Chapter 4

Escaping sets of composition of transcendental entire functions

4.1. Introduction

It is known that if \( f \) and \( g \) are transcendental entire functions then so are \( f \circ g \) and \( g \circ f \). Several results dealing with these two functions have also been studied, for instance Bergweiler and Wang in [15] proved that if \( f \) and \( g \) are transcendental entire functions then \( z \in J(f \circ g) \) if and only if \( g(z) \in J(g \circ f) \). In this chapter we have studied some more properties related to \( f \circ g \) and \( g \circ f \), where \( f \) and \( g \) are transcendental entire functions. In this chapter, we have given many results concerning various escaping sets of composition of transcendental entire functions. In second section, we have proved various theorems stated in first section. In third section, we have discussed permutable transcendental entire functions and gave a result related to 0\( \text{th} \) level of fast escaping set for permutable functions.

Our first result in this direction deals with the escaping set and fast escaping set of composition of transcendental entire functions and also with the residual Julia set. In fact Baker and Dominguez in [9] discussed residual Julia set, which contains those points of \( J(f) \) which do not lie on the boundary of any of Fatou components. The set containing
all these points is called residual Julia set, and is denoted by \( J_r(f) \), and the points in \( J_r(f) \) are called buried points. If \( J_r(f) \neq \emptyset \) then it is completely invariant, dense in \( J(f) \) and uncountably infinite. This set has also been discussed in Chapter 1, \( 1.1.6 \). Our first theorem gives a relation between escaping sets, fast escaping sets and residual Julia set of composition of transcendental entire functions.

**Theorem 4.1.1.** If \( f \) and \( g \) be two transcendental entire functions then following results hold

(a) \( g(I(f \circ g)) = I(g \circ f) \)

(b) \( g(A(f \circ g)) = A(g \circ f) \)

(c) \( g(J_r(f \circ g)) = J_r(g \circ f) \).

Recently Evdoridu \[27\] has defined a subset \( I(f, (a_n)) \) of \( I(f) \) as:

\[
I(f, (a_n)) = \{ z \in \mathbb{C} : |f^n(z)| \geq a_n, \text{ for } n \in \mathbb{N} \}
\]

Above Theorem 4.1.1 also hold good for this set as well, that is, \( g(I((f \circ g), (a_n))) = I((g \circ f), (a_n)) \). In fact this theorem is also true for the set where the iterates are not even escaping, that is the set of points whose iterate always remain bounded. For a transcendental entire function \( f \), the set \( K(f) \) has been studied where

\[
K(f) = \{ z : \text{there exists } R > 0 \text{ such that } |f^n(z)| \leq R \text{ for } n \geq 0 \}. \tag{4.1.1}
\]

This set also has been discussed in Chapter 1, \( 1.2.9 \).

**Theorem 4.1.2.** If \( f \) and \( g \) be two transcendental entire functions then

\[
g(K(f \circ g)) = K(g \circ f).
\]

In the same direction of escaping and fast escaping set we have “levels” of fast escaping set. The level 0 was first considered by Rippon and Stallard (though no name was given to it) in \[57\]. They introduced a subset of \( B(f)(= A(f)) \) of the form

\[
B_D(f) = \{ z : f^n(z) \notin T(f^n(D)), \text{ for } n \in \mathbb{N} \}, \tag{4.1.2}
\]

where \( D \) is any open disc meeting \( J(f) \). Here \( A(f) \) and \( B(f) \) are independent of \( R \) and \( D \) and this is because of the blowing up property of \( J(f) \) and the existence of \( L \in \mathbb{N} \) in
(1.2.5) and also in (1.2.6). Since no such variable $L$ exists in $B_D(f)$, so it will depend upon $D$.

Clearly $B_{D'}(f) \subset B_D(f)$ whenever $D \subset D'$, but converse need not to be true. It is interesting to note that while $B(f)$ is completely invariant, $B_D(f)$ need not (see Lemma 3.1 [57]). However we observe that there exists $M \in \mathbb{N}$ such that

$$f^m(B_D(f)) \subset B_D(f),$$

for $m \geq M$. (4.1.3)

Several other properties of $B_D(f)$ have been studied in [57]. On similar lines to $B_D(f)$, Rippon and Stallard in [59] defined levels of fast escaping sets.

$$A^L_R(f) = \{z : |f^n(z)| \geq M^{n+L}(R,f) \text{ for } n + L \geq 0, n \in \mathbb{N}\},$$

where $R > 0$ is such that $M(r,f) > r$, and $r \geq R$. We denote $A^0_R(f)$ by $A_R(f)$.

Clearly $A^L_R(f) \subset A^{-(L+1)}_R(f)$, $L \in \mathbb{N}$. Note that while $A(f)$ is independent of $R$, $A^L_R(f)$ and particularly $A_R(f)$ is not.

Also we note that while $A(f) = B(f)$, $A_R(f)$ and $B_D(f)$ are two different concepts having rather different properties, for instance, $f(A^L_R(f)) \subset A^{L+1}_R(f) \subset A^L_R(f)$ for all $L \in \mathbb{Z}$, one does not have $f(B_D(f)) \subset B_D(f)$ [59], however we have (4.1.3).

Since we have (4.1.3), thus one would also be interested in knowing what relation can we expect for $m < M$. In this regard we have shown,

$$f^n(B_D(f)) \subset B_{D_n}(f), \quad n \in \mathbb{N}$$

and also

$$f(B_D(f^n)) \subset B_{D_n}(f^n), \quad n \in \mathbb{N},$$

(4.1.5)

where $D_n$ is any open disc in $f^n(D)$ meeting $J(f)$. In fact we have the following theorem.
**Theorem 4.1.3.** Let \( f \) be a transcendental entire function and \( B_D(f) \) is as in (4.1.2) then for all \( n \in \mathbb{N} \),

\[
f^n(B_D(f)) \subset B_{D_n}(f),
\]

where \( D_n \) is any open disc in \( f^n(D) \) intersecting \( J(f) \).

We also have a result for composition of transcendental entire function.

**Theorem 4.1.4.** Let \( f \) and \( g \) be two transcendental entire functions then

\[
g(B_D(f \circ g)) \subset B_{D_1}(g \circ f),
\]

where \( D_1 \) is any open disc in \( g(D) \) intersecting \( J(g \circ f) \).

Our next theorem deals with semi-conjugacy. Bergweiler and Hinkkanen in [16], defined the concept of semiconjugacy. If \( f \) and \( h \) are two transcendental entire functions and \( g : \mathbb{C} \to \mathbb{C} \) is a non constant continuous map such that

\[
g \circ f = h \circ g
\]

then \( f \) is said to be semiconjugated to \( h \) (by \( g \)) and \( g \) is called semiconjugacy.

We shall prove

**Theorem 4.1.5.** If \( f \) and \( h \) be two transcendental entire functions and \( g \) be a polynomial map satisfying (4.1.7) then

\[
g(I(f)) = I(h).
\]

### 4.2. PROOFS OF THEOREMS

In this section we have proved theorems stated above.

For proving Theorem 4.1.1, we need the following result from [15].
LEMMA 4.2.1. Let $f$ and $g$ be two non linear entire functions and $z \in \mathbb{C}$. Then $z \in J(f \circ g)$ if and only if $g(z) \in J(g \circ f)$.

Proof of Theorem 4.1.1 (a). Let $z \in I(f \circ g)$. Then $(f \circ g)^n(z) \to \infty$, as $n \to \infty$.

On rewriting, $f(((g \circ f)^{n-1})g)(z) \to \infty$, as $n \to \infty$. Clearly $((g \circ f)^{n-1})g(z) \to \infty$, as $n \to \infty$, for if not then $\lim_{n \to \infty}((g \circ f)^{n-1})g(z) = p$, where $p \neq \infty$, and so $p$ must be a pole of $f$, contradicting to the fact that $f$ is entire. Thus $g(z) \in I(g \circ f)$. Hence

$$g(I(f \circ g)) \subseteq I(g \circ f). \quad (4.2.1)$$

Now let $z \in I(g \circ f)$ and a similar argument as above shows that $f(I(g \circ f)) \subseteq I(f \circ g)$, and so $g(f(I(g \circ f))) \subseteq g(I(f \circ g))$. Using complete invariance of $I(g \circ f)$ we have

$$I(g \circ f) \subseteq g(I(f \circ g)). \quad (4.2.2)$$

from (4.2.1) and (4.2.2) we have that

$$g(I(f \circ g)) = I(g \circ f).$$

(b). If $z \in A(f \circ g)$, then using (1.2.6), there exists $L \in \mathbb{N}$, such that

$$(f \circ g)^{n+L}(z) \notin T((f \circ g)^n(D)),$$

where $D$ is any open disc meeting $I(f \circ g)$. Thus, $f(((g \circ f)^{n+L-1})g)(z) \notin T(f((g \circ f)^{n-1}))$.

Let us denote $g(D)$ by $D'$. Then clearly by using Lemma 4.2.1, $D'$ is a region intersecting $I(g \circ f)$ and further since $f(T(U)) \subset T(f(U))$ for any region $U$, we have

$$(g \circ f)^{n+L-1}g(z) \notin T(((g \circ f)^{n-1})(D')),$$

for all $n \in \mathbb{N}$.

So we have $g(z) \in A(g \circ f)$. Hence

$$g(A(f \circ g)) \subseteq A(g \circ f). \quad (4.2.3)$$

Next by using same arguments in part (a) above and complete invariance of $A(g \circ f)$, we have the required result.

In order to prove part (c), we need following lemmas.
LEMMA 4.2.2. If \( f \) is any entire map and \( W \) is any domain then

\[
\partial(f(W)) \subset f(\partial(W)).
\]

LEMMA 4.2.3. If \( f \) is any entire map and \( W \) is Fatou component, then

\[
f(\partial(W)) \subset \partial(f(W)).
\]

Proof. Let \( y \in f(\partial(W)) \), then \( y = f(x) \) where \( x \in \partial(W) \). So there exists a sequence \( x_n \in W \) such that \( x_n \to x \). Now \( f(x_n) \in f(W) \) and \( f(x_n) \to f(x) = y \). Therefore, \( y \in \overline{f(W)} \). Here \( y \notin f(W) \), otherwise \( x \) will be a point of Fatou component contradicting the fact that \( x \in \partial(W) \). Therefore, \( y \in \partial(f(W)) \).

(c). We first show that for any \( z \in J_r(fog) \), \( g(z) \in J_r(gof) \). For if not, then \( g(z) \) must belongs to boundary of some of component say, \( V \subset f(gof) \). Thus \( f(g(z)) \in f(\partial V) \subset \partial(f(V)) \), which follows from Lemma 4.2.2 and Lemma 4.2.3. Denoting \( f(g(z)) \) by \( \xi \), there exists a sequence \( z_n \in f(V) \) such that \( z_n \to \xi \) as \( n \to \infty \).

Let \( W \) be the component of \( F(fog) \) containing \( f(V) \), then \( W \setminus f(V) \) contains at most a single point [15]. So leaving this single point if necessary, we may assume that all \( z_n \in W \). As \( z_n \to \xi \), it follows that \( \xi \in \overline{W} \). This is not possible, for if \( \xi \in \partial W \) then \( \xi \notin J_r(fog) \), and by complete invariance of \( J_r(fog) \), \( z \notin J_r(fog) \). Also \( \xi \) is not an interior point of \( W \), for \( W \) is a component of \( F(fog) \) and \( \xi \in J_r(fog) \) by complete invariance. Thus we have shown that \( g(J_r(fog)) \subset J_r(gof) \).

Next on similar lines, \( f(J_r(gof)) \subset J_r(fog) \) and so \( g(f(J_r(gof))) \subset g(J_r(fog)) \), and consequently by complete invariance of \( J_r(gof) \), we have \( J_r(gof) \subset g(J_r(fog)) \). This completes the proof.

Proof of Theorem 4.1.2. Let \( z \in K(f \circ g) \). So there exists a constant \( A \) such that

\[
|(f \circ g)^n(z)| \leq A, \quad \forall n \in \mathbb{N}.
\]  

Therefore \( |f(g \circ f)^{n-1}(g(z))| \leq A, \quad \forall n \in \mathbb{N} \). Let \( B = M(A, g) \). Then clearly \( B \) is also a constant. Now let us take
\[(g \circ f)^{n-1}(g(z)) = \left|g((f \circ g)^n(z))\right| = |g(t)|, \text{ where } t = (g \circ f)^{n-1}(z) \text{ so } |t| \leq A, \text{ using (4.2.4). Thus } (g \circ f)^{n-1}(g(z)) \leq M(A, g) = B. \text{ So } g(z) \in K(g \circ f), \text{ implying,}\]
\[g(K(f \circ g)) \subset K(g \circ f). \quad (4.2.5)\]

Now by taking \(z \in K(g \circ f)\) and repeating the above procedure and using the complete invariance of \(K(g \circ f)\), we have the required result.

Here we note that we have proved much more. In fact if \(z \in K(f \circ g)\) with \(|(f \circ g)^n(z)| \leq A, \forall n \in \mathbb{N}\), then \(g(z) \in K(g \circ f), \text{ with } |(g \circ f)^n(z)| \leq M(A, g)\).

**Proof of Theorem 4.1.3** Let \(z \in B_D(f)\). So \(f^n(z) \notin T(f^n(D))\), for \(n \in \mathbb{N}\).
Thus \(f^{n-1}(f(z)) \notin T(f^{n-1}(f(D))), \text{ for } n \in \mathbb{N}\), and so \(f^{n-1}(f(z)) \notin T(f^{n-1}(D_1)), \text{ for } n \in \mathbb{N}\), where \(D_1\) is any open disc in \(f(D)\) meeting \(J(f)\) which follows as \(J(f)\) is completely invariant under \(f\). Thus \(f(z) \in B_{D_1}(f)\), and so
\[
f(B_D(f)) \subset B_{D_1}(f). \quad (4.2.6)\]

By induction, we have \(f^n(B_D(f)) \subset B_{D_n}(f)\), for \(n \in \mathbb{N}\), where \(D_n\) is any open disc in \(f^n(D)\) meeting \(J(f)\).

Note that by the expanding neighbourhood property of Julia point, there exists \(m \in \mathbb{N}\), such that \(f^m(D) \supset D\) and so Theorem 4.1.3 would also yield (4.1.3).

**Proof of Theorem 4.1.4** Let \(z \in B_D(f \circ g)\). Then
\[
(f \circ g)^n(z) \notin T((f \circ g)^n(D)), \text{ for } n \in \mathbb{N}, \quad (4.2.7)
\]
where \(D\) is any open disc meeting \(J(f \circ g)\), and so \(f(((g \circ f)^{-1})g)(z) \notin T(f((g \circ f)^{-1})g)\).
Clearly
\[
((g \circ f)^{-1})g(z) \notin T(((g \circ f)^{-1})(g(D)).
\]
for if not, then \(f(((g \circ f)^{-1})g(z)) \in f(T(((g \circ f)^{-1})(g(D)))) \subseteq T(f(((g \circ f)^{-1})(g(D)))\)
as \(f(T(U)) \subset T(f(U))\), for any region \(U\).
Thus \((f \circ g)^n(z) \in T((f \circ g)^n(D))\), contradicting (4.2.7).
Thus \(((g \circ f)^{n-1})g(z) \notin T(((g \circ f)^{n-1})(g(D)))\), and so \(((g \circ f)^{n-1})g(z) \notin T(((g \circ f)^{n-1})g(D))\)

where \(D_1\) is any open disc in \(g(D)\) intersecting \(J(g \circ f)\) using Lemma 4.2.1

So we have \((g \circ f)^{n-1}g(z) \in T((g \circ f)^{n-1}(D_1))\), and so \((g \circ f)^{n-1}g(z) \in T((g \circ f)^{n-1}D_1)\).

where \(D_1\) is any open disc in \(g(D)\) intersecting \(J(g \circ f)\) using Lemma 4.2.1.

So we have \(g(z) \in B_{D_1}(g \circ f)\) and hence

\[ g(B_D(f \circ g)) \subset B_{D_1}(g \circ f). \] (4.2.8)

Proof of Theorem 4.1.5 First if \(z \in I(h)\), then \(h^n(z) \to \infty\) as \(n \to \infty\). Also as \(g\) is a polynomial there exists some \(z_0 \in \mathbb{C}\) such that \(g(z_0) = z\) and so \(h^n(g(z_0)) \to \infty\) as \(n \to \infty\). Using (4.1.7) it follows that \(g(f^n(z_0)) \to \infty\) as \(n \to \infty\) and so \(f^n(z_0) \to \infty\) as \(n \to \infty\), consequently \(z \in g(I(f))\).

Next if \(z \in g(I(f))\), then \(z = g(\xi)\) where \(f^n(\xi) \to \infty\) as \(n \to \infty\). As \(g\) is a polynomial, it follows that \(g(f^n(\xi)) \to \infty\) as \(n \to \infty\) so \(h^n(g(\xi)) \to \infty\) as \(n \to \infty\). Thus \(g(\xi) \in I(h)\), so \(g(I(f)) \subseteq I(h)\). This proves the theorem.

4.3. PERMUTABLE TRANSCENDENTAL ENTIRE FUNCTIONS

Permutable transcendental entire functions always keep attracting Mathematicians due to long standing conjecture of Baker that if \(f\) and \(g\) are two permutable transcendental entire functions then their Julia sets are equal. We have discussed this in Chapter 1.

Singh and Wang in [69] discussed the Julia sets for permutable holomorphic functions. They showed that for permutable transcendental entire functions, fast escaping sets of \(f \circ g\) satisfies

\[ A(f \circ g) \subset A(f) \cap A(g). \]

It is reasonable to expect a similar result for the levels of fast escaping sets. In this regard we have proved the following.

Theorem 4.3.1. Let \(f\) and \(g\) be two permutable transcendental entire functions with \(f(0) = g(0) = 0\), then for large \(R\)

\[ A_{6R}(f \circ g) \subset A_R(f) \cap A_R(g). \]
Another result dealing with permutable transcendental entire functions in connection with $K(f)$ and $J_r(f)$ is as follows:

**THEOREM 4.3.2.** Let $f$ and $g$ be two permutable transcendental entire functions then,

(a) $g(K(f)) \subset K(f)$
(b) $K(f) \cup K(g) \subset K(f \circ g)$
(c) $J_r(f) \cup J_r(g) \subset J_r(f \circ g)$.

Motivation for this result was a result of Baker [2] that

\[ J(f) \cup J(g) \subset J(f \circ g). \] \tag{4.3.1}

Note that in above equation (4.3.1), if $J(f)$ and $J(g)$ form spider’s web, then $J(f \circ g)$ also form spider’s web being connected superset of spider’s web [59].

In order to prove Theorem 4.3.1, we need the following lemmas from [18] and [59] respectively.

**LEMMA 4.3.1.** [18] If $f$ and $g$ are two transcendental entire functions with $g(0) = 0$, then for $r > 0$,

\[ M(6r, f \circ g) \geq M(M(r, g), f). \]

**LEMMA 4.3.2.** [59] Let $f$ be a transcendental entire function and let $D = \{z : |z| < R\}$. If $R > 0$ be sufficiently large then for $n \in \mathbb{N}$,

\[ \{z : |z| \leq M^n(R/2, f)\} \subset \{z : |z| \leq M(R, f^n)\} \subset \{z : |z| \leq M^n(R, f)\}. \]

**Proof of Theorem 4.3.1** Let $z \in A_{6R}(f \circ g)$. Then for all $n \in \mathbb{N}$, $|(f \circ g)^n(z)| \geq M^n(6R, f \circ g)$, where $R > 0$ is such that $M(r, f \circ g) > r$ and $r \geq 6R$. So if $R$ is sufficiently large then by Lemma 4.3.2 and 4.3.1 for every $n \in \mathbb{N}$ we have,

\[ |(f^n g^n)(z)| \geq M^n(6R, f \circ g) = M^n(6R, g \circ f) \geq M(6R, g^n \circ f^n) \geq M(M(R, f^n), g^n). \]

Thus

\[ |f^n(g^n(z))| \geq M(M(R, f^n), g^n). \] \tag{4.3.2}
Also

\[ M(|f^n(z)|, g^n) \geq |g^n(f^n(z))| = |f^n(g^n(z))|. \quad (4.3.3) \]

Clearly from (4.3.2) and (4.3.3)

\[ M(|f^n(z)|, g^n) \geq M(M(R, f^n), g^n). \]

Using Maximum modulus principle we obtain

\[ |f^n(z)| \geq M(R, f^n), \text{ for every } n \in \mathbb{N} \]

and so \( z \in A_R(f) \). Since \( f \) and \( g \) are permutable and by symmetry, we have \( z \in A_R(g) \) as well. Thus

\[ A_{6R}(f \circ g) \subset A_R(f) \cap A_R(g). \]

**Corollary 4.3.1.** If in the Theorem 4.3.1, \( A_R(f \circ g) \) form a spider’s web then \( A_R(f) \cup A_R(g) \) also forms spider’s web.

Since \( A_R(f \circ g) \subset A_{6R}(f \circ g) \), as \( 6R > R \), so proof is immediate from above theorem and using the result of Rippon and Stallard [59], that connected superset of a spider’s web is also a spider’s web.

**Proof of Theorem 4.3.2.** (a) Let \( z \in K(f) \), so there exists a constant \( A \) such that \( |f^n(z)| \leq A, \quad \forall n \in \mathbb{N} \) and therefore \( |g(f^n(z))| \leq M(A, g) = B, \quad \forall n \in \mathbb{N} \), here \( B \) is also a constant.

Thus \( |f^n(g(z))| \leq B \), implying that \( g(z) \in K(f) \), so \( g(K(f)) \subset K(f) \).

(b) For second set relation let \( z \in K(f) \cup K(g) \), particularly let \( z \in K(f) \), so by above part we have \( g(z) \in K(f) \). So \( g(g(z)) \in g(K(f)) \subset K(f) \), thus in general

\[ g^n(z) \in K(f), \quad \forall n \in \mathbb{N}. \quad (4.3.4) \]

Now let \( \xi \in K(f) \cup K(g) \), particularly let \( \xi \in K(f) \), so there exists some constant say \( M_\xi \) such that

\[ |f^n(\xi)| \leq M_\xi, \quad \forall n \in \mathbb{N}. \quad (4.3.5) \]
Now for any $m \in \mathbb{N}$, by (4.3.4) $g^m(\xi) \in K(f)$, so $|f^n(g^m(\xi))| \leq M \xi, \ \forall n \in \mathbb{N}$. Here right side is independent of $m$, we have

$$|g^m(\xi)| \leq M \xi, \ \forall m \in \mathbb{N}.$$ 

So $\xi \in K(f \circ g)$, so $K(f) \cup K(g) \subset K(f \circ g)$.

Let $z \in J_f$, so $z$ does not lie along the boundary of any of the Fatou component or we may write

$$z \notin \partial U \ \text{for any} \ U \subset F(f). \quad (4.3.6)$$

Let if possible $z \in \partial V$ for some $V \subset F(f \circ g)$, using (4.3.1), we have $F(f \circ g) \subset F(f) \cap F(g) \subset F(f)$. So $V \subset F(f)$ and $z \in \partial V$, this contradicts (4.3.6). Therefore $z \in J_f$, implying $J_f \subset J_{f \circ g}$ and by symmetry we have $J_g \subset J_{f \circ g}$. Thus

$$J_f \cup J_g \subset J_{f \circ g}.$$