
\]

and if \( \theta_1 > \theta_2 \)

\[
R_{[\theta_1 > \theta_2]} = \frac{\kappa_1 \kappa_2^2 \alpha_1 \beta_1}{(1 + \kappa_2^2)(1 + \kappa_2^2)} \int_{-\infty}^{\theta_2} \left[ 1 - \frac{\beta_1}{\kappa_1} x \right]^{-(\alpha_1+1)} \left[ 1 - \frac{\beta_2}{\kappa_2} y \right]^{-\alpha_2} dz + \frac{\kappa_1 \alpha_1 \beta_1}{(1 + \kappa_2^2)} \left\{ \int_{\theta_2}^{\theta_1} \left[ 1 - \frac{\beta_1}{\kappa_1} x \right]^{-(\alpha_1+1)} \left[ 1 - \frac{1}{(1 + \kappa_2^2)} \left( 1 + \beta_2 \kappa_2 y \right)^{-\alpha_2} \right] dz + \int_{\theta_1}^{\infty} \left[ 1 + \beta_1 \kappa_1 x \right]^{-(\alpha_1+1)} \left[ 1 - \frac{1}{(1 + \kappa_2^2)} \left( 1 + \beta_2 \kappa_2 y \right)^{-\alpha_2} \right] dz \right\},
\]

where \( x = (z - \theta_1) \) and \( y = (z - \theta_2). \)

Thus, the reliability parameter \( R \) can be expressed as

\[ R = R_{[\theta_1 < \theta_2]} I_{[\theta_1 < \theta_2]} + R_{[\theta_1 > \theta_2]} I_{[\theta_1 > \theta_2]} \]  \hspace{1cm} (4.5.4)
where \( I(.) \) is the indicator function. The MLE of the \( R = P(X > Y) \) can be obtained by replacing the parameters \( \theta_1, \theta_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \kappa_1 \) and \( \kappa_2 \) in the expression of \( R \) by their MLE’s. Using Maple program we can evaluate the integrals and compute the maximum likelihood estimator of \( R \).

### 4.6 Conclusion

In this Chapter we have introduced ACL distribution, which is the heavy tailed generalization of the AL distribution. We studied various properties of the ACL and also derived the stress-strength reliability \( Pr(X > Y) \) for ACL.

### References


