APPENDIX – B

B.1 MODEL CALCULATIONS FOR THERMAL ANALYSIS OF ANTISYMMETRIC CROSS-PLY LAMINATED PLATES USING HIGHER-ORDER DISPLACEMENT MODEL - 1

The following graphite epoxy material properties are used for each lamina of the laminated composite, to obtain in-plane stresses and transverse normal and shear stresses of a simply supported cross ply $(0^\circ/90^\circ)$.

\[
\frac{E_1}{E_2} = 25, \quad \frac{G_{12}}{E_2} = 0.5, \quad \frac{G_{13}}{E_2} = 0.2, \quad E_2 = E_3 = 10^6 \text{ N/cm}^2
\]

\[G_{12} = G_{13} \quad \text{and} \quad \mu_{12} = \mu_{23} = \mu_{13} = 0.25\]

\[
\alpha_1 = 1125, \quad \alpha_2 = 1.1 \times 10^{-6} \text{ C}^{-1}, \quad \alpha_2 = 1125 \times 10^{-6} \text{ C}^{-1}
\]

\[K_1 = 36.42 \text{ W m}^{-1} \text{ C}^{-1}, \quad K_2 = 0.96 \text{ W m}^{-1} \text{ C}^{-1}\]

Substituting Eq. (B.1) in Eq. (3.4), the plane stress reduced elastic constants of the \(L^{th}\) lamina are

\[C_{11} = 25.062 \times 10^6, \quad C_{12} = 0.2506 \times 10^6, \quad C_{22} = 1.003 \times 10^6, \quad \ldots \text{B.2}\]

\[C_{33} = G_{12} = 0.5 \times 10^6, \quad C_{44} = G_{23} = 0.2 \times 10^6, \quad C_{55} = G_{13} = G_{12} = 0.5 \times 10^6, \quad \ldots \text{B.2}\]

Substituting Eq. (B.2) into Eq. (3.6) then the plane stress reduced elastic constants of the \(L^{th}\) laminate are:

For \(\theta = 0^\circ\)

\[Q_{11} = 25.062 \times 10^6, \quad Q_{12} = 0.253 \times 10^6, \quad Q_{13} = 0, \quad Q_{22} = 1.003 \times 10^6, \quad Q_{23} = 0, \quad Q_{33} = 0.5 \times 10^6, \quad Q_{44} = 2.8 \times 10^6, \quad Q_{45} = 0, \quad Q_{55} = 0.5 \times 10^6, \quad \ldots \text{B.3}\]

\[\alpha_x = 1 \times 10^{-6}, \quad \alpha_y = 1125 \times 10^{-6}, \quad \alpha_{xy} = 0\]

For \(\theta = 90^\circ\)

\[Q_{11} = 1.025 \times 10^6, \quad Q_{12} = 0.253 \times 10^6, \quad Q_{13} = 0, \quad Q_{22} = 0.0625 \times 10^6, \quad Q_{23} = 0, \quad Q_{33} = 0.5 \times 10^6, \quad Q_{44} = 0.5 \times 10^6, \quad Q_{45} = 0, \quad Q_{55} = 0.2 \times 10^6, \quad \ldots \text{B.4}\]

\[\alpha_x = 1 \times 10^{-6}, \quad \alpha_y = 1125 \times 10^{-6}, \quad \alpha_{xy} = 0\]

from Eq. (3.12c)
layer 1

\[ H_1 = 2.5, \quad H_2 = -3.125, \quad H_3 = 5.2083, \quad H_4 = -9.7656, \quad H_5 = 9.5313, \quad H_6 = -40.69, \quad H_7 = 87.193 \]

\[ \cdots \text{B.5} \]

layer 2

\[ H_1 = 2.5, \quad H_2 = 3.125, \quad H_3 = 5.2083, \quad H_4 = 9.7656, \quad H_5 = 9.5313, \quad H_6 = 40.69, \quad H_7 = 87.193 \]

\[ \cdots \text{B.6} \]

By substituting the Eq. (B.1) - (B.6) in Eq. (3.12 a,b), the A, B, D, D' matrices can be obtained, the elements of A, B, D, D' matrices in to Eq. (5.5a) then \( S_{ij} \) are obtained as:

\[
\begin{align*}
S_{11} &= 1.0685, \quad S_{12} = 0.0593, \quad S_{13} = 0, \quad S_{14} = 1.1873, \quad S_{15} = 0, \quad S_{16} = 2.2260, \quad S_{17} = 0.1235 \\
S_{18} &= 3.7104, \quad S_{21} = 0, \quad S_{22} = 1.0684, \quad S_{23} = 0, \quad S_{24} = -1.1873, \quad S_{25} = 0.1235, \\
S_{26} &= 2.2260, \quad S_{27} = 0, \quad S_{28} = -3.7104, \quad S_{33} = 0.0553, \quad S_{34} = 0.2199, \quad S_{35} = 0.2199, \\
S_{36} &= 0.23562, \quad S_{37} = -0.23562, \quad S_{38} = 1.37445, \quad S_{44} = 1.37445, \quad S_{45} = 3.97602, \quad S_{46} = 0.1235, \\
S_{47} &= 5.5854, \quad S_{48} = 0, \quad S_{49} = 19.2851, \quad S_{49} = 0.46302, \quad S_{55} = 3.97602, \quad S_{56} = 0, \quad S_{57} = -5.5854 \\
S_{58} &= 0.46302, \quad S_{59} = 19.2851, \quad S_{64} = 22.9309, \quad S_{65} = 0.46302, \quad S_{66} = 33.038, \quad S_{67} = 0, \\
S_{77} &= 22.9309, \quad S_{78} = 0, \quad S_{79} = -33.038, \quad S_{88} = 160.31284, \quad S_{89} = 2.06707, \quad S_{99} = 160.312843 \\
\end{align*}
\]

for a given temperature distribution \( T_{mn}(z) \) the values of Eq. (5.5c) are:

\[
\begin{align*}
N^1 &= 128.9706, \quad N^2 = -128.97064, \quad M^1 = -375.7051, \quad M^2 = -375.7051, \quad N'^1 = 403.033266, \\
N'^2 &= -403.033266, \quad M'^1 = -1408.8941, \quad M'^2 = -1408.8941, \\
\end{align*}
\]

\[ \cdots \text{B.7} \]

upon substitution of Eq. (B.7) and (B.8) in Eq. (7.37), the coefficients

\[
\begin{align*}
U_{mn} &= 0.000753, \quad t_{mn}, \quad V_{mn} = -0.000753, \quad t_{mn}, \\
W_{mn} &= 0.004475, \quad t_{mn}, \quad X_{mn} = -0.000637, \quad t_{mn}, \quad Y_{mn} = -0.000637, \quad t_{mn}, \\
U'^*_{mn} &= 0.000055, \quad t_{mn}, \quad V'^*_{mn} = -0.000055, \quad t_{mn}, \quad X'^*_{mn} = -0.000052, \quad t_{mn}, \quad Y'^*_{mn} = -0.000002, \quad t_{mn}. \\
\end{align*}
\]

Substituting the above values in to Eq. (5.3) and substituting the resultants in to strain relations then it is obtained as:
\[ \varepsilon_{x_0} = -9.4652 \times 10^{-5} t_0 \sin \alpha x \sin \beta y, \quad \varepsilon'_{x_0} = 9.4652 \times 10^{-5} t_0 \sin \alpha x \sin \beta y, \]

\[ \varepsilon_{xy0} = 0, \quad k_x = 8.00709 \times 10^{-5} t_0 \sin \alpha x \sin \beta y, \quad k_y = 8.00709 \times 10^{-5} t_0 \sin \alpha x \sin \beta y, \]

\[ k_{xy} = 0, \quad k^*_{xy} = -0.02514 \times 10^{-5} t_0 \sin \alpha x \sin \beta y, \quad k^*_{xy} = -0.02514 \times 10^{-5} t_0 \sin \alpha x \sin \beta y, \]

\[ \varepsilon_{xy0} = 0, \quad \phi_y = 0, \quad \phi_x = 0, \quad \varepsilon_{yxy} = 0, \quad \varepsilon_{xxy} = 0, \quad \phi^*_{x} = 0, \quad \phi^*_{y} = 0 \]

Considering the above strain relations in to the laminate constitutive Eq. (3.5) it is obtained as:

At \( z = + h/2 \) and \( \theta = 0^\circ \)

\[ \sigma_x = 1210.1624 t_0, \quad \sigma_y = -790.6104 t_0 \]

To obtain non-dimensional stress values, multiply the stresses with \( m_3 = \sigma_i / E_2 t_0 \)

\[ \sigma_x \times 10^{-3} / m_3 = 1.21016, \quad \sigma_y \times 10^{-3} / m_3 = -0.79061 \]

At \( z = - h/2 \) and \( \theta = 90^\circ \)

\[ \sigma_x = 1535.4643 t_0, \quad \sigma_y = -620.8441 t_0 \]

The non-dimensional stresses are

\[ \sigma_x \times 10^{-3} / m_3 = 1.53546, \quad \sigma_y \times 10^{-3} / m_3 = -0.8208441 \]

At \( z = 0 \) and at \( (a/2, 0) \)

\[ \tau_{xx} = -37.2646 t_0 \]

The non-dimensional stress is

\[ \tau_{xy} \times 10^{-3} / m_3 = -0.0372646 \]

At \( z = 0 \) and at \( (0, b/2) \)

\[ \tau_{zz} = -37.2646 t_0 \]

The non-dimensional stress is

\[ \tau_{zz} \times 10^{-3} / m_3 = -0.0372646 \]
APPENDIX – D

D.1 MODEL CALCULATIONS FOR TRANSIENT ANALYSIS OF LAMINATED COMPOSITE PLATES

The following graphite epoxy material properties are used for each lamina of the laminated composite, to compute the transient characteristics of the composite laminated plate:

\[ a = b = 25 \text{ cm}, \quad h = 1 \text{ cm} \quad (a/b = 1, \quad a/h = 25) \]

\[ \rho = 8 \times 10^6 \text{ N-s}^2 / \text{cm}^4, \quad E_2 = 2.1 \times 10^6 \text{ N/cm}^2 \]

\[ E_1 = 25 E_2, \quad G_{12} = G_{13} = 0.5 E_2, \quad \gamma_{12} = 0.25 \]

The values of \( \alpha \) and \( \gamma \) in the Newmark integration scheme are taken to be 0.5, which correspond to constant – average acceleration method (stable).

Assume zero initial conditions for velocity and displacement:

\[ u_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \]

\[ \dot{u}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \]

The steps can be summarized as follows:

At time \( t = 0 \) find:

1. For \([K]\) and \([M]\) from Eq. (5.13) where:

\[ S_{11} = 0.4516 \times 10^6, \quad S_{12} = 0.0277 \times 10^6, \quad S_{13} = 0, \quad S_{14} = 0.0999 \times 10^6, \quad S_{15} = 0, \]

\[ S_{16} = 0.0801 \times 10^6, \quad S_{17} = 0.0376 \times 10^6, \quad S_{18} = 0.0023 \times 10^6, \quad S_{19} = 0.0043 \times 10^6, \]

\[ S_{110} = 0.0125 \times 10^6, \quad S_{111} = 0, \quad S_{112} = 0.02 \times 10^6 \]

\[ S_{22} = 0.4516 \times 10^6, \quad S_{23} = 0, \quad S_{24} = 0, \quad S_{25} = -0.0999 \times 10^6, \quad S_{26} = 0.0809, \]

\[ S_{27} = 0.0023 \times 10^6, \quad S_{28} = 0.00376 \times 10^6, \quad S_{29} = -0.0043 \times 10^6, \]

\[ S_{210} = 0, \quad S_{211} = -0.0125 \times 10^6, \quad S_{212} = 0.02 \times 10^6 \]
\[ S_{33} = 0.023 \times 10^6, \quad S_{34} = 0.0924 \times 10^6, \quad S_{35} = -0.0924 \times 10^6, \quad S_{36} = 0. \]
\[ S_{37} = 0.0198 \times 10^6, \quad S_{38} = -0.0197 \times 10^6, \quad S_{39} = -0.0019 \times 10^6. \]
\[ S_{310} = 0.0231 \times 10^6, \quad S_{311} = 0.0231 \times 10^6, \quad S_{312} = 0. \]
\[ S_{44} = 0.7726 \times 10^6, \quad S_{45} = 0.0023 \times 10^6, \quad S_{46} = 0.012 \times 10^6, \]
\[ S_{47} = 0.1699 \times 10^6, \quad S_{48} = 0, \quad S_{49} = 0.02104 \times 10^6. \]
\[ S_{410} = 0.1894 \times 10^6, \quad S_{411} = 0.00035 \times 10^6, \quad S_{412} = 0.00203 \times 10^6. \]
\[ S_{55} = 0.7726 \times 10^6, \quad S_{56} = -0.01202 \times 10^6. \]
\[ S_{57} = 0, \quad S_{58} = -0.1699 \times 10^6, \quad S_{59} = 0.02103 \times 10^6. \]
\[ S_{510} = 0.00035 \times 10^6, \quad S_{511} = 0.1894 \times 10^6, \quad S_{512} = -0.00203 \times 10^6. \]
\[ S_{66} = 2.2513 \times 10^6, \quad S_{67} = 0.0221 \times 10^6, \quad S_{68} = 0.0221 \times 10^6, \quad S_{69} = 0. \]
\[ S_{610} = 0.0039 \times 10^6, \quad S_{611} = -0.0039 \times 10^6, \quad S_{612} = 0.5626 \times 10^6. \]
\[ S_{510} = 0.00035 \times 10^6, \quad S_{511} = 0.1894 \times 10^6, \quad S_{512} = -0.00203 \times 10^6. \]
\[ S_{77} = 0.2506 \times 10^6, \quad S_{78} = 0.000346 \times 10^6, \quad S_{79} = 0.003005 \times 10^6. \]
\[ S_{710} = 0.0611 \times 10^6, \quad S_{711} = 0, \quad S_{712} = 0.0053 \times 10^6. \]
\[ S_{88} = 0.2506 \times 10^6, \quad S_{89} = -0.003005 \times 10^6. \]
\[ S_{810} = 0, \quad S_{811} = -0.0611 \times 10^6, \quad S_{812} = 0.0053 \times 10^6. \]
\[ S_{99} = -0.750089 \times 10^6, \quad S_{910} = 0.005465 \times 10^6, \quad S_{911} = 0.005465 \times 10^6, \quad S_{912} = 0. \]
\[ S_{1010} = 0.0834 \times 10^6, \quad S_{1011} = 0.000062 \times 10^6, \quad S_{1012} = 0.000751 \times 10^6. \]
\[ S_{1111} = 0.0837 \times 10^6, \quad S_{1112} = -0.000751 \times 10^6, \quad S_{1212} = 0.2531 \times 10^6. \]

\[ M_{11} = 8, \quad M_{12} = M_{13} = M_{14} = M_{15} = M_{16} = 0, \quad M_{17} = 0.6667, \]
\[ M_{18} = M_{19} = M_{110} = M_{111} = M_{112} = 0, \]
\[ M_{22} = 8, \quad M_{23} = M_{24} = M_{25} = M_{26} = M_{27} = 0, \]
\[ M_{28} = 0.6667, \quad M_{29} = M_{210} = M_{211} = M_{212} = 0, \]
\[ M_{33} = 8, \quad M_{34} = M_{35} = M_{36} = M_{37} = M_{38} = 0, \]
\[ M_{39} = 0.6667, \quad M_{310} = M_{311} = M_{312} = 0. \]
$M_{44} = 0.6667, M_{45} = M_{46} = M_{47} = M_{48} = M_{49} = 0, M_{410} = 0.1, M_{411} = M_{412} = 0$

$M_{55} = 0.6667, M_{56} = M_{57} = M_{58} = M_{59} = M_{510} = 0, M_{511} = 0.1, M_{512} = 0$

$M_{66} = 0.6667, M_{67} = M_{68} = M_{69} = M_{610} = M_{611} = 0, M_{612} = 0.1$

$M_{77} = 0.1, M_{78} = M_{79} = M_{710} = M_{711} = M_{712} = 0$

$M_{88} = 0.1, M_{89} = M_{810} = M_{811} = M_{812} = 0$

$M_{99} = 0.1, M_{910} = M_{911} = M_{912} = 0$

$M_{1010} = 0.017857, M_{1011} = M_{1012} = 0$

$M_{1110} = 0.017857, M_{1112} = 0.017857$

2. With the initial values of $\{u_0\}$ and $\{\dot{u}_0\}$, compute $\{\ddot{u}_0\}$ from Eq. (9.1)

$\{\ddot{u}_0\} = \begin{bmatrix} 0 & 0 & -0.003 & 0 & 0 & -0.0304 & 0 & 0 & 0.0608 & 0 & 0 & 0.2837 \end{bmatrix}^T \times 10^{-4}$

3. Choosing $\Delta t = 50 \mu s$ and $\alpha = \gamma = 0.5$, compute the following constants:

$a_1 = 25, a_2 = 25, a_3 = 0.0016, a_4 = 0.08, a_5 = 1$

4. Form the effective stiffness matrix $[\hat{K}]$ from Eq. (9.4a)

5. For each time step, calculate effective load $[\hat{F}]$ from Eq. (9.4b) at time $t+\Delta t$.

6. Solve for $u_{t+\Delta t}$ from Eq. (9.4)

$\{u_{50}\} = \begin{bmatrix} 0.0067 & -0.0069 & 0.2430 & -0.0296 & -0.296 & 0.0007 & 0.0001 & -0.00002 & 0.0001 & -0.0011 & -0.0010 \end{bmatrix}^T \times 10^{-3}$

7. Calculate the accelerations and velocities at time $t + \Delta t$ from Eq. (9.3a) and (9.3b)

$\{\ddot{u}_{50}\} = \begin{bmatrix} 0.00601 & 0.0001 & 0.0069 & -0.0005 & -0.0005 & 0.0030 & 0 & -0.0608 & 0 & -0.2837 \end{bmatrix}^T \times 10^{-4}$

$\{\dot{u}_{50}\} = \begin{bmatrix} 0.0266 & -0.0276 & 0.0971 & -0.1183 & -0.1183 & 0.0002 & 0.00004 & -0.0000 & -0.00068 & -0.00045 & -0.00043 \end{bmatrix}^T \times 10^{-5}$

The procedure is repeated for every time step to compute transient characteristics of laminated composite plates.
A.1 Model calculations for a static bending problem using higher order displacement model-1 (for anti-symmetric cross-ply laminated plate)

The following Graphite Epoxy material properties are used for each lamina of the laminated composite, to obtain in-plane stresses and transverse normal and shear stresses of a simply supported cross ply (0°/90°) laminated plate as:

\[
\frac{E_1}{E_2} = 25, \quad \frac{G_{12}}{E_2} = 0.5, \quad \frac{G_{13}}{E_2} = 0.2, \quad E_2 = E_3 = 10^6 \text{ N/cm}^2
\]

\[G_{12} = G_{13} \text{ and } \mu_{12} = \mu_{13} = 0.25\] \hspace{1cm} \text{.....A.1}

Substituting Eq. (A.1) in Eq. (3.4), the plane stress reduced elastic constants of the \(L^{th}\) lamina are:

\[
C_{11} = 25.062 \times 10^6, \quad C_{12} = 0.2506 \times 10^6, \quad C_{22} = 1.003 \times 10^6, \quad C_{33} = G_{12} = 0.5 \times 10^6, \quad C_{44} = G_{13} = 0.2 \times 10^6, \quad C_{55} = G_{11} = G_{12} = 0.5 \times 10^6,
\]

\text{.....A.2}

Substituting Eq. (A.2) into Eq. (3.6) then the plane stress reduced elastic constants of the \(L^{th}\) laminate are:

For \(\theta = 0^\circ\)

\[
Q_{11} = 25.062 \times 10^6, \quad Q_{12} = 0.253 \times 10^6, \quad Q_{13} = 0, \quad Q_{22} = 1.003 \times 10^6, \quad Q_{33} = 0.5 \times 10^6, \quad Q_{44} = 0.2 \times 10^6, \quad Q_{45} = 0, \quad Q_{55} = 0.5 \times 10^6,
\]

\text{.....A.3}

For \(\theta = 90^\circ\)

\[
Q_{11} = 1.025 \times 10^6, \quad Q_{12} = 0.253 \times 10^6, \quad Q_{13} = 0, \quad Q_{22} = 0.0625 \times 10^6, \quad Q_{33} = 0.5 \times 10^6, \quad Q_{44} = 0.2 \times 10^6, \quad Q_{45} = 0, \quad Q_{55} = 0.5 \times 10^6,
\]

\text{.....A.4}

from Eq. (3.12c) \(H_i\) are:

Layer 1

\(H_1 = 2.5, \quad H_2 = -3.125, \quad H_3 = 5.2083, \quad H_4 = -9.7656, \quad H_5 = 19.5313, \quad H_6 = -40.69, \quad H_7 = 87.193\)

\text{.....6.5(e)}

Layer 2

\(H_1 = 2.5, \quad H_2 = 3.125, \quad H_3 = 5.2083, \quad H_4 = 9.7656, \quad H_5 = 19.5313, \quad H_6 = 40.69, \quad H_7 = 87.193\)

\text{.....6.5(f)}
By substituting Eq. (A.1 - A.4) in Eq. (3.12 a,b), the A, B, D, D' matrices can be obtained, considering the elements of A, B, D, D' matrices in Eq. (5.5a) then the S_{ij} is obtained as:

\[
S_{11} = 1.6695, \ S_{12} = 0.0926, \ S_{13} = 0, \ S_{14} = 1.8552, \ S_{15} = 0, \ S_{16} = 3.4781, \ S_{17} = 0.1929 \\
S_{18} = 5.7975, \ S_{19} = 0, \ S_{22} = 1.6695, \ S_{23} = 0, \ S_{24} = 0, \ S_{25} = -1.8552, \ S_{26} = 0.1929, \\
S_{27} = 0.4781, \ S_{28} = 0, \ S_{29} = -5.7975, \ S_{33} = 0.0864, \ S_{34} = 0.2749, \ S_{35} = 0.2749, \\
S_{36} = 0.2945, \ S_{37} = -0.2945, \ S_{38} = 1.7181, \ S_{39} = 1.7181, \ S_{44} = 5.2281, \ S_{45} = 0.1929, \\
S_{46} = 7.6725, \ S_{47} = 0, \ S_{48} = 23.9808, \ S_{49} = 0.7235, \ S_{55} = 5.2281, \ S_{56} = 0, \ S_{57} = -7.6725, \\
S_{58} = 0.7235, \ S_{59} = 23.9808, \ S_{66} = 27.6267, \ S_{67} = 0.7235, \ S_{68} = 41.7343, \ S_{69} = 0, \\
S_{77} = 27.6267, \ S_{78} = 0, \ S_{79} = -41.7343, \ S_{88} = 181.276, \ S_{89} = 3.2298, \ S_{99} = 181.276.
\]

By substituting the load and above values in the Eq. (6.1) and solving the coefficients of displacements in Eq. (5.3) is obtained as:

\[
U_{mn} = 2.286 \times 10^6 Q_{mn}, \ V_{mn} = -2.2888 \times 10^6 Q_{mn} \\
W_{mn} = 26.2840 \times 10^6 Q_{mn}, \ X_{mn} = -1.8247 \times 10^6 Q_{mn}, \ Y_{mn} = -1.8291 \times 10^6 Q_{mn}, \\
U^{*}_{mn} = 0.1011 X 10^{-6} Q_{mn}, \ V^{*}_{mn} = -0.1013 X 10^{-6} Q_{mn}, \ X^{*}_{mn} = -0.0951 X 10^{-6} Q_{mn}, \\
Y^{*}_{mn} = -0.0947 X 10^{-6} Q_{mn}.
\]

Substituting the above values in to Eq. (5.3) and substituting the resultant in to strain relations then it is obtained as:

\[
\varepsilon_{x0} = -2.286 \times 10^{-6} q_0 \alpha \sin \alpha \sin \beta y, \ \varepsilon_{y0} = 2.286 \times 10^{-6} q_0 \beta \sin \alpha \sin \beta y, \\
\varepsilon_{xy0} = 0, \ \kappa_{x} = 1.8247 \times 10^{-6} q_0 \alpha \sin \alpha \sin \beta y, \\
\kappa_{y} = 1.8291 \times 10^{-6} q_0 \beta \sin \alpha \sin \beta y, \ \kappa_{xy} = 0, \\
\kappa_{x}^{*} = 0.0951 \times 10^{-6} q_0 \alpha \sin \alpha \sin \beta y, \\
\kappa_{y}^{*} = 0.0947 \times 10^{-6} q_0 \beta \sin \alpha \sin \beta y, \ \kappa_{xy}^{*} = 0 \\
\varepsilon_{x0}^{*} = -0.1011 \times 10^{-6} q_0 \alpha \sin \alpha \sin \beta y, \ \varepsilon_{y0}^{*} = 0.1013 \times 10^{-6} q_0 \beta \sin \alpha \sin \beta y, \\
\varepsilon_{xy0}^{*} = 0, \ \phi_{y} = 10^{-6} q_0 \sin \alpha \cos \beta y, (-1.8291 + 26.274 \beta), \\
\phi_{x} = 10^{-6} q_0 \cos \alpha \sin \beta y, (-1.8291 + 26.274 \alpha), \\
\varepsilon_{y20} = -2 \times 0.1013 \times 10^{-6} q_0 \sin \alpha \cos \beta y, \\
\varepsilon_{x20} = 0.2022 \times 10^{-6} q_0 \cos \alpha \sin \beta y, \\
\phi_{x}^{*} = 3 \times -0.0951 \times 10^{-6} q_0 \cos \alpha \sin \beta y, \\
\phi_{y}^{*} = 3 \times -0.0947 \times 10^{-6} q_0 \sin \alpha \cos \beta y,
\]
Considering the above strain relations in laminate constitutive Eq. (3.5).

For the top layer at $z = \frac{h}{2}$ and $\theta = 0^\circ$

$\sigma_x = 12.7182 \, q_0, \quad \sigma_y = 1.53667 \, q_0$

To obtain non-dimensional stress values, multiply the stresses with $m_4 = \frac{h^2}{q_0} a^2$

$\sigma_x \times m_4 = 0.79488, \quad \sigma_y \times m_4 = 0.09604$

For the bottom layer at $z = -\frac{h}{2}$ and $\theta = 90^\circ$

$\sigma_y = -12.7182 \, q_0, \quad \sigma_x = -1.53667 \, q_0$

The non-dimensional stresses are

$\sigma_y \times m_4 = -0.79488, \quad \sigma_x \times m_4 = -0.09604$

At the mid-plane $z = 0$ and at $\left( \frac{a}{2}, 0 \right)$

$\tau_{xz} = 1.1498 \, q_0, \quad \tau_{yz} = 1.1104 \, q_0$

The non-dimensional stresses are

$\tau_{yz} \, m_5 = 0.28745, \quad \tau_{xz} \, m_5 = 0.2776$

The three dimensional elastic solutions can be obtained by considering $\sigma_x', \sigma_y, \tau_{xy}$ in Eq. (3.30b), the following non-dimensional stress values are obtained.

$\tau_{xz}' = 0.3094, \quad \tau_{yz}' = 0.2987$
The following Graphite Epoxy material properties are used for each lamina of the laminated composite, to obtain the free vibrations of a simply supported cross ply ($0/90^\circ$) laminated plate.

\[
\frac{E_1}{E_2} = 3, \quad E_2 = E_3 = 10^6 \text{ N/cm}^2
\]
\[
G_{12} = G_{13} = 0.6E_2, \quad G_{23} = 0.5E_2 \quad \text{and} \quad \mu_{12} = \mu_{13} = \mu_{23} = 0.25
\]

Substituting Eq. (C.1) in Eq. (3.4), the plane stress reduced elastic constants of the $L^{th}$ lamina are:

\[
C_{11} = 3.063830 \times 10^6, \quad C_{12} = 0.255319 \times 10^6, \quad C_{22} = 1.021277 \times 10^6, \quad C_{33} = 0.6 \times 10^6, \quad C_{44} = 0.5 \times 10^6, \quad C_{55} = 0.6 \times 10^6
\]

Substituting Eq. (C.2) in Eq. (3.6) then the plane stress reduced elastic constants of the $L^{th}$ laminate are:

For $\theta = 0^\circ$

\[
Q_{11} = 3.063830 \times 10^6, \quad Q_{12} = 0.255319 \times 10^6, \quad Q_{13} = 0, \quad Q_{22} = 1.021277 \times 10^6, \quad Q_{23} = 0, \quad Q_{33} = 0.6 \times 10^6, \quad Q_{44} = 0.5 \times 10^6, \quad Q_{55} = 0, \quad Q_{55} = 0.6 \times 10^6
\]

For $\theta = 90^\circ$

\[
Q_{11} = 1.021277 \times 10^6, \quad Q_{12} = 0.255319 \times 10^6, \quad Q_{13} = 0, \quad Q_{22} = 0.06383 \times 10^6, \quad Q_{23} = 0, \quad Q_{33} = 0.6 \times 10^6, \quad Q_{44} = 0.6 \times 10^6, \quad Q_{55} = 0, \quad Q_{55} = 0.6 \times 10^6
\]

From Eq. (3.12c)

layer 1 ($\theta = 90^\circ$)

\[
H_1 = 2, \quad H_2 = -2, \quad H_3 = 2.6667, \quad H_4 = -4, \quad H_5 = 6.4,
\]

\[
H_6 = -10.6667, \quad H_7 = 18.2857
\]

layer 2 ($\theta = 0^\circ$)

\[
H_1 = 2, \quad H_2 = 2, \quad H_3 = 2.6667, \quad H_4 = 4, \quad H_5 = 6.4,
\]

\[
H_6 = 10.6667, \quad H_7 = 18.2857
\]
By substituting the Eq. (C.1) - (C.6) in Eq. (3.12 a,b), the A, B, D, D' matrices can be obtained, considering the elements of A, B, D, D' matrices in to Eq. (8.1) then $S_0$ and $M_0$ are obtained as:

$$S_{11} = 0.260810 \times 10^6, S_{12} = 0.084417 \times 10^6, S_{13} = 0, S_{14} = 0.100796 \times 10^6, S_{15} = 0,$$

$$S_{16} = 0.347746 \times 10^6, S_{17} = 0.112555 \times 10^6, S_{18} = 0.201592 \times 10^6, S_{19} = 0,$$

$$S_{22} = 0.260810 \times 10^6, S_{23} = 0, S_{24} = 0, S_{25} = -0.100796, S_{26} = 0.112555 \times 10^6,$$

$$S_{27} = 0.347746 \times 10^6, S_{28} = 0, S_{29} = -0.201592 \times 10^6, S_{33} = 0.108566 \times 10^6,$$

$$S_{35} = 0.345576 \times 10^6, S_{33} = 0.345576 \times 10^6, S_{36} = 0.062832 \times 10^6, S_{37} = -0.062832 \times 10^6,$$

$$S_{40} = 1.382301 \times 10^6, S_{41} = 1.382301 \times 10^6, S_{44} = 2.547746 \times 10^6, S_{45} = 0.112555 \times 10^6,$$

$$S_{46} = 0.601592 \times 10^6, S_{47} = 0, S_{48} = 9.634591 \times 10^6, S_{49} = 0.270133 \times 10^6,$$

$$S_{55} = 2.347746 \times 10^6, S_{56} = 0, S_{57} = -0.601592 \times 10^6, S_{58} = 0.270133 \times 10^6,$$

$$S_{59} = 9.634591 \times 10^6, S_{66} = 12.567924 \times 10^6, S_{67} = 0.270133 \times 10^6, S_{68} = 2.937578,$$

$$S_{78} = 0, S_{77} = 12.567924 \times 10^6, S_{76} = 0, S_{79} = -2.937578 \times 10^6,$$

$$S_{88} = 65.744544 \times 10^6, S_{89} = 0.771809 \times 10^6, S_{99} = 65.744544 \times 10^6.$$

\[ \cdots \text{C.7} \]

$$M_{11} = 4, M_{12} = M_{13} = M_{14} = M_{15} = 0, M_{16} = 5.3333, M_{17} = M_{18} = M_{19} = 0,$$

$$M_{22} = 4, M_{23} = M_{24} = M_{25} = M_{26} = 0, M_{27} = 5.3333, M_{28} = M_{29} = 0,$$

$$M_{33} = 4, M_{34} = M_{35} = M_{36} = M_{37} = M_{38} = M_{39} = 0, M_{44} = 5.3333, M_{45} = M_{46} = M_{47} = 0,$$

$$M_{48} = 12.8, M_{49} = 0, M_{53} = 5.3333, M_{54} = M_{55} = M_{56} = 0, M_{57} = 12.8,$$

$$M_{58} = 12.8, M_{59} = 0, M_{67} = 12.8, M_{77} = M_{78} = M_{79} = 0,$$

$$M_{88} = 36.571429, M_{89} = 0, M_{99} = 36.571429 \text{ C.8}.$$

Upon substitution Eq (C.7) and (C.8) in characteristic Eq. (8.3), the solution of the non-dimensional fundamental frequency is obtained using the Mat lab as:

$$\bar{\omega} = \left( \frac{\omega a^2}{h} \right) \sqrt{\frac{\rho}{E_s}} = \left( \frac{62.161 \times 20^2}{4} \right) \times \sqrt{\frac{1}{10^6}} = 6.2161 \text{ \hspace{1cm} \cdots \text{C.9}}$$