CHAPTER-XI
PROBABILITY OF RUIN IN THE PRESENCE OF
INTEREST EARNINGS AND TAX PAYMENTS

The influence of interest earnings and tax payments in the classical risk model has been
a theme of recent research. In this context, a remarkable result has been the Albrecher –
Hipp tax identity (Albrecher-Hipp, 2007). In their framework, tax is paid at a fixed rate
\( \gamma \in [0,1] \) whenever the insurer is in profitable position. The modified surplus at time \( t \)
is written as \( U_\gamma(t) \) and the corresponding ruin and the non-ruin probabilities are denoted
by \( \psi_\gamma(u) \) and \( \varphi_\gamma(u) \) respectively.

They derived the following simple formula for \( \varphi_\gamma(u) \) assuming that the insurer is in a
profitable condition immediately after time 0

According to (1.1) of Wei, (2009), the result of Albrecher-Hipp, (2007) is given as

\[
\varphi_\gamma(u) = [\varphi(u)]^{\frac{1}{1-\gamma}}
\]

(11.1)

In this paper, we have computed the ruin probability in the presence of interest earning
and tax payments using the results as derived in (Wei, 2009)

We give a brief introduction to the general frame work of the derivation details of which
can be found in Wei, 2009.

(11.1) Description of the Model:

In the set up formulated in Wei, (2009) the surplus process at time \( t, U_g(t) \) satisfies the
following stochastic differential equation
where $C_1(.)$ and $C_2(.)$ are two positive functions and $M_g(t) = \max_{0 \leq s \leq t} \{U_g(s)\}$ denote the running maximum of the Surplus process. Whenever, the surplus is at the running maximum, the company is according to the terminology of Albrecher and Hipp (2007) is in a profitable situation. Let $\psi_g(u)$ denote the ruin probability for this process as a function of the initial surplus $u$.

In this context an important quantity that is been introduced is $q(x)$. This is a conditional probability that as the surplus process up crosses the level $x$ for the first time, there is a claim at that instant and $q(x)$ denotes the probability that ruin occurs before the surplus return to level $x$. Also for $x \geq u \geq 0$, let $h(u,x)$ denote the probability the surplus process $\{U_g(t): t \geq 0\}$ having initial value $u$ will reach the level $x$ before possible ruin. Consequently it can be realized that $1 - q(x)$ gives the probability that the surplus stays non negative before it returns to the level $x$.

According to the proposition (2.1) of Wei, 2009, we have

$$h(u,x) = \exp\left\{ - \int_u^x \frac{\lambda q(y)}{C_2(y)} \, dy \right\}$$

(11.3)

Also, it can be noted that $h(u, \infty) = \phi_g(u)$ thereby giving rise to the corollary (2.1) of Wei, 2009 as

$$\phi_g(u) = \exp\left\{ - \int_u^\infty \frac{\lambda q(y)}{C_2(y)} \, dy \right\}$$

(11.4)
\[ q(u) = \frac{\phi'_g(u) C_2(u)}{\phi_g(u) \lambda} \]  \hfill (11.5)

Hence once \( C_2(u) \) is known, \( \phi_g(u) \) can be determined if \( q(u) \) is known.

The results that we shall be using have been derived by Wei, 2009, considering the special cases \( C_1(x) = c + \delta x \) and \( C_2(x) = c + \delta x(1 - \gamma(x)) \) with \( \delta > 0 \) interpreted as a constant force of interest and \( \gamma(x) \epsilon [0,1) \) as a surplus dependent tax rate. For this special case, let \( \psi_{\delta,\gamma}(u) \) and \( \phi_{\delta,\gamma}(u) \) denote the corresponding ruin and non-ruin probabilities respectively.

Hence, if \( C_1(x) = c + \delta x \) and \( C_2(x) = c + \delta x(1 - \gamma(x)) \) with \( \delta > 0 \), we have from (11.4)

\[ \phi_{\delta,\gamma}(u) = \exp \left\{ - \int_u^\infty \frac{\lambda q(y)}{(c + \delta y)(1 - \gamma(y))} dy \right\} \]  \hfill (11.6)

If \( \gamma(y) = 0 \), then

\[ \phi_\delta(u) = \exp \left\{ - \int_u^\infty \frac{\lambda q(y)}{(c + \delta y)} dy \right\} \]  \hfill (11.7)

which leads to the following

\[ q(u) = \frac{\phi'_\delta(u) (c + \delta u)}{\phi_\delta(u) \lambda} \]  \hfill (11.8)

Note that \( \phi_\delta(u) \) denotes the probability of survival under the interest rate and Sundt and Teugels, 1995 deals with the estimates of the probability of ruin under interest rates and an explicit expression for \( \phi_\delta(u) \) has been derived for exponentially distributed claim severity. Using this, \( q(u) \) for exponential distribution was derived from (11.8) in Wei,
2009 and thereafter an explicit expression for $\phi_{\delta,\gamma}(u)$ for exponential distribution was derived using (11.6). For rest of the distributions, explicit expressions for $\phi_{\delta,\gamma}(u)$ don’t exist.

From (11.6), we have

$$
\psi_{\delta,\gamma}(u) = 1 - \exp\left\{ - \int_u^\infty \frac{\lambda q(y)}{(c + \delta y)(1 - \gamma(y))} dy \right\}
$$

(11.9)

Therefore, as $u \to \infty$ (11.9) implies

$$
\psi_{\delta,\gamma}(u) \sim \int_u^\infty \frac{\lambda q(x)}{(c + \delta x)(1 - \gamma(x))} dx
$$

(11.10)

(11.2) **Probability of Ruin for the Sub-exponential Distributions in the presence of Interest earnings and Tax Payments**

We are primarily concerned with the results of Wei, 2009 which deals with sub-exponential distributions

By definition, a distribution $F$ on $[0, \infty)$ is said to be subexponential if $\bar{F}(x) = 1 - F(x) > 0$ for all $x > 0$ and

$$
\lim_{x \to \infty} \frac{\bar{F}^{n*}(x)}{\bar{F}(x)} = n
$$

(11.11)

For the computation of the probability of ultimate ruin in the presence of interest earnings and tax payments for sub-exponential distributions, the main result of Wei, (2009) that will be used for this purpose is cited below:

For a distribution $F$ on $[0, \infty)$ with $\bar{F}(x) > 0$ for all $x > 0$, define
With

\[ F^*(v) = \lim_{x \to +\infty} \sup \frac{F(vx)}{F(x)} \text{ for } v > 1 \]  \hspace{1cm} (11.13)

**The Main Result**: 

Suppose that the claim size distribution \( F \) and its equilibrium distribution \( F_1 \) are subexponential and that \( J_*(F) \) defined by (11.12) satisfies \( 1 < J_*(F) \leq \infty \), then

\[
\psi_{s,y}(u) \sim \int_u^\infty \frac{\hat{F}(x)}{(c + \delta x)(1 - \gamma(x))} dx \hspace{1cm} (11.14)
\]

For obtaining a slight insight into its proof found in Wei, 2009, the lemma (4.1) stated herein is used to show that for a sub-exponential \( q(x) \sim \tilde{F}(x) \).

Hence using this in (11.10), the main result stated in (11.14) is obtained.

We have taken the tax structure as used in Wei, (2009) for our computation. As a matter in practice, the interest rate is subjected to the choice of investment made by the insurance company and the tax rate is governed by the fiscal policies of the country concerned.

The tax structure used is

\[
\gamma(x) = \begin{cases} 
0.10, & 0 < x \leq 10^4 \\
0.18, & 10^4 < x \leq 10^5 \\
0.30, & 10^5 < x \leq 10^6 \\
0.50, & x > 10^6 
\end{cases} \hspace{1cm} (11.15)
\]
An illustrative value for the rate of interest has been taken as $\delta = 0.05$. As has been used in this work, the illustrative value of $\lambda$ has been taken as $\lambda = 32.427$

(11.3) Computation of the probability of Ruin for the Sub-Exponential distributions in the presence of interest earnings and tax payments

(11.3.1) For the Pareto distribution,

$$F(x) = 1 - \left(\frac{\lambda_1}{\lambda_1 + x}\right)^\alpha$$

$$\psi_{\delta,y}(u) \sim \int_u^\infty \frac{\lambda \tilde{F}(x)}{(c + \delta x)(1 - y(x))} \, dx$$

$$= \int_u^{10^4} \frac{\lambda}{(c + \delta x)} \left(\frac{\lambda_1}{\lambda_1 + x}\right)^\alpha \, dx$$

$$+ \int_{10^4}^{10^5} \frac{\lambda}{(c + \delta x)} \left(\frac{\lambda_1}{\lambda_1 + x}\right)^\alpha \left(1 - \frac{18}{100}\right) \, dx$$

$$+ \int_{10^5}^{10^6} \frac{\lambda}{(c + \delta x)} \left(\frac{\lambda_1}{\lambda_1 + x}\right)^\alpha \left(1 - \frac{30}{100}\right) \, dx$$

$$+ \int_{10^6}^\infty \frac{\lambda}{(c + \delta x)} \left(\frac{\lambda_1}{\lambda_1 + x}\right)^\alpha \left(1 - \frac{50}{100}\right) \, dx$$

(11.16)

Changing the scale by substituting

$$x - u = y$$
We have

\[ \psi_{x,y}(u) \sim \frac{1}{0.9} \lambda \lambda_1^{\alpha} \int_0^{10^4-u} \frac{1}{\{c + \delta(y + u)\}\{\lambda_1 + (y + u)\}} dy \]

+ \frac{1}{0.82} \lambda \lambda_1^{\alpha} \int_{10^5-u}^{10^6-u} \frac{1}{\{c + \delta(y + u)\}\{\lambda_1 + (y + u)\}} dy

+ \frac{1}{0.70} \lambda \lambda_1^{\alpha} \int_{10^6-u}^{\infty} \frac{1}{\{c + \delta(y + u)\}\{\lambda_1 + (y + u)\}} dy

+ \frac{1}{0.50} \lambda \lambda_1^{\alpha} \int_{10^6-u}^{\infty} \frac{1}{\{c + \delta(y + u)\}\{\lambda_1 + (y + u)\}} dy \quad (11.17) \]

The above integral was computed numerically

(11.3.2) For the Weibul distribution,

\[ F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\beta}, \quad x > 0, \theta > 0, \beta > 0 \]

\[ \psi_{x,y}(u) \sim \int_u^{\infty} \frac{\lambda F(x)}{(c + \delta x)(1 - \gamma(x))} dx \]

\[ = \int_u^{10^4} \frac{\lambda}{(c + \delta x)} \left(\frac{90}{100}\right) dx + \int_{10^5}^{10^6} \frac{\lambda}{(c + \delta x)} \left(\frac{90}{100}\right) dx + \int_{10^6}^{\infty} \frac{\lambda}{(c + \delta x)} \left(\frac{90}{100}\right) dx \]

+ \int_{10^6}^{\infty} \frac{\lambda}{(c + \delta x)} \left(\frac{90}{100}\right) dx \quad (11.18) \]

Changing the scale, we have
Here also the integral was computed numerically

(11.3.3) For Log–Normal Distribution

Here

\[ F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_0^x \frac{e^{-\frac{1}{2\sigma^2}(\log y - \mu)^2}}{y} \, dy \]

\[ \psi_{\delta, y}(u) \sim \int_u^\infty \lambda \left\{ 1 - \frac{1}{\sqrt{2\pi}\sigma} \int_0^x \frac{e^{-\frac{1}{2\sigma^2}(\log y - \mu)^2}}{y} \, dy \right\} \frac{1}{(c + \delta x)(1 - y(x))} \, dx \] (11.20)

Changing the scale, we have

\[ \psi_{\delta, y}(u) \sim \frac{\lambda}{0.90} \int_0^{10^4 - u} \frac{1 - plnorm(y + u, \mu, \sigma)}{c + \delta(y + u)} \, dy \]

\[ + \frac{\lambda}{0.82} \int_{10^4 - u}^{10^5 - u} \frac{1 - plnorm(y + u, \mu, \sigma)}{c + \delta(y + u)} \, dy \]

\[ + \frac{\lambda}{0.70} \int_{10^5 - u}^{10^6 - u} \frac{1 - plnorm(y + u, \mu, \sigma)}{c + \delta(y + u)} \, dy \]

\[ + \frac{\lambda}{0.50} \int_{10^6 - u}^{\infty} \frac{1 - plnorm(y + u, \mu, \sigma)}{c + \delta(y + u)} \, dy, \text{ where plnorm} \]

\[ = \frac{1}{\sqrt{2\pi}\sigma} \int_0^x \frac{1}{y} e^{-\frac{1}{2\sigma^2}(\log y - \mu)^2} \, dy \] (11.21)
The above integral was computed numerically

(11.3.4) For the Burr XII distribution

\[ F(x) = 1 - \left\{ 1 + \left( \frac{x}{\varphi} \right)^{\tau} \right\}^{-\alpha}, \quad y > 0, \alpha > 0, \tau > 0, \varphi > 0 \]

Therefore,

\[
\psi_{\delta, \gamma}(u) \sim \int_{u}^{\infty} \frac{\lambda \left\{ 1 + \left( \frac{x}{\varphi} \right)^{\tau} \right\}^{-\alpha}}{(c + \delta x)(1 - \gamma(x))} \, dx
\]

\[
= \int_{u}^{10^4} \frac{\lambda \left\{ 1 + \left( \frac{x}{\varphi} \right)^{\tau} \right\}^{-\alpha}}{(c + \delta x) \left( \frac{90}{100} \right)} \, dx + \int_{10^4}^{10^5} \frac{\lambda \left\{ 1 + \left( \frac{x}{\varphi} \right)^{\tau} \right\}^{-\alpha}}{(c + \delta x) \left( \frac{82}{100} \right)} \, dx + \int_{10^5}^{10^6} \frac{\lambda \left\{ 1 + \left( \frac{x}{\varphi} \right)^{\tau} \right\}^{-\alpha}}{(c + \delta x) \left( \frac{70}{100} \right)} \, dx
\]

\[
+ \int_{10^6}^{\infty} \frac{\lambda \left\{ 1 + \left( \frac{x}{\varphi} \right)^{\tau} \right\}^{-\alpha}}{(c + \delta x) \left( \frac{50}{100} \right)} \, dx
\]

(11.22)

Changing the scale, we have

\[
\psi_{\delta, \gamma}(u) \sim \int_{u}^{10^4-u} \frac{\lambda \left\{ 1 + \left( \frac{y + u}{\varphi} \right)^{\tau} \right\}^{-\alpha}}{(c + \delta (y + u)) \left( \frac{90}{100} \right)} \, dy
\]

\[
+ \int_{10^4-u}^{10^5-u} \frac{\lambda \left\{ 1 + \left( \frac{y + u}{\varphi} \right)^{\tau} \right\}^{-\alpha}}{(c + \delta (y + u)) \left( \frac{82}{100} \right)} \, dy + \int_{10^5-u}^{10^6-u} \frac{\lambda \left\{ 1 + \left( \frac{y + u}{\varphi} \right)^{\tau} \right\}^{-\alpha}}{(c + \delta (y + u)) \left( \frac{70}{100} \right)} \, dy
\]

\[
+ \int_{10^6-u}^{\infty} \frac{\lambda \left\{ 1 + \left( \frac{y + u}{\varphi} \right)^{\tau} \right\}^{-\alpha}}{(c + \delta (y + u)) \left( \frac{50}{100} \right)} \, dy
\]

(11.23)

This integral was also computed numerically
(11.4) Probability of Ruin under Interest Rate:

It needs to be noted that the above result used for determining the Probability of ultimate ruin in the presence of interest earnings and tax payments is valid only for Sub-exponential distributions and for two of our other distributions under consideration, viz Mixture of 3 exponential and Gamma, which don’t belong to the class of sub-exponential distributions, we tried identifying alternative ways to evaluate their probabilities of ultimate ruin under interest earnings and tax payments. However, owing to some limitations, we could only identify some results which determines this probability under interest rate irrespective of whether the claim severity is sub-exponential or not and the following description as extracted from Sundt and Teugels, 1995 is aimed at a preliminary description of the underlying model required for estimating ruin probabilities under the influence of interest rates.

In studying the influence of interest upon the probability of ruin, let us assume that in addition to the premium income, the company also receives interest on its surplus with a constant force \( \delta \). Let \( U_\delta(t) \) denote the value of the surplus at time \( t \). Under all other assumptions which are in effect for the classical risk model, the Stochastic differential equation for the probability of ultimate ruin under interest rate can be casted in the following form (See Sundt and Teugels, 1995)

\[
dU_\delta(t) = cdt + U_\delta(t)\delta dt - dS_t
\]

i.e.

\[
U_\delta(t) = ue^{\delta t} + \omega_\delta(t) - \int_0^t e^{\delta(t-u)} dS_u
\]

with
\[ u = U(0) \]

and

\[ \bar{\omega}_t^{(\delta)} = \int_0^t e^{\delta v} dv = \begin{cases} t, & \text{if } \delta = 0 \\ \frac{e^{\delta t} - 1}{\delta}, & \text{if } \delta > 0 \end{cases} \]

As derived in section (3.1) of the Sundt and Teugels, 1995, the integral equation for the non-ruin probability is given by

\[ \phi_\delta(u) = \frac{c}{c + \delta u} \phi_\delta(0) + \frac{1}{c + \delta u} \int_0^u \phi_\delta(u - y)\{\delta + \lambda[1 - F(y)]\}dy \quad (11.26) \]

In the same reference, the solution to the integral equation is given in terms of a first order differential equation as follows

\[ -\delta \gamma_\delta(s) + [c - \rho \zeta(s)]\gamma_\delta(s) = K_\delta \zeta(s) \quad (11.27) \]

With initial condition

\[ \lim_{s \to \infty} \gamma_\delta(s) = 0 \quad (11.28) \]

Where

\[ K_\delta = \frac{\rho \phi_\delta(0)}{\psi_\delta(0)}, \rho = \lambda p_1 \text{ and } \gamma_\delta(s) = \int_0^\infty e^{-\delta v} dG_\delta(v) \quad (11.27) \]

\[ G_\delta(v) \] is an auxiliary distribution function given by

\[ G_\delta(v) = \frac{\phi_\delta(u) - \phi_\delta(0)}{1 - \phi_\delta(0)} \quad (11.28) \]

\[ \zeta(s) = \int_0^\infty e^{-su} dF_1 \quad (11.29) \]

also equation (14) of section (3.4) of Sundt and Teugels, 1995 gives

\[ \phi_\delta(0) = \left\{ c \int_0^\infty \exp(-cz + \rho \int_0^z \zeta(\delta w) dw) dz \right\}^{-1} \quad (11.30) \]
(11.4.1) Bounds for the Probability of Ultimate Ruin under Interest Rate

We now state the main result for calculating the bounds of $\phi_\delta(u)$ recursively as mentioned in section (3.4) of Sundt and Teugels, 1995

Main Result: Discretizing the integral in (11.26) gives the following bounds to $\phi_\delta(u)$

For any $h > 0$ and $= 1,2,...,$ we have

$$\phi_\delta^{(h-)}(hk) \leq \phi_\delta(hk) \leq \phi_\delta^{(h+)}(hk)$$  \hspace{1cm} (11.31)

Where,

$$\phi_\delta^{(h+)}(hk) = \frac{1}{c + \delta hk} \left\{ c \phi_\delta(0) + \sum_{j=1}^{k} \phi_\delta^{(h+)}(h(k-j)) f_j^{(h)} \right\}$$  \hspace{1cm} (11.32)

$$\phi_\delta^{(h-)}(hk) = \frac{1}{c + \delta hk - f_1^{(h)}} \left\{ c \phi_\delta(0) + \sum_{j=1}^{k-1} \phi_\delta^{(h-)}(h(k-j)) f_j^{(h)} \right\}$$  \hspace{1cm} (11.33)

$$f_k^{(h)} = \int_{h(k-1)}^{hk} \left[ \delta + \lambda(1 - F(y)) \right] dy$$

(11.34)

To start the recursion an explicit expression for $\phi_\delta(0)$ is given by (11.30)

Let us recollect that the equilibrium distribution is given by

$$F_1(x) = \frac{1}{p_1} \int_0^x \{ 1 - F(y) \} dy$$

And $\zeta(s)$ is the Laplace transform of $F_1$ given by

$$\zeta(s) = \int_0^\infty e^{-sx} dF_1(u)$$  \hspace{1cm} (11.35)

(11.4.2) Determining $\phi_\delta(0)$ for Mixture of 3 Exponentials Distribution

Here

$$\zeta(s) = \frac{1}{p_1} \int_0^\infty e^{-sx} \{ w_1 e^{-\lambda_1 x} + w_2 e^{-\lambda_2 x} + w_3 e^{-\lambda_3 x} \} dx$$
Therefore,

\[-cz + \rho \int_{0}^{z} \zeta(\delta w) dw\]

\[= -cz + \rho \frac{1}{p_1} \int_{0}^{z} \left\{ \frac{w_1}{\lambda_1 + \delta w} + \frac{w_2}{\lambda_2 + \delta w} + \frac{w_3}{\lambda_3 + \delta w} \right\} dw\]

\[= -cz + \rho \frac{1}{p_1 \delta} \left\{ w_1 \log \left(1 + \frac{\delta z}{\lambda_1}\right) + w_2 \log \left(1 + \frac{\delta z}{\lambda_2}\right) + w_3 \log \left(1 + \frac{\delta z}{\lambda_3}\right) \right\} \quad (11.37)\]

Hence

\[\exp \left(-cz + \rho \int_{0}^{z} \zeta(\delta w) dw\right)\]

\[= \exp (-cz)\]

\[+ \frac{\rho}{p_1 \delta} \left\{ w_1 \log \left(1 + \frac{\delta z}{\lambda_1}\right) + w_2 \log \left(1 + \frac{\delta z}{\lambda_2}\right) + w_3 \log \left(1 + \frac{\delta z}{\lambda_3}\right) \right\}\]

\[= \exp \left\{ -cz + \log \left(1 + \frac{\delta z}{\lambda_1}\right)^{\frac{\rho w_1}{p_1 \delta}} + \log \left(1 + \frac{\delta z}{\lambda_2}\right)^{\frac{\rho w_2}{p_1 \delta}} + \log \left(1 + \frac{\delta z}{\lambda_3}\right)^{\frac{\rho w_3}{p_1 \delta}} \right\}\]

\[= e^{-cz} \left(1 + \frac{\delta z}{\lambda_1}\right)^{\frac{\rho w_1}{p_1 \delta}} \left(1 + \frac{\delta z}{\lambda_2}\right)^{\frac{\rho w_2}{p_1 \delta}} \left(1 + \frac{\delta z}{\lambda_3}\right)^{\frac{\rho w_3}{p_1 \delta}} \quad (11.38)\]

\[\int_{0}^{\infty} \exp (-cz + \rho \int_{0}^{z} \zeta(\delta w) dw) dz\]

\[= \int_{0}^{\infty} \left\{ e^{-cz} \left(1 + \frac{\delta z}{\lambda_1}\right)^{\frac{\rho w_1}{p_1 \delta}} \left(1 + \frac{\delta z}{\lambda_2}\right)^{\frac{\rho w_2}{p_1 \delta}} \left(1 + \frac{\delta z}{\lambda_3}\right)^{\frac{\rho w_3}{p_1 \delta}} \right\} dz \quad (11.39)\]

Let

\[\frac{\rho w_1}{p_1 \delta} = A, \frac{\rho w_2}{p_1 \delta} = B, \frac{\rho w_3}{p_1 \delta} = C\]
Hence,

\[
\int_0^\infty \exp (-cz + \rho z) \int_0^z \xi(\delta w)dw \, dz
\]

\[
= \int_0^\infty e^{-cz}(1 + az)^\lambda (1 + az)^\eta (1 + az)^\delta \, dz
\]

(11.40)

This equation has to be evaluated numerically and its value is inserted in (11.30) to obtain \( \phi(z) \).

Also

\[
f_k^{(h)} = \int_{h(k-1)}^{h(k)} [\delta + \lambda(1 - F(y))] \, dy
\]

\[
= \int_{h(k-1)}^{h(k)} \{\delta + \lambda(w_1 e^{-\lambda_1 y} + w_2 e^{-\lambda_2 y} + w_3 e^{-\lambda_3 y})\} \, dy
\]

\[
= \delta \left\{ h + \frac{w_1}{\lambda_1} [e^{-\lambda_1 h(k-1)} - e^{-\lambda_1 h(k+1)}] + \frac{w_2}{\lambda_2} [e^{-\lambda_2 h(k-1)} - e^{-\lambda_2 h(k+1)}] + \frac{w_3}{\lambda_3} [e^{-\lambda_3 h(k-1)} - e^{-\lambda_3 h(k+1)}] \right\}
\]

(11.41)

(11.5) Results and Discussions

From Table nos: 11.1, 11.2, 11.3 and 11.4, it is observed that the probability of ultimate ruin (under interest earning and tax payments) is decreasing with an increase in the initial surplus, which is as expected. As we have mentioned in the earlier chapters, the variation in the initial surplus is of theoretical interest. In practice, the decision on the initial surplus of an insurance company is its internal decision and amounts considerably larger than those displayed are used. Among the four fitted sub-exponential distributions
which we have used, for Burr XII distribution, the probability of ultimate ruin is found to be least whereas for Pareto distribution, it is found to be highest.

An occurrence of some amount of error in the result is inevitable owing to the numerical integrations being carried out to evaluate the integrals in equations (11.17), (11.19), (11.21) and (11.23). We have used the integrate function in R to evaluate these integrals and error accumulated for each of them is not more than 1e-15. It needs to be noted that in evaluating the probability of ultimate ruin (under interest earnings and tax payments) for Log Normal distribution, numerical error might have been a bit higher because here two numerical integrations were carried out in nested order. A look in to the Log Normal case reveals evaluation of the final integral (numerically) has taken plnorm function as an input, which is actually the distribution function of the Log Normal computed numerically in R. This simultaneous execution of two numerical integrals has lead to a higher accumulation of error. However, the overall error margin is still as low as 1e-10.

The general impression is that the interest earnings should lower the chance of the probability of ultimate ruin that is, it has negative contribution towards ruin whereas the tax payments should inflate the probability of ultimate ruin, i.e. it has positive contribution towards ruin. It needs to be mentioned that the results as stated in Wei, 2009 which were used to evaluate the probability of ultimate ruin under interest earnings and tax payments are applicable only when the insurance company is in profitable situation in terms of a criterion as mentioned in the same reference. In our case, the verification of this condition was technically impossible.
No generalization could be made as to the net impact of interest earnings and tax payments on the probability of ultimate ruin and it depends on the actual values of the parameters governing the Surplus process. For the each of the distributions under consideration, the probability of ultimate ruin was evaluated through a variety of techniques in the absence of interest rate and tax payments in the earlier chapters. Hence, there remains scope for comparison between the values of ultimate ruin probabilities obtained in the presence and absence of interest earnings and tax payments since both the situations are determined by the use of the common values of $\lambda$, the intensity parameter and $\theta$, the security loading factor.

In our case, the net impact of interest earnings and tax payments on the probability of ultimate ruin is found to be positive that is, as a whole, probability of ultimate ruin under the presence of interest earnings and tax payments got increased as compared to that obtained in the absence of interest earnings and tax payments. It further needs mentioning that the interest rate used is purely illustrative and the tax structure is extracted from Wei, 2009, although an tax structure prevalent in India during the time data was collected would have been more realistic. Additionally, it needs to be stated that the interest rate that has been used, is illustrative, mainly because the information on the real value of $\delta$ is almost impossible to be obtained for it is difficult to record information on the investment decision of an insurance company that too, for the surpluses generated out of a specific portfolio.

It can be noted that the computation of the probability of ultimate ruin under interest earnings and tax payments carried out in section (11.2) is valid only when the claim severity distribution belongs to the family of sub-exponential distributions and the four distributions for which it was evaluated namely the Pareto distribution, Weibull
distribution, Log normal distribution and Burr XII distribution belong to the sub-
exponential family. Hence the light tailed distributions under consideration viz the
Mixture of 3 exponentials and the Gamma distribution need different treatment for the
computation of the probability of ultimate ruin under interest earnings and tax
payments. We have only succeeded in evaluating the bounds to the probability of
ultimate ruin under interest rate for mixture of 3 exponentials using the results as cited
in (11.31) to (11.34). We could find no result which simultaneously deals with the
influence of interest rate and tax payments on the probability of ultimate ruin for
mixture of 3 exponentials.

As pointed out earlier, in computing the probability of ultimate ruin under interest
earnings and tax payments, the real difficulty lies in computing $q(y)$ from which an
approximate expression for $\psi_{\delta,y}(u)$ can be obtained using equation (11.10). But for
obtaining, $q(y)$, an expression for $\phi_\delta(u)$ (equation (11.8)) is needed which is rather
difficult to be obtained. Hence, we just tried evaluating the upper and lower bounds for
the probability of ultimate ruin under the influence of interest rate with no consideration
of tax payments. Again to start the recursion scheme for evaluating the bounds, we need
an expression for $\phi_\delta(u)$, which is also associated with computational difficulties, main
difficulty being the computation of the integral $\int_0^z \zeta(\delta w)dw$ which implies integration
of the Laplace transform of the claim severity distribution in the interval $[0,z]$ and
then, this is to be used in evaluating the integral $\int_0^\infty \exp(-cz + \rho \int_0^z \zeta(\delta w)dw)dz$. For
mixture of three exponentials, $\int_0^z \zeta(\delta w)dw$ does have a closed form expression whereas
for Gamma distribution, it doesn’t even has a closed form expression. Numerical
integration of $\int_0^z \zeta(\delta w)dw$ and then putting this output as an input in evaluating
\[ \int_{0}^{\infty} \exp (-cz + \rho \int_{0}^{z} \zeta(\delta w)dw)dz \]

numerically for Gamma distribution led to absurd results which compelled us to exclude Gamma distribution from being one of the claim severity distributions for which the probability of ultimate ruin under interest rate is sought.

As we have mentioned, that a closed form for \( \int_{0}^{\infty} \zeta(\delta w)dw \) exists for Mixture of 3 Exponentials but computation of \( \int_{0}^{\infty} \exp (-cz + \rho \int_{0}^{z} \zeta(\delta w)dw)dz \) gave rise to many computational issues, resolution of which are beyond the scope of this work. For example when we tried finding this expression for the mixture of three exponentials that we have fitted, it yielded the value \textit{NAN} (\textit{that is a number beyond the range of representable numbers in R.}) A little diagnosis lead to the identification of the following cause behind it. The expression \( \exp (-cz + \rho \int_{0}^{z} \zeta(\delta w)dw) \) appearing in the integrand of \( \int_{0}^{\infty} \exp (-cz + \rho \int_{0}^{z} \zeta(\delta w)dw)dz \) can be split as the product of \( \exp (-cz) \) and \( \exp (\rho \int_{0}^{\infty} \zeta(\delta w)dw) \). Now for the fitted mixture of 3 exponentials, \( c \) is very large at \( c = 841094.9 \). Hence \( \exp (-cz) \) has a very low value tending to zero whereas for the estimated values of the parameters of Mixture of 3 exponentials and the assumed value of \( \lambda \) at \( \lambda = 32.427 \), \( \exp (\rho \int_{0}^{\infty} \zeta(\delta w)dw) \) was found to yield very high values tending to infinity for all values of \( z \). Hence the product of two quantities one tending to zero and the other tending to infinity led to the occurrence of a value such as \textit{NAN}.

Hence, just to illustrate the application of the results obtained in (11.4) for finding the bounds of the Probability of ultimate ruin for Mixture of 3 exponentials in presence of interest rates, instead of our fitted mixture of 3 exponentials, we have used an illustrative Mixture of 3 exponentials with \( \lambda_1 = 1.066956, \lambda_2 = 7.979466, \lambda_3 = \)
The upper and lower bounds to the probability of ultimate ruin in the presence of interest rate for this mixture of 3 exponentials is shown in 11.5. Both the bounds are decreasing with increasing initial surplus which is to be expected considering the fact that higher initial surplus tends to constraints the surplus process to reach the negative threshold and higher the initial surplus, higher is the interest earning.

Table no: 11.1 Probability of ultimate ruin for Log Normal Distribution with parameters $\hat{\mu} = 9.327069$ and $\hat{\sigma} = 0.9254849$ under the tax structure given by (11.15) and rate of interest $\delta=0.05$.

<table>
<thead>
<tr>
<th>Value of the initial surplus $u$ (in Rs)</th>
<th>$\psi_{\delta,Y}(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.8548930</td>
</tr>
<tr>
<td>20</td>
<td>0.8543974</td>
</tr>
<tr>
<td>30</td>
<td>0.8539018</td>
</tr>
<tr>
<td>40</td>
<td>0.8534062</td>
</tr>
<tr>
<td>50</td>
<td>0.8529106</td>
</tr>
<tr>
<td>60</td>
<td>0.8524150</td>
</tr>
<tr>
<td>70</td>
<td>0.8519194</td>
</tr>
<tr>
<td>80</td>
<td>0.8514239</td>
</tr>
<tr>
<td>90</td>
<td>0.8509283</td>
</tr>
<tr>
<td>100</td>
<td>0.8504327</td>
</tr>
<tr>
<td>200</td>
<td>0.8494415</td>
</tr>
<tr>
<td>500</td>
<td>0.8479547</td>
</tr>
<tr>
<td>1000</td>
<td>0.8058814</td>
</tr>
</tbody>
</table>
Table no: 11.2 Probability of ultimate ruin for Weibull Distribution with parameters \( \hat{\theta} = 18058.838357 \) and \( \hat{\beta} = 1.0196673 \) under the tax structure given by (11.15) and rate of interest \( \delta=0.05 \).

<table>
<thead>
<tr>
<th>Value of the initial surplus u (in Rs)</th>
<th>( \psi_{\delta,Y}(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.8534887</td>
</tr>
<tr>
<td>20</td>
<td>0.8530119</td>
</tr>
<tr>
<td>30</td>
<td>0.8525354</td>
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<tr>
<td>40</td>
<td>0.8520591</td>
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<tr>
<td>50</td>
<td>0.8515830</td>
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<td>0.8511072</td>
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<td>70</td>
<td>0.8506316</td>
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<td>80</td>
<td>0.8501563</td>
</tr>
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<td>90</td>
<td>0.8496812</td>
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<td>100</td>
<td>0.8492063</td>
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<tr>
<td>200</td>
<td>0.8482574</td>
</tr>
<tr>
<td>500</td>
<td>0.8468357</td>
</tr>
<tr>
<td>1000</td>
<td>0.8074712</td>
</tr>
</tbody>
</table>

Table no: 11.3 Probability of ultimate ruin for the Pareto Distribution with parameters \( \hat{\alpha} = 96819.07 \) and \( \hat{\beta} = 6.447139 \) under the tax structure given by (11.15) and rate of interest \( \delta=0.05 \).

<table>
<thead>
<tr>
<th>Value of the initial surplus u (in Rs)</th>
<th>( \psi_{\delta,Y}(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.8547157</td>
</tr>
<tr>
<td>20</td>
<td>0.8542353</td>
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<td>30</td>
<td>0.8537552</td>
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<td>0.8518382</td>
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<td>0.8513597</td>
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<td>90</td>
<td>0.8508816</td>
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<td>0.8504037</td>
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<td>200</td>
<td>0.8494490</td>
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<tr>
<td>500</td>
<td>0.8480194</td>
</tr>
<tr>
<td>1000</td>
<td>0.8086724</td>
</tr>
</tbody>
</table>
Table no: 11.4 Probability of ultimate ruin for the Burr XII Distribution with parameters $\hat{\phi} = 1.047651e + 06$ and $\hat{a} = 1.670876e + 05$ and $\hat{t} = 8.657284e - 01$ under the tax structure given by (11.15) and rate of interest $\delta = 0.05$.

<table>
<thead>
<tr>
<th>Value of the initial surplus $u$ (in Rs)</th>
<th>$\psi_{\delta \gamma}(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3.246241e-26</td>
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<tr>
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<td>3.089485e-28</td>
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<td>3.086375e-30</td>
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<td>3.223717e-32</td>
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<td>4.649844e-38</td>
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<tr>
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<tr>
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<td>1.635798e-47</td>
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</tr>
<tr>
<td>1000</td>
<td>5.995740e-191</td>
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</tbody>
</table>

Table no: 11.5 Probability of ultimate ruin for Mixture of 3 Exponentials with $\lambda_1 = 1.066956, \lambda_2 = 7.979466, \lambda_3 = 1.005759, w_1 = 0.0454989, w_2 = 0.7096710, w_3 = 0.2448301$ and $\lambda = 5.89$ and rate of interest $\delta = 0.05$.

<table>
<thead>
<tr>
<th>Value of the initial surplus $u$ (in Rs)</th>
<th>Lower bound to $\psi_\delta(u)$</th>
<th>Upper bound to $\psi_\delta(u)$</th>
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</thead>
<tbody>
<tr>
<td>10</td>
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<td>0.7260401</td>
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<td>0.7260205</td>
<td>0.7260398</td>
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<td>0.7260205</td>
<td>0.7260398</td>
</tr>
<tr>
<td>40</td>
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<td>0.7260398</td>
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<td>0.7260398</td>
</tr>
<tr>
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<td>0.7260205</td>
<td>0.7260398</td>
</tr>
<tr>
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<td>0.7260398</td>
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</table>