Chapter 2

On some properties of spider’s web and maximum term of transcendental entire functions

2.1. INTRODUCTION

In this chapter we have discussed some results associated with the spider’s web, which was first considered by Rippon and Stallard [59]. We have given example for which some results hold good for fast escaping set, but not for the levels of fast escaping set. In second section we have proved various results associated with spider’s web and levels of fast escaping sets. In third section we have proved that if triangulation is applied on a spider’s web, then it again forms a spider’s web. In forth section we have consider bounded Fatou components and spider’s web and gave a result associated with composition of transcendental entire functions. In fifth section we have obtained some results connecting the maximum term of transcendental functions with spider’s web and we have defined a set $B_R(f)$ involving maximum term, which is analogues to $A_R(f)$ and have given few theorems where $B_R(f)$ forms spider’s web.

In [59], Rippon and Stallard defined subsets of $A(f)$ called levels of $A(f)$, as defined in Chapter 1 (4.1.2). This discovery lead to simplification of proofs of several earlier
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results and new results concerning the properties of $A(f)$ and also helped in putting a step ahead towards Eremenko’s conjecture [1.2.1] by proving that $I(f)$ has at least one unbounded component namely $A(f)$, which has all of its components unbounded known as levels. Here

$$A(f) = \bigcup_{L \in \mathbb{N}} A^{-L}_{R}(f)$$

(2.1.1)

and thus $A^{-L}_{R}(f) \subset A^{-{(L+1)}}_{R}(f)$, for $L \in \mathbb{N}$.

For $0^{th}$ level $A_{R}(f)$, Rippon and Stallard [59] proved the following:

**Theorem A.** Let $f$ be a transcendental entire function and $n \in \mathbb{N}$. If $R > 0$ is sufficiently large then

$$A_{R}(f) \subset A_{R}(f^n) \subset A_{R/2}(f).$$

Here we have supplemented the above result by proving the following theorem, which deals with the levels of higher order.

**Theorem 2.1.1.** Let $f$ be a transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$, for $r \geq R$ and $L, n \in \mathbb{N}$, then

$$A_{R}^{nL}(f) \subset A_{R}^{n}(f^{L}) \subset A_{R/2}^{nL}(f)$$

and also

$$A_{R}^{nL}(f) \subset A_{R}^{n}(f^{n}) \subset A_{R/2}^{nL}(f).$$

An interesting observation of the above theorem is that, if $p$ is composite number having two different factorizations, say $p = p_{1} \cdot q_{1} = p_{2} \cdot q_{2}$, then

$$A_{R}^{p}(f) \subset A_{R}^{p_{1}}(f^{q_{1}}) \subset A_{R/2}^{p}(f)$$

and also

$$A_{R}^{p}(f) \subset A_{R}^{p_{2}}(f^{q_{2}}) \subset A_{R/2}^{p}(f).$$
Note also that $A_{R}^{p_{1}}(f^{q_{1}})$ need not equal $A_{R}^{p_{2}}(f^{q_{2}})$, for instance, considering $f(z) = e^{z}$, then the above two quantities are not equal.

Rippon and Stallard [59] discovered infinite spider’s web, which has been defined in Chapter 1 1.2.2.

They showed that for sufficiently large $R$, if $A_{R}(f)^{c}$ has a bounded component then each of $A_{R}(f), A(f), I(f)$ is a spider’s web. Several other examples of functions having $A_{R}(f)$ as spider’s web were also given. We observed that if cut a spider’s web by an infinite arc, it may be divided in to countable number of components. For this, let $A_{n} = \{z : |z| = n\}$ and $B_{1} = \{z : |z| > 1, \arg z = \pi/4\}$, then

$$S = (\bigcup_{n \in \mathbb{N}} A_{n}) \cup B_{1}$$

is a spider’s web and $S \setminus B_{1}$ contains countable number of components.

![Figure 2.1.1. Removal of arc yielding countable components of spider’s web](image)

Here we note that a bounded set can not be a spider’s web. For this, let us suppose $S$ is a bounded set and suppose it also form a spider’s web, then there exits a sequence of bounded simply connected domain $G_{n}$, such that $G_{n} \subset G_{n+1}$, for $n \in \mathbb{N}, \partial G_{n} \subset S$ and $\bigcup_{n \in \mathbb{N}} G_{n} = \mathbb{C}$. Since $S$ is bounded so there exits a positive constant $K$, such that $|z| \leq K$, for all $z \in S$. Also $\bigcup_{n \in \mathbb{N}} G_{n} = \mathbb{C}$ and $G_{n}$ are bounded, so there exists a $G_{t}(\text{say})$ whose boundary contains $z_{0}$ such that $|z_{0}| > K$, thus boundary of $G_{t}$ will not be a part
of $S$, which is contradiction to $\partial G_n \subset S$, for $n \in \mathbb{N}$.

Another observation we made that if $S$ is a spider’s web and a curve $\gamma \to \infty$, then $S \cup \gamma$ is a spider’s web. In fact if $\gamma$ is any curve intersecting $S$ then also $S \cup \gamma$ is a spider’s web.

Following are few results associated with spider’s web.

**Theorem 2.1.2.** Let $f$ be a transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$, for $r \geq R$ and $L, n \in \mathbb{N}$, then $A^{nL}_R(f)$ is a spider’s web if and only if $A^n_R(f^L)$ is a spider’s web.

**Theorem 2.1.3.** Let $S_i, i = 1, 2, 3, \ldots n$ be spider’s web, then $\bigcup_{i=1}^n S_i$ is also a spider’s web.

Note that intersection of two spider’s web need not form spider’s web as it need not to be connected. Also continuous image of a spider’s web need not form spider’s web, for this let $f : \mathbb{C} \to \mathbb{C}$ be a non constant continuous map defined by

\[
f(0) = 0
\]

\[
f(re^{i\theta}) = re^{i\theta} \quad \text{for } 0 < r \leq 1, \ 0 < \theta \leq 2\pi
\]

\[
f(re^{i\theta}) = e^{i\theta} \quad \text{for } r > 1, \ 0 < \theta \leq 2\pi
\]

Then clearly $f$ is a continuous map and if $S$ be any spider’s web, then $f(S)$ is not a spider’s web being bounded. However if we take $f$ to be continuous open map, then we have following theorem.

**Theorem 2.1.4.** Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be open continuous map, which takes a bounded domain to bounded domain. If $S$ be a spider’s web, then $f(S)$ is also a spider’s web.

In particular if $f$ is an entire function without any picard exceptional value, then also above theorem holds.

**Theorem 2.1.5.** If $E$ be a spider’s web then all the components of $E^c$ are bounded.

### 2.2. Proofs of Theorems on Levels of Fast Escaping Sets and Spider’s Web

For the proof of Theorem 2.1.1 we need following lemmas.
2.2. PROOFS OF THEOREMS ON LEVELS OF FAST ESCAPING SETS AND SPIDER’S WEB

LEMMA 2.2.1. Let \( f \) be a transcendental entire function and let \( D = \{ z : |z| < R \} \).
If \( R > 0 \) be sufficiently large then
\[
\{ z : |z| \leq M^n(R/2, f) \} \subset \{ z : |z| \leq M(R, f^n) \} \subset \{ z : |z| \leq M^n(R, f) \}
\] (2.2.1)

LEMMA 2.2.2. For any \( L, K \in \mathbb{N} \) and sufficiently large \( R \),
\[
M^{KL}(R, f) \geq M^K(R, f^L).
\] (2.2.2)

We have proved this by using induction. For \( k = 1 \), result is true by above Lemma 2.2.1.
Let us assume that
\[
M^{KL}(R, f) \geq M^K(R, f^L).
\] (2.2.3)

Finally, \( M^{(K+1)L}(R, f) = M^{(KL+L)}(R, f) \geq M^{(KL+1)}(R, f) \), as \( M \) is increasing function of \( r \) and Maximum Modulus theorem. Next
\[
M^{(KL+1)}(R, f) = M(M^{KL}(R, f), f) \geq M(M^K(R, f^L), f) = M^{(K+1)}(R, f^L),
\]
using Equation (2.2.3).

**Proof of Theorem 2.1.1** Let \( R > 0 \) be such that \( M(r, f) > r \), for \( r \geq R, L, n \in \mathbb{N} \).
If \( z \in A_R^{nL}(f) \), then
\[
|f^m(z)| \geq M^{m+nL}(R, f), \text{ for } m \in \mathbb{N},
\]
thus \( |(f^L)^m(z)| = |f^{mL}(z)| \geq M^{mL+nL}(R, f) = M^{(m+n)L}(R, f) \geq M^{m+n}(R, f^L), \) for \( m \in \mathbb{N} \) by Lemma 2.2.2. Hence \( z \in A_R^{nL}(f^L) \), and so
\[
A_R^{nL}(f) \subset A_R^{n}(f^L).
\] (2.2.4)

Now let \( z \in A_R^{n}(f^L) \), then
\[
|(f^L)^m(z)| \geq M^{m+n}(R, f^L) \geq M^{(m+n)L}(R/2, f), \text{ for } m \in \mathbb{N}
\]
by Lemma 2.2.1. So we must have
\[
|f^m(z)| \geq M^{m+nL}(R/2, f), \text{ for } m \in \mathbb{N}
\]
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Hence \( z \in A_R^{nL/2}(f) \). Thus

\[
A_R^n(f^L) \subset A_R^{nL/2}(f).
\]

(2.2.5)

From (2.2.4) and (2.2.5) we have

\[
A_R^{nL}(f) \subset A_R^n(f^L) \subset A_R^{nL/2}(f).
\]

The second set relation follows on similar lines.

**Remark 2.2.3.** Theorem A cannot be generalised for each level. That is, the set relation

\[
A_R^L(f) \subset A_R^L(f^m) \subset A_R^{L/2}(f)
\]

need not hold.

In order to disprove the above set relation, we have to show that following inequality doesn’t hold.

\[
M^{mn+L}(R, f) \geq M^{n+L}(R, f^m).
\]

(2.2.6)

For this, take \( m = 2 \), consider \( f(z) = e^z \) and take \( n = 1, L = 2 \), in left and right side of above equation separately. Here \( M(R, f) = e^r, M^2(R, f) = e^{e^r} \), ... as \( f^2(z) = e^{e^z} \). 

Left side yields,

\[
M^{2+1+2}(R, f) = M^4(R, f) = e^{3r}.
\]

And right side is

\[
M^{1+2}(R, f^2) = e^{(6)r}.
\]

So Equation (2.2.6) is not true, where \( e^{(k)r} = e^{(e^{(k-1)r})} \).

For proving Theorem 2.1.2 we need following lemmas:

**Lemma 2.2.4.** [59] Let \( f \) be a transcendental entire function. Let \( R > 0 \) be such that \( M(r, f) > r \), for \( r \geq R \) and \( L \in \mathbb{Z} \).

(a) If \( G \) is a bounded component of \( A_R^L(f)^c \) then \( \partial G \subset A_R^L(f) \) and \( f^n \) is a proper map of \( G \) on to a bounded component of \( A_R^{n+L}(f)^c \), for each \( n \in \mathbb{N} \).

(b) If \( A_R^L(f)^c \) has a bounded component, then \( A_R^L(f) \) is a spider’s web and hence every component of \( A_R^L(f)^c \) is bounded.

(c) \( A_R(f) \) is a spider’s web if and only if \( A_R^L(f) \) is a spider’s web.

(d) Let \( R' > R \) then \( A_R(f) \) is a spider’s web if and only if \( A_{R'}(f) \) is a spider’s web.
As an consequence of Lemma 2.2.4 we have the following Lemma:

**Lemma 2.2.5.** Let $f$ be a transcendental entire function. Let $R > 0$ be such that $M(r,f) > r$, for $r \geq R$ and $L \in \mathbb{Z}$.

(a) If $A_{R}^{L}(f)^{c}$ has a bounded component then, $A_{R}^{m}(f)$ is a spider’s web, for all $m \geq L + 1$.

(b) Let $R' > R$ then $A_{R}^{L}(f)$ is a spider’s web if and only if $A_{R'}^{L}(f)$ is a spider’s web.

**Proof (a)** If $A_{R}^{L}(f)^{c}$ has a bounded component then so has $A_{R}^{L+1}(f)^{c}$, by Lemma 2.2.4(a). So $A_{R}^{L+1}(f)$ is spider’s web, by Lemma 2.2.4(b). In a similar way we can prove that $A_{R}^{L+2}(f), A_{R}^{L+3}(f), \ldots$ all will be spider’s web. In general $A_{R}^{m}(f)$, for all $m \geq L + 1$, will be spider’s web.

(b) Proof is immediate from 2.2.4 (c) and (d).

**Proof of Theorem 2.1.2.** Firstly suppose that $A_{R}^{n}(f)$ is a spider’s web. It follows from Lemma 2.2.4(b), that each component of $A_{R}^{n}(f)^{c}$ is bounded.

We know by Theorem 2.1.1 that $A_{R}^{n}(f) \subset A_{R}^{n}(f^{L})$. So each component of $A_{R}^{n}(f^{L})^{c}$ is bounded. Thus by Lemma 2.2.4(b), $A_{R}^{n}(f^{L})$ is a spider’s web. Conversely, let us suppose that $A_{R}^{n}(f^{L})$ is a spider’s web. If $R' > 2R$ for $R$ sufficiently large, then we have by Theorem 2.1.1 $A_{R'}^{n}(f^{L}) \subset A_{R'}^{n}(f^{L})$. Now from Lemma 2.2.5(b), $A_{R'}^{n}(f^{L})$ is a spider’s web. Hence by Lemma 2.2.4(b), it follows that every component of $A_{R'}^{n}(f^{L})^{c}$ is bounded. So every component of $A_{R'/2}^{nL}(f)^{c}$ is bounded. So by Lemma 2.2.4(b), $A_{R'/2}^{nL}(f)$ is a spider’s web. Hence by Lemma 2.2.5(b), $A_{R}^{nL}(f)$ is a spider’s web.

**Remark 2.2.6.** Here we also noted that for $L, m \in \mathbb{N}$, $A_{R}^{L}(f)$ is a spider’s web if and only if $A_{R}^{L}(f^{m})$ is a spider’s web, which can easily be derived from Lemma 2.2.4(c) and Theorem 8.4 [59].

For proving the next theorem we have introduced a new notation. If $E$ is a spider’s web, then by Definition 1.2.2, there exist sequence of bounded simply connected domains $G_{n}$ with $G_{n} \subset G_{n+1}$, for $n \in \mathbb{N}, \partial G_{n} \subset E$, for $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} G_{n} = \mathbb{C}$. It is quite possible that there might exist more than one sequence of such domains say $(H_{m})_{m \in \mathbb{N}}$.
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In order to distinguish the spider’s web with corresponding domains we shall use the notation \((E, G_n)_{n \in \mathbb{N}}\) and \((E, H_m)_{m \in \mathbb{N}}\) respectively.

**Proof of Theorem 2.1.3** Using induction, it is sufficient to prove the theorem for two spider’s web. Let \((S, G_n)_{n \in \mathbb{N}}\) and \((T, H_m)_{m \in \mathbb{N}}\) be two spider’s webs.

Now consider \(G_1\). Then there exits some \(H_m\), such that \(G_1 \cap H_m \neq \emptyset\). This is possible since \(\bigcup_{m \in \mathbb{N}} H_m = \mathbb{C}\). Let \(k\) be the smallest positive integer such that \(G_1 \cap H_k \neq \emptyset\).

Let \(\mathcal{T}(D)\) denotes the union of domain \(D\) with all it’s bounded complementary components, and define \(K_n = \mathcal{T}(G_n \cup H_{k+(n-1)})\), \(n = 1, 2, \ldots\)

Then clearly, \(\bigcup_{n=1}^{\infty} K_n \supset \bigcup_{n=1}^{\infty} G_n = \mathbb{C}\), and for \(n = 1, 2, \ldots\), \(G_n\) and \(H_{k+(n-1)}\) are simply connected domains and \(G_n \cap H_{k+(n-1)} \supset G_1 \cap H_k \neq \emptyset\), so \(\mathcal{T}(G_n \cup H_{k+(n-1)})\) is simply connected domain. Thus \(K_n\) is simply connected as well as bounded domain being union of two bounded domains. Further

\[ K_{n+1} = \mathcal{T}(G_{n+1} \cup H_{k+n}) \supset \mathcal{T}(G_n \cup H_{k+(n-1)}) = K_n \]  
\[ \partial K_n = \partial \mathcal{T}((G_n \cup H_{k+(n-1)})) \subset \partial (G_n) \cup \partial (H_{k+(n-1)}) \subset S \cup T. \]

Thus \((S \cup T, K_n)_{n \in \mathbb{N}}\) is also a spider’s web.

For proving the Theorem 2.1.4, we shall need the following lemma.

**Lemma 2.2.7.** (a) If \(A_1\) and \(A_2\) are domains such that \(A_1 \subset A_2\), then

\[ \mathcal{T}(A_1) \subset \mathcal{T}(A_2), \]

hence

\[ \mathcal{T}(A_1) \cup \mathcal{T}(A_2) = \mathcal{T}(A_1 \cup A_2). \]

(b) Let \((A_n)_{n \in \mathbb{N}}\) be a family of domains, with \(A_1 \subset A_2 \subset A_3 \subset \ldots\), then

\[ \bigcup_{n \in \mathbb{N}} \mathcal{T}(A_n) = \mathcal{T}(\bigcup_{n \in \mathbb{N}} A_n). \]
Proof. For (a), Let $z \in \mathcal{T}(A_1)$. If $z \in A_1$ then $z \in A_2 \subset \mathcal{T}(A_2)$.

If $z$ be in bounded complementary component of $A_1$, it is sufficient to show that $z$ does not belong to any unbounded complementary component of $A_2$. For suppose, it does. Then there exists an arc in $A_2^c$ joining $z$ to $\infty$. As $A_1 \subset A_2$, so this arc lies in $A_1^c$. Consequently $z$ lies in unbounded complementary component of $A_1$. This contradiction proves (a). The other results are simple set theoretic consequences of (a).

Note: The conditions imposed on $A_1, A_2$ in Lemma 2.2.7 are necessary, for instance, let

$$A_1 = \{(x, y) : -2 \leq x \leq 1, -2 \leq y \leq -1\} \cup \{(x, y) : -2 \leq x \leq 1, 1 \leq y \leq 2\} \cup \{(x, y) : -2 \leq x \leq -1, -1 \leq y \leq 1\}$$

and

$$A_2 = \{(x, y) : 0 \leq x \leq 2, -2 \leq y \leq -1\} \cup \{(x, y) : 0 \leq x \leq 2, 1 \leq y \leq 2\} \cup \{(x, y) : 1 \leq x \leq 2, -1 \leq y \leq 1\}.$$

Then

$$\mathcal{T}(A_1) = A_1$$

and

$$\mathcal{T}(A_2) = A_2,$$

whereas

$$\mathcal{T}(A_1 \cup A_2) = \{(x, y) : -2 \leq x \leq 2, -2 \leq y \leq 2\}.$$ 

So $\mathcal{T}(A_1) \cup \mathcal{T}(A_2) \neq \mathcal{T}(A_1 \cup A_2)$.

Proof of Theorem 2.1.4 Let $(S, S_n)_{n \in \mathbb{N}}$ be a spider's web. Let $H_n = \mathcal{T}(f(S_n))$ and denote $H = f(S)$.

Here clearly $H_n$ are bounded simply connected domains. Now $\partial H_n = \partial(\mathcal{T}(f(S_n))) \subset f(\partial S_n) \subset f(S) = H$. Further $H_n = \mathcal{T}(f(S_n)) \subset \mathcal{T}(f(S_{n+1})) = H_{n+1}$, for $n \in \mathbb{N}$, by Lemma 2.2.7. Hence in order to show that $f(S)$ is a spider’s web it only remains that $\bigcup_{n=1}^\infty H_n = \mathbb{C}$.

For this consider any $z \in \mathbb{C}$, then there exists some $\zeta \in \mathbb{C}$ such that $z = f(\zeta)$, $f$. Also as $\bigcup_{n \in \mathbb{N}} S_n = \mathbb{C}$, it follows that $\zeta \in S_n$, for some $n \in \mathbb{N}$ and consequently $z = f(\zeta) \in f(S_n) \subset \mathcal{T}(f(S_n)) = H_n$. Thus $\bigcup_{n \in \mathbb{N}} H_n = \mathbb{C}$.
So \((f(S), H_n)_{n \in \mathbb{N}}\) is a spider’s web.

Note that the restriction used on \(f\) is necessary. For example if we take spider’s web as

\[ S = \bigcup_{n=1}^{\infty} \{ z = ne^{i\theta}, 0 \leq \theta \leq 2\pi \} \cup \{ z : |z| \geq 1, \text{IM}(z) = 0 \} \]

and define a function as: \(f(z) = \frac{1}{z}\) for \(|z| > \frac{1}{2}\).
Then \(f(S) \subseteq (|z| \leq 1)\). So \(f(S)\) is not a spider’s web, being bounded.

**Proof of Theorem 4.1.5** Let \(E^c\) has an unbounded component say \(K\). So by definition of spider’s web, there exists a sequence of bounded simply connected domains such that for some \(n \in \mathbb{N}\), we must have \(G_n \cap K \neq \emptyset\). Since \(G_n^j\)s are bounded simply connected domains so their boundaries are jordan closed curves. Therefore \(\partial G_n \cap K \neq \emptyset\). This contradicts that \(\partial G_n \subset E\) and \(K \subset E^c\).

### 2.3. Triangulation on Spider’s Web

As it was mentioned by Rippon and Stallard that connected superset of a spider’s web is also a spider’s web, as a consequence we applied triangulation on spider’s web which results again spider’s web. The concept of triangulation has several interesting properties and it has been used in [11], pp 83. Below is the Definition of triangulation.

**Definition 2.3.1.** Let \(S\) be a plane domain together with it’s boundary, provided this boundary consists of simple closed curves. A triangulation \(T\) of \(S\) is a partition of \(S\) in to finite number of mutually disjoint subsets called vertices, edges and faces with following properties.

(i) Each vertex is a point of \(S\).

(ii) For each edge \(e\), there is a homeomorphism \(\phi\) of a closed interval \([a,b]\) in \(\mathbb{R}\) in to \(S\) which maps the open interval \((a,b)\) on to \(e\), and the end points \(a, b\) to the vertices of \(T\).

(iii) For each edge \(e\), there is a homeomorphism \(\phi\) of a closed triangle \(Q\) in \(\mathbb{C}\) in to \(S\) which maps the edges and vertices of \(Q\) on to edges and vertices of \(T\), and such that \(f\) is the \(\phi\) image of interior of \(Q\).
In particular, if triangulation $T$ consists of $F$ faces, $E$ edges and $V$ vertices respectively then the Euler characteristic $\chi(S)$ of $S$ is

$$\chi(S) = F - E + V. \quad (2.3.1)$$

If domain is simply connected then $\chi(S) = 1$.

**Definition 2.3.2.** Let $S$ and $T$ be two spider's web. We say that $S \subset T$, if there exist a sequences $G_n, H_m, n, m \in \mathbb{N}$ such that $(S, G_n)_{n \in \mathbb{N}}$ and $(T, H_m)_{m \in \mathbb{N}}$ are spider's webs and each $G_n$ is a finite union of $H_m$.

In following Theorem we have applied triangulation on spider's web and prove that it still remain a spider's web without being affected, by different ways, we do triangulation.

**Theorem 2.3.1.** Let $(S, G_n)_{n \in \mathbb{N}}$ be a spider's web, let $T$ be a triangulation of $G_n$. Then $(S', H_m)_{m \in \mathbb{N}}$ is also a spider's web, where $S'$ is $S \cup T$ and $H_m$ are suitably defined domains.

*Proof:* Let $(S, G_n)_{n \in \mathbb{N}}$ is a spider's web. Now constructing a triangulation $T$ on $S$ and denote it as $S'$. Let us apply $T$ on particularly some $G_k, k \in \mathbb{N}$, we add a vertex and three edges hence three faces. Here $\chi(H_k) = 1$, being simply connected. After applying $T$, $\chi(G_k) = F - E + V = 3 - 3 + 1 = 1$. So Euler characteristic is invariant. Next let us denote three faces as $G_{k_1}, G_{k_2}$ and $G_{k_3}$.

Rename as $H_1 = G_{k_1}, H_2 = G_{k_2} \cup H_1, H_3 = G_{k_3} \cup H_2$ and $G_{k_4} = H_4, G_{k_5} = H_5 \ldots$ Then $(S', H_n)$ forms a spider's web and is different from $(S, G_n)$, as in this we have not considered $\{G_i\}_{i \leq k_1}$.

**Remark:** We can also use those domains which we have left in above theorem. Let $S$ be as in above theorem and $S'$ is also same as we obtained by triangulation. Now $S' \cup \{G_i\}_{i \leq k_1}$ is again a spider's web, call it as $E$, then clearly $S \subset E$ and $S' \subset E$.

**Theorem 2.3.2.** Let $(S, G_n)_{n \in \mathbb{N}}$ be a spider's web. Let $T$ be a triangulation of $G_n$ with faces $F_1, F_2, F_3$. Then $(S, H_n)$ where $H_i = G_i \forall i = 1 \ldots n - 1, H_n = G_{n-1} \cup$
$F_1, H_{n+1} = H_n \cup F_2, H_{n+2} = H_{n+1} \cup F_3, H_{n+3} = G_{n+1}, H_{n+4} = G_{n+2}$ is a spider’s web $S \subset S_1$.

A proof similar as of Theorem 2.3.1 may work for this theorem also.

2.4. BOUNDED FATOU COMPONENTS AND SPIDER’S WEB

Regularity conditions and growth on transcendental entire function always play an important role in transcendental dynamics. In this section we discuss few results related to growth and regularity of transcendental entire function. Various regularity conditions have been discussed in Chapter 1, like $\psi$-regular, log-regular and strong regular. Before proceeding towards this section we recall well known definitions of order $\rho_f$ and lower order $\lambda_f$ of entire functions given by:

\[
\rho_f = \lim_{r \to \infty} \frac{\log \log M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \lim_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.
\]

Rippon and Stallard proved the following:

**Lemma 2.4.1.** [59] Let $f$ be a transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$, for $r \geq R$ and let $A_R(f)$ be a spider’s web, then $f$ has no unbounded Fatou components.

Sixsmith gave following result which is associated with regularity condition.

**Lemma 2.4.2.** [71] Let $f$ be a transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Then $A_R(f)$ is a spider’s web if for some $m > 1$,

(a) there exists $R_0 > 0$ such that for all $r \geq R_0$ there is a simply connected domain $G = G(r)$ with

\[
B(0, r) \subset G \subset B(0, r^m) \quad \text{and} \quad |f(z)| \geq M(r, f), \text{ for } z \in \partial G, \quad (2.4.1)
\]

and

(b) $f$ has a regular growth in the sense that there exists a sequence $(r_n)_{n \geq 0}$ with

\[
r_n > M^n(R, f) \quad \text{and} \quad M(r_n, f) \geq r_{n+1}^m, \text{ for } n \geq 0. \quad (2.4.2)
\]
LEMMA 2.4.3. [71] Let $f_1, f_2, \ldots, f_k$ be non constant transcendental entire functions. Suppose that for all $j \in \{1, 2, \ldots, k\}$, $f_j$ satisfies Lemma 2.4.2(a) with $m = m_j > 1$. Let $g = f_1 \circ f_2 \circ \cdots \circ f_k$, then $g$ satisfies Lemma 2.4.2(a) with $m = m_1 m_2 \cdots m_k$.

LEMMA 2.4.4. [31] If $f$ is a transcendental entire function with finite order and positive lower order then $f$ is log -regular and hence satisfies Lemma 2.4.2(b).

LEMMA 2.4.5. [71] If $f$ is a transcendental entire function with order less than 1/2, then $f$ satisfies Lemma 2.4.2(a) for some $m > 1$.

LEMMA 2.4.6. [71] Let $f_1, f_2, \ldots, f_k$ be non constant transcendental entire functions. Suppose that for some $j \in \{1, 2, \ldots, k\}$, $f_j$ is log-regular. Then $g = f_1 \circ f_2 \circ \cdots \circ f_k$ is also log-regular.

THEOREM 2.4.1. Let $h = f_1 \circ f_2 \circ f_3 \circ \cdots \circ f_n$ where $f_i (i = 1, 2, \ldots, n)$ be transcendental entire functions, each having order less than 1/2. If there is a number $j \in \{1, 2, 3, \ldots, n\}$, such that $f_j$ has positive lower order, then $A_R(h)$ is a spider’s web.

Proof: Here $h = f_1 \circ f_2 \circ f_3 \circ \cdots \circ f_n$. It is given that each $f_i$ has order less than 1/2. So by Lemma 2.4.5 each $f_i$ will satisfy Lemma 2.4.2(a) with $m = m_i > 1$. Hence $h$ satisfies Lemma 2.4.2(a) by Lemma 2.4.3 with $m = m_1 m_2 \cdots m_n$. Now it is also given that $\lambda_j > 0$ and $f_j$ has finite order, for some $j \in \{1, 2, 3, \ldots, n\}$. So $f_j$ is log regular, by Lemma 2.4.4. Hence $h$ is log regular by Lemma 2.4.6. So $h$ satisfy Lemma 2.4.2(b) by Lemma 2.4.4. So $A_R(h)$ is a spider’s web by Lemma 2.4.2.

Remark: A.P. Singh in [70] considered the class of transcendental entire functions $\mathcal{F} = \bigcup_{k \geq 1} \mathcal{F}_k$ where $\mathcal{F}_k = \{f : \log \log M(r, f) \geq \log r^{1/k}\}$. If $f \in \mathcal{F}_1$, then $f$ has positive lower order and hence if $f$ also has an order less than half, then $A_R(f)$ is a spider’s web from Lemma 2.4.4 and 2.4.5. Hence by Lemma 2.4.1 $f$ has no unbounded components. Theorem B of [70] is an immediate consequence to the above theorem, by Lemma 2.4.1.
2.5. **Maximum term of transcendental entire functions and Spider's Web**

Let $f$ be a transcendental entire function such that
\[ \sum_{n=1}^{\infty} f(z) = a_n z^n, \]
then $\mu(r, f) = \max_{n \geq 0} |a_n| r^n$ is called as the maximum term of $f(z)$ for $|z| = r$.

Note that $\mu(r, f)$ and $M(r, f)$ are connected by following equation [64], for $0 \leq r < R$,
\[ \mu(r, f) \leq M(r, f) \leq \frac{R}{R-r} \mu(R, f) \tag{2.5.1} \]
also
\[ \rho_f = \limsup_{r \to \infty} \frac{\log \log \mu(r, f)}{\log r} \tag{2.5.2} \]
and
\[ \lambda_f = \liminf_{r \to \infty} \frac{\log \log \mu(r, f)}{\log r}. \tag{2.5.3} \]

Following are some important properties of $\mu(r, f)$, many of these are similar to that of maximum modulus of transcendental entire function.

**Theorem 2.5.1.** (a) $\mu(r, f)$ is an increasing function of $r$, in fact $\mu(r, f) \to \infty$ as $r \to \infty$.
(b) $\mu(r, f) \leq M(r, f)$ and hence $\mu^n(r, f) \leq M^n(r, f)$, for $n \in \mathbb{N}$.
(c) $\log \frac{\mu(r, f)}{\log r} \to \infty$ as $r \to \infty$.
(d) $\frac{\mu(kr, f)}{\mu(r, f)} \to \infty$ as $r \to \infty$ where $k > 1$.
(e) $\mu^n(r, f) \geq \mu(r, f^n)$ for $n \geq 1$.
(f) $\mu(r^c, f) \geq \mu(r, f)^c$ for $c \geq 1$.

Here we have proved a few of them.

**Proof.** (b) First part obtained by (2.5.1) and for second part we have used induction.
Since it is true for $n = 1$, let
\[ \mu^k(r, f) \leq M^k(r, f). \tag{2.5.4} \]
Now consider
\[ \mu^{k+1}(r, f) \]
\[ = \mu(\mu^k(r, f), f) \]
\[ \leq \mu(M^k(r,f),f) \]
\[ \leq M(M^k(r,f),f) = M^{k+1}(r,f) \]
using equations (2.5.4) and (2.5.1).

(c) Using equation (2.5.1), we have
\[ \mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f), \text{ for } R > r. \]
Let \( R = 2r \), so
\[ M(r,f) \leq 2\mu(2r,f), \text{ or we may write,} \]
\[ \mu(r,f) \geq 1/2M(r/2,f). \]
Thus
\[ \log(\mu(r,f)) \geq \log(1/2) + \log(M(r/2,f)). \]

Therefore
\[
\frac{\log(\mu(r,f))}{\log r} \geq \frac{\log(1/2)}{\log r} + \frac{\log(M(r/2,f))}{\log r} \\
\geq \frac{\log(1/2)}{\log r} + \frac{\log(M(r/2,f))}{\log(r/2) + \log(r/2) + \log(M(r/2,f))} \\
\geq \frac{\log(1/2)}{\log r} + \frac{\log(r/2)^2}{\log(r/2)[1 + \log^2 r/2]}. 
\]
by taking \( \lim r \to \infty \), we have \( \frac{\log(\mu(r,f))}{\log r} \to \infty \) as \( \frac{\mu(r,f)}{r} \to \infty \) as \( r \to \infty \).

(d) Here \( \mu(r,f) \geq 1/2M(r/2,f) \), so
\[ \mu(kr,f) \geq 1/2M(kr/2,f). \tag{2.5.5} \]
Also \( \mu(r,f) \leq M(r,f) \), so
\[ \frac{1}{\mu(r,f)} \geq \frac{1}{M(r,f)}. \tag{2.5.6} \]
From (2.5.5) and (2.5.6) we have
\[
\frac{\mu(kr,f)}{\mu(r,f)} \geq \frac{M(kr/2,f)}{2M(r,f)} \geq \frac{M(kr/2,f)}{2M(r/2,f)} + \frac{M(r/2,f)}{M(r,f)}. 
\]
Since R.H.S. \( \to \infty \) as \( r \to \infty \), and \( k > 1 \), therefore \( \frac{\mu(kr,f)}{\mu(r,f)} \to \infty \) as \( r \to \infty \).

(e) This can easily be proved using induction.

(f) This can be proved by using (2.5.1) and a well known property of maximum modulus that \( M(r^c,f) \geq M(r,f)^c \) for \( c > 1 \).
Define a set

\[ B_R(f) = \{ z \in I(f) : |f^n(z)| \geq \mu^n(R, f), n \in \mathbb{N} \}. \] (2.5.7)

Then clearly \( A_R(f) \subset B_R(f) \). Also by definition this set is completely invariant, closed and \( \partial B_R(f) = J(f) \).

Most of the results which do follow for \( A_R(f) \) also follow for \( B_R(f) \).

**Lemma 2.5.1.** \( B_R(f) \subset A_{R'}(f) \) for some \( R' < R \).

**Proof.** Let \( z \in B_R(f) \), so

\[ |f^n(z)| \geq \mu^n(R, f) \] (2.5.8)

since \( \frac{M(r, f)}{\mu(r, f)} \to \infty \) as \( r \to \infty \), we may write \( \mu(r, f) = kM(r, f) \), where \( k \in (0, 1) \). So

\[ \mu(r, f) = kM(r, f) \geq \frac{1}{k} M(kr, f) \geq r. \]

Therefore \( \mu(r, f) \geq M(kr, f) \geq kr \) implying \( \mu^n(r, f) \geq M^n(kr, f) \), so

\[ \mu^n(R, f) \geq M^n(R', f) \text{ where } R' < R. \] (2.5.9)

From (2.5.8) and (2.5.9) we have \( z \in A_{R'}(f) \).

Here in fact \( B_R(f) = A_{R'}(f) \), by the definition and a result of Rippon and Stallard in [59], that \( A_{R'}(f) \subset A_R(f) \) for some \( R' < R \).

**Theorem 2.5.2.** If \( B_R(f)^c \) has a bounded component then \( B_R(f) \) form a spider's web.

**Proof.** Let \( G \) be a bounded component of \( B_R(f)^c \). Since \( A_R(f) \subset B_R(f) \), so proof of above theorem can be easily follows from Theorem 1.4, Lemma 7.1 [59], Lemma 2.5.1 and using the concept that connected superset of a spider’s web is again a spider’s web. It should be noted that in this case all of \( A_R(f), A(f) \) and \( I(f) \) are also spider's webs, by Theorem 1.4 [59].

**Theorem 2.5.3.** If \( f \) be a transcendental entire function, \( m \in \mathbb{N} \) and \( R > 0 \) be sufficiently large then

\[ B_R(f) \subset B_R(f^m) \subset B_{R/2}(f). \]
**Proof.** Let $z \in B_R(f)$. Thus $|f^n(z)| \geq \mu^n(R, f), n \in \mathbb{N}$, or we may write, $|f^{nm}(z)| \geq \mu^{nm}(R, f), n, m \in \mathbb{N}$, which in turn gives $|f^{nm}(z)| \geq \mu^n(R, f^m)$, by using Theorem 2.5.1(e). Therefore $z \in B_R(f^m)$ and so

$$B_R(f) \subset B_R(f^m).$$ \hspace{1cm} (2.5.10)

Now let $z \in B_R(f^m)$, so

$$|f^{nm}(z)| \geq \mu^n(R, f^m).$$ \hspace{1cm} (2.5.11)

Also $\mu(R, f^n) \geq \mu^n(R/2, f) \geq \mu(R/2, f^n)$, so we have

$$\mu(R, f^n) \geq \mu(R/2, f^n).$$ \hspace{1cm} (2.5.12)

From (2.5.11) and (2.5.12), we have $|f^n(z)| \geq \mu^n(R/2, f), n \in \mathbb{N}$, so $z \in B_{R/2}(f)$, therefore

$$B_R(f^m) \subset B_{R/2}(f).$$ \hspace{1cm} (2.5.13)

From (2.5.10) and (2.5.13), we have the result.

**Theorem 2.5.4.** If $B_R(f)$ has a multiply connected Fatou component then it forms a spider’s web.

**Proof.** Proof of above theorem follows from Theorem 2.5.2.

**Theorem 2.5.5.** (a) Let $R' > R$, then $B_R(f)$ is a spider’s web if and only if $B_{R'}(f)$ is a spider’s web.

(b) $B_R(f)$ is a spider’s web if and only if $B_R(f^m)$ is a spider’s web.

**Proof.** Proof of above theorem follows from definition 2.5.7, Theorem 2.5.1 and Theorem 2.5.3.