Chapter 4

$z_J$-ideals in lattices
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4.1 Introduction

The concept of $z$-ideals, which are both algebraic and topological objects, were first introduced by Kohls \cite{44} and played a fundamental role in studying the ideal theory of $C(X)$, the ring of continuous real-valued functions on a completely regular Hausdorff space $X$; see Gillman and M. Jerison \cite{21}. An ideal $I$ of a commutative ring $R$ is called a $z$-ideal if whenever any two elements of $R$ are contained in the same set of maximal ideals and $I$ contains one of them, then it also contains the other (see L. Gillman and M. Jerison \cite{21} for an equivalent definition). Mason \cite{50} studied $z$-ideals in general commutative rings. He proved that maximal ideals, minimal prime ideals and some other important ideals in commutative rings are $z$-ideals (see \cite{50}, p. 281). As a generalization of $z$-ideals, the concept of $z^0$-ideals is introduced and studied in $C(X)$. Note that in \cite{33}, Huijsmans and de Pagter studied $z^0$-ideals under the name of $d$-ideals in Riesz spaces. Speed \cite{63} introduced and studied the concept of Baer
ideals in a commutative Baer ring which are essentially \( z^0 \)-ideals (equivalently, \( d \)-ideals) and characterized regular rings and quasi-regular rings. Jayaram [35], Anderson, Jayaram and Phiri [6] defined this concept (Baer ideals) for lattices and multiplicative lattices respectively. Since \( z \)-ideal and \( z^0 \)-ideals (Baer ideals or \( d \)-ideals) are closely related in commutative rings, hence it is natural to study the analogues concept of \( z \)-ideals in lattices.

Hence, in this chapter, we introduce and study \( z \)-ideal and \( z_J \) ideal, a generalization of \( z \)-ideal, in bounded lattices and obtained some characterizations.

### 4.2 \( z \)-ideals

Let \( \mu \) as well as \( \text{Max}(L) \) denotes the set of all maximal ideals in a lattice \( L \) and let \( \mu(a) = \{ M \in \mu \mid a \in M \} \) for \( a \in L \).

For \( a \in L \), the intersection of all maximal ideals in \( L \) containing \( a \) is denoted by \( M_a \), that is, \( M_a = \bigcap \mu(a) \).

**Through out this chapter, \( L \) denotes a lattice with 1.**

Now, we define the concept of a \( z \)-ideal.

**Definition 4.2.1.** Let \( L \) be a lattice. An ideal \( I \) of \( L \) is a \textit{z-ideal} if

\[ \mu(b) \subseteq \mu(a) \text{ and } b \in I \text{ implies } a \in I. \]

**Lemma 4.2.2.** Every maximal ideal is a \( z \)-ideal.

\textit{Proof.} Let \( M \) be a maximal ideal and \( \mu(a) \subseteq \mu(b), a \in M \). Since \( a \in M \) implies \( M \in \mu(a) \). But \( \mu(a) \subseteq \mu(b) \) gives \( M \in \mu(b) \). Thus \( b \in M \). Hence \( M \) is a \( z \)-ideal. \( \square \)

The following result gives which ideals are not \( z \)-ideals.
Lemma 4.2.3. Let $M$ be a unique maximal ideal of a lattice $L$. Let $I$ be an ideal of $L$ such that $I \nsubseteq M$. Then $I$ is not a $z$-ideal.

Proof. Since $I \nsubseteq M$, there exists $x \in M$ such that $x \notin I$. Let $i \in I$. Since $M$ is the unique maximal ideal, we have $\mu(i) = \mu(x)$, but $x \notin I$. Thus $I$ is not a $z$-ideal. □

The following result is well known.

Lemma 4.2.4. Every maximal ideal of a 1-distributive (hence distributive) lattice is a prime ideal.

Definition 4.2.5. Let $L$ be a lattice with the smallest element $0$. The lattice $L$ is called semi-complemented if for any element $a \in L$ (with $a \neq 1$, if 1 exists) there exists a nonzero element $b \in L$ such that $a \land b = 0$. Dually, we can define a dual semi-complemented lattice.

First we characterize dual semi-complemented lattices in terms of maximal ideals.

Theorem 4.2.6. A lattice $L$ with 0 is dual semi-complemented if and only if $\bigcap_{M \in \text{Max}(L)} M = \{0\}$.

Proof. Let $L$ be a dual semi-complemented lattice. Suppose on the contrary that $\bigcap_{M \in \text{Max}(L)} M \neq \{0\}$. Let $a \in \bigcap_{M \in \text{Max}(L)} M$ and $a \neq 0$. Since $L$ is dual semi-complemented, there exists $b \neq 1$ such that $a \lor b = 1$. This implies that $b \notin \bigcap_{M \in \text{Max}(L)} M$. Since $b \neq 1$, there exists a maximal ideal say $M_1$ such that $b \in M_1$. Since $a \in \bigcap_{M \in \text{Max}(L)} M$ implies $a \in M_1$. Thus $1 = a \lor b \in M_1$, a contradiction. Hence $\bigcap_{M \in \text{Max}(L)} M = \{0\}$.
Conversely, suppose that \( \bigcap_{M \in \text{Max}(L)} M = \{0\} \). Let \( 0 \neq a \in L \). Since \( a \neq 0 \) implies \( a \notin \bigcap_{M \in \text{Max}(L)} M = \{0\} \). Then there exists a maximal ideal \( M_1 \) such that \( a \notin M_1 \). Therefore \( (a] \lor M_1 = L \) implies \( 1 \in (a] \lor M_1 \). Hence \( 1 = a \lor b \) for some \( b \in M_1 \). Clearly, \( b \neq 1 \). Thus \( L \) is dual semi-complemented.

**Lemma 4.2.7.** Let \( L \) be a dual semi-complemented lattice. Then \( (0] \) is a \( z \)-ideal.

*Proof.* Let \( \mu(a) \subseteq \mu(b) \) and \( a \in I = (0] \). Then \( a = 0 \) and \( \mu(a) = \mu(0) = \text{Max}(L) \). So \( \mu(b) = \text{Max}(L) \) implies \( b \in \bigcap_{M \in \text{Max}(L)} M = \{0\} \), by Theorem 4.2.6. Thus \( b = 0 \) implies. Hence \( (0] \) is a \( z \)-ideal. \( \square \)

**Remark 4.2.8.** Note that in non dual semi-complemented lattice \( L \) the ideal \( (0] \) is not a \( z \)-ideal, see the lattices \( L \) depicted in Figure 4.2.1.

![Figure 4.2.1: A non dual semi-complemented lattice](image)

**Lemma 4.2.9.** Let \( L \) be a 1-distributive lattice and \( a, b \in L \). Then the following statements hold.

1. \( M_{a \land b} = M_a \cap M_b \).
2. If $\mu(b) \subseteq \mu(a)$ then $\mu(b \land c) \subseteq \mu(a \land c)$ for any $c \in L$.

Proof. 1) Let $x \in M_{a \land b}$ and $x \notin M_a \cap M_b$. Without loss of generality, assume that $x \notin M_1$ for a maximal ideal $M_1$ containing $a$. But then $x \in M_{a \land b} \subseteq M_1$, a contradiction. Hence $M_{a \land b} \subseteq M_a \cap M_b$. Now, let $x \in M_a \cap M_b$ and $x \notin M_{a \land b}$. Then there is a maximal ideal, say $M_2$ such that $a \land b \in M_2$ but $x \notin M_2$. Since $L$ is 1-distributive, by Lemma 4.2.4 $M_2$ is prime. This gives that $a \in M_2$ or $b \in M_2$. Without loss of generality, assume that $a \in M_2$. But then $x \in M_a \subseteq M_2$, a contradiction. Hence $M_a \cap M_b \subseteq M_{a \land b}$. Thus $M_{a \land b} = M_a \cap M_b$.

2) Let $M$ be a maximal ideal containing $b \land c$, i.e., $M \in \mu(b \land c)$. By Lemma 4.2.4 $M$ is a prime ideal. Therefore $b \in M$ or $c \in M$. If $c \in M$, $a \land c \in M$, and we are through. Now, let $b \in M$. Then $M \in \mu(b) \subseteq \mu(a)$, we have $a \in M$. This gives $a \land c \in M$. Thus $M = \mu(a \land c)$. 

Remark 4.2.10. Note that the assertion of Lemma 4.2.9 need not be true, if we drop 1-distributivity. Consider the lattice $L$ depicted in Figure 4.2.2. Clearly, $L$ is not 1-distributive. In this lattice, $M_a = (d] = M_b$. Hence $M_a \cap M_b = (d]$ and $M_{a \land b} = (0]$. Thus $M_{a \land b} \nsubseteq M_a \cap M_b$. Also $\mu(a) = \mu(d) = (d]$ but $\mu(a \land b) \neq \mu(d \land b)$.

![Figure 4.2.2: $M_{a \land b} \nsubseteq M_a \cap M_b$](image-url)
Lemma 4.2.11. Every ideal \( I \) is contained in the least \( z \)-ideal namely,
\[
I_z = \bigcap \{ J \supseteq I \mid J \text{ is a } z\text{-ideal} \}.
\]

Proof. Let \( \mu(b) \subseteq \mu(a) \), \( b \in I_z \). Let \( J_1 \) be an arbitrary \( z \)-ideal such that \( J_1 \supseteq I \). Since \( b \in J_1 \) and \( J_1 \) is a \( z \)-ideal with \( \mu(a) \supseteq \mu(b) \), we have \( a \in J_1 \). Thus \( a \in I_z \). Hence \( I_z \) is a \( z \)-ideal.

Now, let \( J \) be any \( z \)-ideal containing \( I \). We claim that \( I_z \subseteq J \). Let \( x \in I_z \). Then clearly, \( x \in J \). Thus \( I_z \subseteq J \).

Remark 4.2.12. Consider the lattice \( L \) depicted in Figure 4.2.3. Let \( I = (a) \) be an ideal of \( L \). Then \( I_z = (c) \supsetneq I \).

![Figure 4.2.3: Example of a lattice with \( I \subseteq I_z \)](image)

Lemma 4.2.13. Let \( L \) be a lattice and \( I \) and \( J \) be any two ideals of \( L \). Then the following statements hold.

1. If \( I \subseteq J \) then \( I_z \subseteq J_z \).

2. \( (I_z)_z = I_z \).

Proof. (1) Let \( I \subseteq J \) and \( x \in I_z = \bigcap_{K \supseteq I} K \), where \( K \) is a \( z \)-ideal. If \( x \notin J_z \), then there exists a \( z \)-ideal \( Q_1 \) such that \( x \notin Q_1 \) and \( J \subseteq Q_1 \). This together with \( I \subseteq J \), we have \( I \subseteq Q_1 \). But \( x \in I_z \) and \( I \subseteq Q_1 \) for a \( z \)-ideal \( Q_1 \), we have \( x \in Q_1 \), a contradiction. Hence \( I_z \subseteq J_z \).
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(2) Clearly, $I_z \subseteq (I_z)_z$. Now, let $x \in (I_z)_z = \bigcap_{Q \supseteq I_z} Q$, where $Q$ is a $z$-ideal. But $I_z$ is the least $z$-ideal containing $I_z$. Therefore $x \in I_z$. Hence $(I_z)_z = I_z$. □

**Lemma 4.2.14.** Let $L$ be a lattice and $a, b \in L$. Then $a \in M_b$ if and only if $M_a \subseteq M_b$ if and only if $\mu(b) \subseteq \mu(a)$.

**Proof.** Let $M_a \subseteq M_b$. Since $a \in M_a$ implies that $a \in M_b$. Now, suppose that $a \in M_b = \bigcap_{b \in M \in \mu} M$, and $x \in M_a = \bigcap_{a \in M \in \mu} M$. Let $M_1$, be any maximal ideal with $b \in M_1$. Then $a \in M_1$, as $a \in M_b$. This gives $x \in M_1$. Hence $x \in M_b$. It is clear that $M_a \subseteq M_b$ if and only if $\mu(b) \subseteq \mu(a)$. □

In the following result, we characterize $z$-ideals in lattices.

**Lemma 4.2.15.** Let $I$ be an ideal of a 1-distributive lattice $L$. Then the following statements are equivalent.

1. $I$ is a $z$-ideal.

2. If $\mu(a) = \mu(b)$ and $b \in I$ implies $a \in I$.

3. $M_a \subseteq I$ for all $a \in I$.

4. If $M_b \subseteq M_a$ and $a \in I$ implies $b \in I$.

**Proof.** (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): Let $x \in M_a$. Then by Lemma 4.2.14 $M_x \subseteq M_a$. Hence $M_x = M_x \cap M_a = M_{a \land x}$ by Lemma 4.2.9. This gives $\mu(x) = \mu(x \land a)$. If $a \in I$, then $a \land x \in I$. By (2), $x \in I$. 

(3) $\Rightarrow$ (4): Let $M_b \subseteq M_a$ and $a \in I$. Then $M_b \subseteq M_a \subseteq M_{a \land x}$ by (3). Hence $\mu(b) \subseteq \mu(a \land x)$. By definition, $\mu(a) = \mu(a \land x)$. Therefore $\mu(b) \subseteq \mu(a)$.

(4) $\Rightarrow$ (1): Let $I$ be an ideal such that $M_a \subseteq I$ for all $a \in I$. Then $\mu(a) = \mu(b) \Rightarrow b \in I$. If $a \in I$, then $a \land x \subseteq I$. Hence $\mu(x) = \mu(a \land x)$. Therefore $\mu(x) = \mu(a) = \mu(b)$. Thus $b \in I$.

(1) $\Rightarrow$ (4): Let $I$ be a $z$-ideal. Then $M_a \subseteq I$ for all $a \in I$. If $M_b \subseteq M_a$ and $a \in I$, then $b \in I$.
(3) ⇒ (4): Let \( a \in I \). Then by (3), \( M_a \subseteq I \). Now if \( M_b \subseteq M_a \), then \( b \in M_b \subseteq I \).

(4) ⇒ (1): Follows from Lemma 4.2.14. 

Now, we prove a separation theorem for \( z \)-ideals.

**Theorem 4.2.16.** Let \( L \) be a distributive lattice. If \( I \cap F = \emptyset \) for a \( z \)-ideal \( I \) and for a filter \( F \) in \( L \), then there exists a prime \( z \)-ideal \( P \) containing \( I \) and disjoint from \( F \).

**Proof.** Consider \( \mathcal{F} = \{ J \mid J \text{ is a } z \text{-ideal containing } I \text{ and } J \cap F = \emptyset \} \).

Since \( I \in \mathcal{F} \), \( \mathcal{F} \neq \emptyset \). Let \( \mathcal{C} \) be a chain in \( \mathcal{F} \) and \( M = \bigcup_{J \in \mathcal{C}} J \). Clearly, \( M \) is an ideal. Now to show that \( M \) is a \( z \)-ideal, consider \( \mu(a) \subseteq \mu(b) \) and \( a \in M \). Then \( a \in J_i \) for some \( i \). But \( J_i \) is a \( z \)-ideal, therefore \( b \in J_i \). Hence \( b \in M \). Thus \( M \) is a \( z \)-ideal. By Zorn’s Lemma, there exists a maximal element \( P \) of \( \mathcal{F} \). Clearly, \( P \) is a \( z \)-ideal with \( P \cap F = \emptyset \).

We claim that \( P \) is a prime ideal. Let \( a \land b \in P \) and \( a, b \notin P \). Then \( (P \lor (a]) \cap F \neq \emptyset \) and \( (P \lor (b]) \cap F \neq \emptyset \). Let \( x \in (P \lor (a]) \cap F \) and \( y \in (P \lor (b]) \cap F \). Then \( x \leq p_1 \lor a \) and \( y \leq p_2 \lor b \) for some \( p_1, p_2 \in P \).

This gives \( x \leq p_3 \lor a \) and \( y \leq p_3 \lor b \), where \( p_3 = p_1 \lor p_2 \). Therefore \( x \land y \leq (p_3 \lor a) \land (p_3 \lor b) = p_3 \lor (a \land b) \in P \). Thus \( x \land y \in P \). Also \( x \land y \in F \) gives that \( P \cap F \neq \emptyset \), a contradiction. Hence \( P \) is a prime \( z \)-ideal. 

It is known that behavior of ideals is influenced by a behavior of prime ideals. The following result is an example of such a behavior.

**Theorem 4.2.17.** Let \( L \) be a distributive lattice. Then every prime ideal is a \( z \)-ideal if and only if every ideal is a \( z \)-ideal.
Proof. Suppose every prime ideal is a $z$-ideal. Let $I$ be any ideal and $\mu(b) \subseteq \mu(a)$, $b \in I$. Suppose $a \notin I$. By Theorem 4.2.16, there exists a prime ideal $P \supseteq I$ and $a \notin P$. Clearly, $b \in P$ and $\mu(b) \subseteq \mu(a)$ implies $a \in P$ (since $P$ is a $z$-ideal), a contradiction. Thus $a \in I$ proving $I$ is a $z$-ideal.

Theorem 4.2.18. Let $L$ be a 1-distributive, SSC lattice such that $\bigcap_{M \in \text{Max}(L)} M = \{0\}$, then every principal ideal is a $z$-ideal.

Proof. Let $I = (x)$ be an ideal of an SSC lattice $L$. Let $\mu(b) \subseteq \mu(a)$ and $b \in I$. Now, $b \in I = (x)$ and suppose $a \notin I = (x)$. Then there exists $c \neq 0$ such that $c \leq a$ and $c \land x = 0$. This gives $b \land c = 0$. Thus $\text{Max}(L) = \mu(b \land c) \subseteq \mu(a \land c)$, by Lemma 4.2.9. Then $c = a \land c \in \bigcap_{M \in \text{Max}(L)} M = \{0\}$. Therefore $c = a \land c = 0$, a contradiction. Thus $a \in I$. Hence $I$ is a $z$-ideal.

Lemma 4.2.19. In a 1-distributive, dual semi-complemented lattice $L$, $a^\perp = \bigcap\{M \in \text{Max}(L) | a \notin M\}$ for any $a \in L$.

Proof. Let $x \in a^\perp$. Then $a \land x = 0$. Let $M \in \text{Max}(L)$. Since $L$ is 1-distributive, we have $M$ is prime. If $a \notin M$ then $x \in M$. Thus $a^\perp \subseteq \bigcap\{M \in \text{Max}(L) | a \notin M\}$.

Conversely, suppose that $x \in \bigcap\{M \in \text{Max}(L) | a \notin M\}$ and $x \notin a^\perp$, i.e., $x \land a \neq 0$. Hence $x \land a \notin \bigcap_{M \in \text{Max}(L)} M = \{0\}$ by Theorem 4.2.6. Therefore there exists a maximal ideal $M_1$ such that $a \land x \notin M_1$. But then $a \notin M_1 \in \text{Max}(L)$ with $x \notin M_1$, a contradiction to $x \in \bigcap\{M \in \text{Max}(L) | a \notin M\}$. Thus $a^\perp = \bigcap\{M \in \text{Max}(L) | a \notin M\}$.
We recall the following definition.

**Definition 4.2.20.** For an ideal $I$ and a prime ideal $P$ of a lattice $L$, we define the set $I(P)$ as follows: $I(P) = \{ x \in L \mid x \land y \in I \text{ for some } y \in L \setminus P \}$. If $I = \{0\}$ then $I(P)$ is denoted by $O(P)$. If $L$ is a 0-distributive lattice then $O(P)$ is an ideal. An ideal $I$ of a lattice $L$ with 0 is said to be **Baer ideal** if $a \in I$ implies $a^\perp \subseteq I$ and is said to be **closed ideal** if $I = I^\perp$. An ideal $I$ of a lattice $L$ with 0 is called a **0-ideal** if there exists a proper filter $F$ such that $I = F^0$, where $F^0 = \{ x \in L \mid x \land y = 0 \text{ for some } y \in F \}$. An ideal $I$ of a lattice $L$ is called a **dense ideal** if $I^\perp = \{0\}$. An ideal $I$ of a lattice $L$ is called a **non-dense ideal** if $I^\perp \neq \{0\}$.

**Theorem 4.2.21.** Let $L$ be a 1-distributive lattice with 0 such that $\bigcap_{M \in \text{Max}(L)} M = \{0\}$. If $I$ is an ideal of $L$ satisfying any one of the following conditions then $I$ is a $z$-ideal.

1. if $I$ is a non-dense prime ideal;
2. if $I$ is a closed ideal;
3. if $I$ is 0-ideal;
4. if $I = O(P)$ for any prime ideal $P$;
5. if $I = A^\perp$ for any subset $A$ of $L$.

**Proof.** (1) Let $I$ be the non-dense prime ideal and $\mu(b) \subseteq \mu(a)$, $b \in I$. Now, let $I^\perp \neq \{0\}$. Then there exists a nonzero $x \in I^\perp$ this implies that $x \land i = 0$ for all $i \in I$. In particular, $x \land b = 0$. Since $\mu(b) \subseteq \mu(a)$ by Lemma 4.2.9, we have $\text{Max}(L) = \mu(b \land x) \subseteq \mu(a \land x)$. Thus $a \land x \in M$
for all $M \in \text{Max}(L)$. Thus $a \land x = 0$, as $\bigcap_{M \in \text{Max}(L)} M = \{0\}$ which yields $a \land x \in I$. Since $I$ is a prime ideal, $a \in I$ or $x \in I$. If $x \in I$ then $x \in I \cap I^\perp = \{0\}$, a contradiction. Thus $a \in I$. Hence $I$ is a z-ideal.

(2) Let $I$ is a closed ideal, i.e., $I = I^{\perp\perp}$ and $\mu(b) \subseteq \mu(a)$, $b \in I$. Now $b \in I = I^{\perp\perp}$ implies $b \land i = 0$ for all $i \in I^\perp$. Since $\mu(b) \subseteq \mu(a)$ implies $\mu(b \land i) \subseteq \mu(a \land i)$ for $i \in I^\perp$, by Lemma 4.2.9. Thus $\mu(a \land i) = \text{Max}(L)$. Hence $a \land i \in \bigcap_{M \in \text{Max}(L)} M = \{0\}$. Therefore $a \land i = 0$ for all $i \in I^\perp$ implies $a \in I^{\perp\perp} = I$. Hence $I$ is a z-ideal.

(3) Let $I$ is a 0-ideal. Then $I = F^0 = \{x \in L \mid x \land y = 0 \text{ for some } y \in F\}$ for some filter $F$ of $L$. Let $\mu(b) \subseteq \mu(a)$ and $b \in I$. Since $b \in I = F^0$ implies $b \land y = 0$ for some $y \in F$. Now, $\mu(b) \subseteq \mu(a)$, by Lemma 4.2.9. $\text{Max}(L) = \mu(b \land y) \subseteq \mu(a \land y)$. Hence $a \land y \in M$ for all $M \in \text{Max}(L)$. Thus $a \land y \in \bigcap_{M \in \text{Max}(L)} M = \{0\}$. Hence $a \land y = 0$ for some $y \in F$. Thus $a \in F^0 = I$. Hence $I$ is a z-ideal.

(4) Let $I = O(P) = \{x \in L \mid x \land y = 0 \text{ for some } y \notin P\}$ for a prime ideal $P$ of $L$. Then $F = L \setminus P$ is a filter. This gives that $I = F^0$. The result follows from (3).

(5) Let $I = A^\perp = \{x \in L \mid x \land a = 0 \text{ for all } a \in A\}$ and $\mu(b) \subseteq \mu(a)$, $b \in I$. Now $b \in I = A^\perp$ implies $b \land c = 0$ for all $c \in A$. Since $\mu(b) \subseteq \mu(a)$ implies $\mu(b \land c) \subseteq \mu(a \land c)$, by Lemma 4.2.9. Using the similar techniques as that of (3), we get $a \land c = 0$ for all $c \in A$. Thus $a \in A^\perp = I$. Hence $I$ is a z-ideal. \[\square\]

**Remark 4.2.22.** In view of Theorem 4.2.6 and Theorem 4.2.21, it is clear that in a dual semi-complemented lattice every closed ideal is a z-ideal. However the assertion is not true, if we drop the condition of
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dual semi-complemented lattices. Consider a semi-complemented lattice \( L \) depicted in Figure 4.2.4. In this lattice, the ideals \((a]\) and \((b]\) are closed ideals but not \( z \)-ideals.

![Diagram of lattice L with elements 0, a, b, c, and 1 labeled]

Figure 4.2.4: \( z \)-ideal and closed ideal are distinct

As mentioned in the introduction, the concept of Baer ideals (equivalently \( z^0 \)-ideals) and \( z \)-ideals are related in commutative rings with unity. In the following remark, we show that in general lattices they are not related.

**Remark 4.2.23.** From the following two figures, it is clear that neither a \( z \)-ideal is a Baer nor a Baer ideal is a \( z \)-ideal. Consider the lattices \( L_1 \) and \( L_2 \) depicted in Figure 4.2.5 (a) and Figure 4.2.5 (b). In Figure 4.2.5 (a), the ideal \( I = (b] \) is a Baer ideal but not a \( z \)-ideal whereas in Figure 4.2.5 (b), the ideal \( J = (x] \) is a \( z \)-ideal but not a Baer ideal.
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However in a 0-1-distributive lattice under the additional condition, i.e., \( \bigcap_{M \in \text{Max}(L)} M = \{0\} \), we prove that every Baer ideal is a \( z \)-ideal. For this purpose we need following two results.

**Theorem 4.2.24** (Thakare and Pawar [67]). In a 0-distributive lattice \( L \) the pseudocomplement of any ideal \( I \) is the intersection of all minimal prime ideals not containing \( I \).

Let \( \text{Min}(L) \) denotes the set of all minimal prime ideals in \( L \).

**Lemma 4.2.25.** Let \( L \) be a 0-distributive lattice. Then for \( a \in L \), \( a^\perp\perp = \bigcap\{P \in \text{Min}(L) | a \in P\} \).

*Proof.* Put \( I = a^\perp \) then \( I^\perp = a^\perp\perp \). By Theorem 4.2.24, \( I^\perp = a^\perp\perp = \bigcap\{P \in \text{Min}(L) | a^\perp \notin P\} \). By Theorem 3.2.16 we have, \( a^\perp\perp = \bigcap\{P \in \text{Min}(L) | a^\perp \notin P\} = \bigcap\{P \in \text{Min}(L) | a \in P\} \).

**Lemma 4.2.26.** Let \( L \) be a 0-1-distributive lattice such that \( \bigcap_{M \in \text{Max}(L)} M = \{0\} \). Then every Baer ideal is a \( z \)-ideal.
Proof. By Lemma 4.2.25, $a^\perp\perp = \bigcap\{P \in Min(L) | a \in P\} = P_a$ (say). Now, $I$ is a Baer and $a \in I$ implies $a^\perp\perp = P_a \subseteq I$. Suppose $\mu(a) = \mu(b)$, $a \in I$ and $b \notin I$. Then $b \notin P_a$. Hence there exists a minimal prime ideal, say $Q$, such that $b \notin Q$. Moreover, $a \in Q$ implies $a^\perp \notin Q$. Then $(b \cap a^\perp \neq \{0\} = \bigcap_{M \in \text{Max}(L)} M$. Then there exists a maximal ideal $M$ such that $(b \cap a^\perp \notin M$. Clearly, $a^\perp \notin M$. By 1-distributivity, $M$ is a prime ideal. Since $(a \cap a^\perp = \{0\} \subseteq M$ and $a^\perp \notin M$, we have $a \in M$. Thus there exists a maximal ideal $M$ such that $a \in M$ and $b \notin M$. Hence $\mu(a) \neq \mu(b)$, a contradiction to $\mu(a) = \mu(b)$. So $b \in I$. Thus $I$ is a $z$-ideal.

Remark 4.2.27. Let $L$ be a 0-1-distributive lattice $L$ (that is, 0-distributive as well as 1-distributive). If $\bigcap_{M \in \text{Max}(L)} M = \{0\}$, then $z$-ideal need not be a Baer ideal.

For this, consider a lattice $L = \{X \subseteq \mathbb{N} | X$ is an infinite set $\} \cup \{\emptyset\}$. Clearly, $L$ is a 0-distributive lattice under set inclusion and $\bigcap_{M \in \text{Max}(L)} M = \{0\}$. Let $I = (\mathbb{N} - \{1\}$ then $(\mathbb{N} - \{1\})^\perp = \{0\}$ implies $(\mathbb{N} - \{1\})^\perp = \{0\}^\perp = L$. Hence $(\mathbb{N} - \{1\})^\perp \notin (\mathbb{N} - \{1\}$. Therefore $(\mathbb{N} - \{1\}$ is not a Baer ideal. Since every maximal ideal is $z$-ideal, we have $(\mathbb{N} - \{1\}$ is a $z$-ideal.

Lemma 4.2.28. Let $L$ be a 0-1-distributive lattice such that every Baer ideal is a $z$-ideal. Then $\bigcap_{M \in \text{Max}(L)} M = \{0\}$.

Proof. Suppose every Baer ideal is a $z$-ideal and $\bigcap_{M \in \text{Max}(L)} M \neq \{0\}$. Let...
\( a \in \bigcap_{M \in \text{Max}(L)} M \) implies \( \mu(a) = \mu(0) \). Since \((0] \) is a Baer ideal, by the hypothesis \((0] \) is a \( z \)-ideal. Then \( \mu(0) = \mu(a) \) and \( 0 \in (0] \) implies \( a \in (0] \), a contradiction. Hence \( \bigcap_{M \in \text{Max}(L)} M = \{0\} \).

We conclude this section by combining Lemma 4.2.26, Lemma 4.2.28, and Theorem 4.2.6, we have the following result.

**Theorem 4.2.29.** Let \( L \) be a 0-1-distributive lattice. Then the following statements are equivalent.

1. Every Baer ideal is a \( z \)-ideal.
2. \( L \) is a dual semi-complemented.
3. \( \bigcap_{M \in \text{Max}(L)} M = \{0\} \).

### 4.3 \( z_J \)-ideals

Now, we extend the definition of \( z \)-ideal to \( z_J \)-ideal on similar lines of Alibad, Azarpanah and Taherifar [5].

**Definition 4.3.1.** Let \( I \) and \( J \) be two ideals of a lattice \( L \). The ideal \( I \) is said to be a \( z_J \)-ideal if \( M_a \cap J \subseteq I \) for all \( a \in I \).

Clearly, if \( J \subseteq I \), then \( I \) is always a \( z_J \)-ideal and hence an ideal \( I \) is always a \( z_I \)-ideal. Further, if \( J = L \), then \( z_L \)-ideal is nothing but \( z \)-ideal.

**Lemma 4.3.2.** If \( I \) is a \( z \)-ideal, then \( I \) is a \( z_J \)-ideal for any ideal \( J \) of a lattice \( L \).
Proof. Let \( a \in I \) and \( x \in M_a \cap J \). Then \( M_x \subseteq M_a \). This together with \( I \) is a \( z \)-ideal, we have \( x \in I \). Hence \( I \) is a \( z_J \)-ideal. 

Remark 4.3.3. The following example shows that a \( z_J \)-ideal is not a \( z \)-ideal. Consider the lattice \( L \) depicted in Figure 4.3.1. Here \((b]\) is not a \( z \)-ideal, but it is \( z_J \)-ideal for \( J = (a] \).

\[ 
\begin{array}{c}
\circ 1 \\
\circ a \\
\circ \circ c \\
\circ b \\
\circ 0 \\
L
\end{array}
\]

Figure 4.3.1: The ideal \((b]\) is not a \( z \)-ideal but it is a \( z_J \) ideal for \( J = (a] \)

We recall the following definitions.

Definition 4.3.4. A prime ideal \( P \) of a lattice \( L \) is said to be **minimal prime ideal containing an ideal** \( I \), if \( I \subseteq P \) and there exists no prime ideal \( Q \) such that \( I \subsetneq Q \subsetneq P \).

The set of all prime ideals in a lattice \( L \) is denoted by \( \text{Spec}(L) \). The set of all minimal prime ideals containing an ideal \( I \) is denoted by \( \text{Min}(I) \). It is clear that a lattice is 0-distributive if and only if \( I = \{0\} \) is a semiprime ideal.

Lemma 4.3.5. Let \( I \) be a semiprime ideal and \( J \) be any ideal of a 1-distributive lattice \( L \). Then the following statements hold.

1. If \( I \) is a \( z_J \)-ideal (\( z \)-ideal) and \( P \in \text{Min}(I) \), then \( P \) is also a \( z_J \)-ideal (\( z \)-ideal).
2. A prime ideal $P$ in $L$ is a $z_J$-ideal if and only if $P$ is either a $z$-ideal or $J \subseteq P$.

Proof. (1) Let $P$ be a minimal prime ideal containing $I$. Suppose $x \in P$. We claim that $M_x \cap J \subseteq P$. Since $x \in P$, by Lemma 3.2.16(4), there exists $y \notin P$ such that $x \wedge y \in I$. Since $I$ is a $z_J$-ideal, therefore we have $M_{x \wedge y} \cap J = M_x \cap M_y \cap J \subseteq I \subseteq P$, by Lemma 4.2.9. But $M_y \notin P$, and $P$ is prime ideal gives $M_x \cap J \subseteq P$. Thus $P$ is a $z_J$-ideal.

(2) Let $P$ be a prime $z_J$-ideal such that $J \nsubseteq P$. Suppose $M_a \subseteq M_b$ and $b \in P$. Since $P$ is a $z_J$-ideal, $M_b \cap J \subseteq P$. This together with $J \nsubseteq P$ gives $M_b \subseteq P$. But then $M_a \subseteq M_b$ gives $a \in M_a \subseteq P$. Thus $P$ is a $z$-ideal.

Conversely, if $J \subseteq P$ then clearly, $P$ is a $z_J$-ideal. Now, suppose $J \nsubseteq P$ and $P$ is $z$-ideal. By Lemma 4.3.2, $P$ is a $z_J$-ideal. 

Proposition 4.3.6. Let $I$ be a semiprime ideal, $J$ be an ideal and $P$, $Q$ are prime ideals of a $1$-distributive lattice $L$. Then the following statements hold.

1. If $I \cap P$ is a $z_J$-ideal, then either $I$ or $P$ is a $z_J$-ideal.

2. If $P \cap Q$ is a $z_J$-ideal and $P$ and $Q$ are not comparable then $P$ and $Q$ are $z_J$-ideals.

Proof. (1) If $I \subseteq P$, then clearly, $I$ is a $z_J$-ideal. Now, suppose that $I \nsubseteq P$ and $b \in P$. Take $a \in I \setminus P$. Hence $a \wedge b \in I \cap P$. Since $I \cap P$ is a $z_J$-ideal, we have $M_{a \wedge b} \cap J \subseteq I \cap P$. By Lemma 4.2.9, $M_a \cap M_b \cap J \subseteq P$. Since $P$ is a prime and $M_a \nsubseteq P$, we get $M_b \cap J \subseteq P$. Hence $P$ is a $z_J$-ideal.

(2) Follows from (1).
Lemma 4.3.7 (Rav [58]). Every semiprime ideal $I$ of a lattice $L$ is an intersection of minimal prime ideals containing $I$.

Lemma 4.3.8. Let $I = (a]$ be a principal ideal of a lattice $L$. Then $I_z = M_a$.

Proof. Let $I = (a]$. Obviously $I_z \subseteq M_a$. Now, we will show that $M_a \subseteq I_z$. Let $x \in M_a$ and $J$ be any arbitrary $z$-ideal containing $I = (a]$. Since $J$ is $z$-ideal and $x \in M_a$, we have $x \in J$. Thus $x \in I_z$. So $M_a = I_z$. □

Lemma 4.3.9. Let $I$ and $J$ be two ideals of a distributive lattice $L$. Then $(I \cap J)_z = I_z \cap J_z$.

Proof. Clearly, $I_z \cap J_z$ is a $z$-ideal containing $I \cap J$. To prove $(I \cap J)_z = I_z \cap J_z$, it is enough to show that $I_z \cap J_z$ is the smallest $z$-ideal containing $I \cap J$. To see this, let $K$ be a $z$-ideal and $I \cap J \subseteq K$. Since $L$ is distributive, $K$ is semiprime. By Lemma 4.3.7, $K = \bigcap_{P \in \text{Min}(K)} P$. Since for each $P \in \text{Min}(K)$, we have $I \cap J \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. By Lemma 4.3.5, each $P \in \text{Min}(K)$ is a $z$-ideal. Using this fact along with $I_z$ is the smallest $z$-ideal containing $I$, we have $I_z \cap J_z \subseteq P$, therefore $I_z \cap J_z \subseteq \bigcap_{P \in \text{Min}(K)} P = K$. □

Remark 4.3.10. Note that the assertion of Lemma 4.3.9 is not true in non-distributive lattices. Consider the non-distributive lattice depicted in Figure 4.2.2 on page 68. Let $I = (a]$ and $J = (b]$. Then $I_z = J_z = (d]$ and $(I \cap J)_z = (0]_z = \{0\}$. Thus $(I \cap J)_z \nsubseteq I_z \cap J_z$.

In the following result, we characterize $z_J$-ideals in lattices.
4.3 \( z_J \)-ideals

**Theorem 4.3.11.** Let \( I \) be a semiprime ideal of a lattice \( L \). Let \( J \) be an ideal of \( L \). Then the following statements are equivalent.

1. \( I \) is a \( z_J \)-ideal.

2. \( I \cap J \subseteq I \) (equivalently, \( I \cap J = I \cap J \)).

3. There is a \( z \)-ideal \( K \) containing \( I \) exists, and \( K \cap J \subseteq I \).

4. For each \( a \in I \) and \( b \in J \) if \( M_b \subseteq M_a \), then \( b \in I \).

**Proof.** (1)\( \Rightarrow \) (2): Let \( I \) be a semiprime \( z_J \)-ideal. By Lemma 4.3.7, \( I = \bigcap_{P \in \text{Min}(I)} P \). Hence \( I_z = ( \bigcap_{P \in \text{Min}(I)} P \bigcap) \subseteq ( \bigcap_{P \in \text{Min}(I)} P_z \bigcap) \). By Lemma 4.3.5, \( P_z = P \) or \( J \subseteq P \). Hence either in the case, we have \( I_z \cap J = ( \bigcap_{P \in \text{Min}(I)} P \bigcap) \cap J = ( \bigcap_{P \in \text{Min}(I)} P_z \bigcap) \cap J = I \cap J \subseteq I \).

(2)\( \Rightarrow \) (3): Take \( K = I_z \).

(3)\( \Rightarrow \) (4): Let \( a \in I \) and \( b \in J \) with \( M_b \subseteq M_a \). By (3), there exists a \( z \)-ideal \( K \) containing \( I \) such that \( K \cap J \subseteq I \). Then by Lemma 4.2.15, \( M_a \subseteq K \). Clearly, \( b \in M_b \subseteq M_a \). Hence \( b \in M_a \cap J \subseteq K \cap J \subseteq I \). Thus \( b \in I \).

(4)\( \Rightarrow \) (1): Let \( a \in I \) and \( x \in M_a \cap J \). Then by Lemma 4.2.14, \( M_x \subseteq M_a \).

Now by (4), we get \( x \in I \). Thus \( I \) is a \( z_J \) ideal.

**Remark 4.3.12.** From Theorem 4.3.11, it is easy to obtain Lemma 4.2.15 by replacing an ideal \( J \) by a lattice \( L \).

**Lemma 4.3.13.** The following statements hold in any lattice \( L \).

1. If \( I = I_1 \cap I_2 \), \( J = J_1 \cap J_2 \), and \( I_1 \) is a \( z_{J_1} \)-ideal, \( I_2 \) is a \( z_{J_2} \)-ideal, then \( I \) is a \( z_J \)-ideal.
2. If $J \subseteq K$ and $I$ is a $z_k$-ideal, then $I$ is also a $z_J$-ideal.

3. An intersection of $z_J$-ideals ($z$-ideals) is a $z_J$-ideal ($z$-ideal).

4. If $I \subseteq J$, $I$ is a $z_J$-ideal and $J$ is a $z_K$-ideal, then $I$ is a $z_K$-ideal.

Proof. 1) Let $c \in I = I_1 \cap I_2$. Since $I_1, I_2$ are $z_{J_1}$-ideal and $z_{J_2}$-ideal respectively, we have $M_c \cap J_1 \subseteq I_1$ and $M_c \cap J_2 \subseteq I_2$. This gives $M_c \cap J_1 \cap J_2 \subseteq I_1 \cap I_2$. Thus $M_c \cap J \subseteq I$ as required.

2) Easy to prove.

3) Obvious.

4) Let $a \in I \subseteq J$. Since $I$ is a $z_J$-ideal and $J$ is a $z_K$-ideal, we have $M_a \cap J \subseteq I$ and $M_a \cap K \subseteq J$. This gives $M_a \cap K \subseteq M_a \cap J \subseteq I$. Thus $I$ is a $z_K$-ideal.

We have the following result, by Part (3) of Lemma 4.3.13 and Lemma 4.2.2.

**Lemma 4.3.14.** The Jacobson radical $J = \bigcap_{M \in \text{Max}(L)} M$ is a $z$-ideal and is contained in every $z$-ideal.

Let $Id_z(L)$ denotes the set of all $z$-ideals of a lattice $L$. Then we have the following result.

**Theorem 4.3.15.** Let $L$ be a bounded lattice. Then $Id_z(L)$ is a complete lattice

**Lemma 4.3.16.** Let $L$ be a 1-distributive lattice. Then $I$ is a $z_J$-ideal if and only if $I \cap J$ is a $z_J$-ideal.

**Proof.** Let $a \in I \cap J$. Then $a \in I$ and $I$ is a $z_J$-ideal give $M_a \cap J \subseteq I$. This further yield $M_a \cap J \subseteq I \cap J$. Thus $I \cap J$ is a $z_J$-ideal.
Conversely, assume that $a \in I$ and $x \in M_a \cap J$. Then $a \land x \in I \cap J$. Since $I \cap J$ is a $z_J$-ideal, we have $M_{a \land x} \cap J \subseteq I \cap J$. By Lemma 4.2.9, $M_{a \land x} = M_a \cap M_x$. Since $x \in M_a$, we have $M_x \subseteq M_a$. Thus $M_x \cap J \subseteq I \cap J$. Now, $x \in M_x \cap J \subseteq I \cap J \subseteq I$, gives $x \in I$. Thus $M_a \cap J \subseteq I$. Hence $I$ is a $z_J$-ideal.

We close this chapter with the following result.

**Theorem 4.3.17.** Let $I$, $J$ and $K$ be ideals of a lattice $L$. Then the following statements hold.

1. An ideal $I$ of a distributive lattice $L$ is a $z_J$-ideal if and only if $I$ is a $z_{I \lor J}$-ideal.

2. If $J$ is a $z$-ideal, then $I$ is a $z_J$-ideal if and only if $I \cap J$ is a $z$-ideal.

3. $I \cap J$ is both $z_I$-ideal and $z_J$-ideal if and only if $I$ is a $z_J$-ideal and $J$ is a $z_I$-ideal; provided that $L$ is a 1-distributive lattice.

4. If $M$ is a maximal ideal in a distributive lattice $L$, then $I \cap M$ is a $z$-ideal if and only if $I$ is a $z$-ideal.

5. Let $L$ be a distributive lattice. Then $I_z \cap J$ is the smallest $z_J$-ideal containing $I \cap J$.

6. If $I$ is a semiprime ideal and $I \subseteq K$, $I_z = K_z$, $I$ is a $z_J$-ideal then $K$ is also a $z_J$-ideal.

**Proof.** (1) Let $I$ be a $z_J$-ideal. Then $M_a \cap J \subseteq I$ for all $a \in I$. Clearly, $M_a \cap I \subseteq I$. Since $L$ is distributive, the ideal lattice $Id(L)$ is distributive,
we have $M_a \cap (I \vee J) \subseteq I$ for all $a \in I$. Hence $I$ is a $z_{J\vee I}$-ideal. The converse is obvious.

(2) Let $I$ be a $z_J$-ideal and $\mu(b) \subseteq \mu(a)$, $b \in I \cap J$. Since $\mu(b) \subseteq \mu(a)$ and $b \in J$ implies that $a \in J$ (as $J$ is a $z$-ideal). Now $a \in M_a \cap J$ and $M_a \cap J \subseteq I$ (since $I$ is a $z_J$-ideal). Thus $a \in I \cap J$. Hence $I \cap J$ is a $z$-ideal. The converse is obvious.

(3) Follows from Lemma 4.3.16.

(4) Let $I \cap M$ be a $z$-ideal. If $I \subseteq M$ implies that $I = I \cap M$. Suppose $I \not\subseteq M$, then by (3), $I$ is a $z_M$-ideal. By Part (1) $I$ is a $z_{M\vee I}$-ideal, i.e., $z_L$ ideal. But $z_L$ ideals are nothing but $z$-ideals. Hence $I$ is a $z$-ideal. The converse is obvious.

(5) Clearly, $I_z \cap J$ is a $z_J$-ideal containing $I \cap J$. Now, suppose that $K$ is a $z_J$-ideal containing $I \cap J$. Hence $I_z \cap J = I_z \cap J_z \cap J = (I \cap J)_z \cap J \subseteq K_z \cap J \subseteq K$, by Lemma 4.3.9 and Theorem 4.3.11.

(6) If $I$ is a $z_J$-ideal then $I_z \cap J \subseteq I$. Hence $K_z \cap J = I_z \cap J \subseteq I \subseteq K$. Hence $K$ is a $z_J$-ideal. \qed