APPENDIX 'A'

THE OPTIMAL PARTIAL CORRELATION COEFFICIENTS

In this section, the derivation of the optimal partial correlation coefficients for a lattice filter is derived. The partial correlation coefficients for optimum AR, MA, and ARMA lattice filters have been described in detail by Karlsson and Hayes[10]. These derivations are done in a geometrical framework and use a lattice structure where the forward and backward partial correlation coefficients are complex conjugates of each other.

As shown below, this geometrical procedure can also be used to derive the optimal partial correlation coefficients for a lattice filter where the forward and backward coefficients are not the complex conjugates of each other. The optimal forward partial correlation coefficients for stage m can be defined as the coefficient required to project \( e_{fo} \left( \frac{k}{m-1} \right) \) on to the span of \( e_{bo} \left( \frac{k-1}{m-1} \right) \). For the filter structure used in this thesis, such a projection would be \( -K_{fo}^{*} \left( \frac{k}{m} \right) e_{bo} \left( \frac{k-1}{m-1} \right) \). Similarly, the optimal forward error at the m th stage of a lattice filter can be described as the error after this projection and thus can be described using the partial correlation coefficients as

\[
e_{fo} \left( \frac{k}{m} \right) = e_{fo} \left( \frac{k}{m-1} \right) + K_{fo}^{*} \left( \frac{k}{m} \right) e_{bo} \left( \frac{k-1}{m-1} \right)
\]

The partial correlation coefficients conjugates have been used in this formulation to keep this treatment consistent with the notation used earlier in the thesis. Similarly, the projection of \( e_{bo} \left( \frac{k-1}{m-1} \right) \) on to \( e_{fo} \left( \frac{k}{m-1} \right) \) can be defined as the quantity \( -K_{bo}^{*} \left( \frac{k}{m} \right) e_{fo} \left( \frac{k}{m-1} \right) \) to obtain the optimal backward error after stage m as.
From these projection equation, using the inner product space defined by
\[
\langle a, b \rangle = e^{t^* a}
\]  
(A.3)

And using the fact that the error after such a projection is orthogonal to the input, we have
\[
\langle e_{ro} \left( \frac{k}{m} \right), e_{so} \left( \frac{k-1}{m-1} \right) \rangle = 0
\]  
(A.4)
\[
\langle e_{so} \left( \frac{k}{m} \right), e_{ro} \left( \frac{k}{m-1} \right) \rangle = 0
\]  
(A.5)

From this orthogonality, we get,
\[
\left( e_{ro} \left( \frac{k}{m-1} \right) + K_{so}^{*} \left( \frac{k}{m} \right) e_{so} \left( \frac{k-1}{m-1} \right) \right) e_{ro} \left( \frac{k-1}{m-1} \right) = 0
\]  
(A.6)
\[
\left( e_{so} \left( \frac{k-1}{m-1} \right) + K_{bo}^{*} \left( \frac{k}{m} \right) e_{ro} \left( \frac{k}{m-1} \right) \right) e_{ro} \left( \frac{k}{m-1} \right) = 0
\]  
(A.7)

Which leads to
\[
K_{so}^{*} \left( \frac{k}{m} \right) = -\frac{\langle e_{ro} \left( \frac{k}{m-1} \right), e_{so} \left( \frac{k-1}{m-1} \right) \rangle}{\| e_{so} \left( \frac{k-1}{m-1} \right) \|^2}
\]  
(A.8)
\[
K_{bo}^{*} \left( \frac{k}{m} \right) = -\frac{\langle e_{so} \left( \frac{k-1}{m-1} \right), e_{ro} \left( \frac{k}{m-1} \right) \rangle}{\| e_{ro} \left( \frac{k}{m-1} \right) \|^2}
\]  
(A.9)

Which can be written as
\[
K_{ro}^{*} \left( \frac{k}{m} \right) = -\frac{\langle e_{so} \left( \frac{k-1}{m-1} \right), e_{ro} \left( \frac{k}{m-1} \right) \rangle}{\| e_{so} \left( \frac{k-1}{m-1} \right) \|^2}
\]  
(A.10)
Now defining

\[ K_{so}^*(k, m) = -\frac{\varepsilon \left( e_{so} \left( \frac{k}{m-1} \right) e_{fo} \left( \frac{k-1}{m-1} \right) \right)}{\varepsilon \left( e_{so} \left( \frac{k}{m-1} \right) \right)^2} \]  \hspace{1cm} (A.11)

Now defining

\[ \Delta_0 \left( \frac{k}{m-1} \right) = \varepsilon \left( e_{so} \left( \frac{k-1}{m-1} \right) e_{fo} \left( \frac{k}{m-1} \right) \right) \]  \hspace{1cm} (A.12)

\[ F_0 \left( \frac{k}{m-1} \right) = \varepsilon \left( e_{fo} \left( \frac{k}{m-1} \right) \right)^2 \]  \hspace{1cm} (A.13)

\[ B_0 \left( \frac{k-1}{m-1} \right) = \varepsilon \left( e_{so} \left( \frac{k-1}{m-1} \right) \right)^2 \]  \hspace{1cm} (A.14)

Thus, \( K_{fo} \left( \frac{k}{m} \right) \) and \( K_{so} \left( \frac{k}{m} \right) \) can now be expressed as

\[ K_{fo} \left( \frac{k}{m} \right) = -\frac{\Delta_0 \left( \frac{k}{m-1} \right)}{B_0 \left( \frac{k-1}{m-1} \right)} \]  \hspace{1cm} (A.15)

\[ K_{so} \left( \frac{k}{m} \right) = -\frac{\Delta_0^* \left( \frac{k}{m-1} \right)}{F_0 \left( \frac{k}{m-1} \right)} \]  \hspace{1cm} (A.16)
APPENDIX "B"

THE OPTIMAL FORWARD AND BACKWARD ERRORS FOR A CHIRP SIGNAL

In this section, the optimal forward and backward errors are described for a generalized lattice filter. These quantities have been derived in detail in [25], and the results are included here for the sake of completeness. These are used in the derivation of the optimal partial correlation coefficients of the least squares lattice filter as well as in the calculation of the input for the general m-stage of a recursive least squares lattice filter.

The optimal forward and backward errors of a lattice filter can be written as

\[ e_{fo}\left(\frac{k-r}{m}\right) = x(k-r) - w_{fo}\left(\frac{k-r}{m}\right) \chi_m(k-r) \]  
\[ e_{so}\left(\frac{k-r}{m}\right) = x(k-r) - w_{so}\left(\frac{k-r}{m}\right) \chi_m(k-r) \]  

Where the optimal forward and backward transversal weight vectors are given by

\[Phi_m(k)w_{fo}\left(\frac{k}{m}\right) = \theta_f\left(\frac{k}{m}\right)\]
\[Phi_m(k)w_{so}\left(\frac{k-1}{m}\right) = \theta_b\left(\frac{k-1}{m}\right)\]

With the quantities \(\Phi_m\), \(\theta_f\) and \(\theta_b\) as defined previously.

For the chirp signal in particular, this formulation leads to the optimal transversal weight vectors given as under

\[w_{fo}\left(\frac{k-r}{m}\right) = \left(\frac{\rho}{1+\rho m}\right)V_m^{k-r}D_m\]
\[w_{so}\left(\frac{k-r-1}{m}\right) = a_{m-k}^*b_{m+1}^*\left(\frac{\rho}{1+m\rho}\right)V_m^{k-r}D_m\]

The generalized optimal forward and backward errors can be expressed as, [25]
For the calculation of the relevant partial correlation coefficients, the cross-correlation between the forward and backward errors as well as the forward and backward energy are of interest. These can be found to be

\[ e_{fo}(\frac{k-r}{m}) = x(k-r) - \left( \frac{\rho}{1+m\rho} \right) D_m^T V_{m} x_m(k-r) \]  
(B.7)

\[ e_{bo}(\frac{k-r-1}{m}) = x(k-r-m-1) - a_n^* b_n^* \left( \frac{\rho}{1+m\rho} \right) D_m^T V_{m} x_m(k) \]  
(B.8)

For the calculation of the relevant partial correlation coefficients, the cross-correlation between the forward and backward errors as well as the forward and backward energy are of interest. These can be found to be

\[ \mathbb{E}\left[ e_{fo}(\frac{k-r}{m}) e_{bo}^*(\frac{k-r-1}{m}) \right] = P \cdot \frac{b_n a_n^*}{1+m\rho} \]  
(B.9)

\[ \mathbb{E}\left[ \left| e_{bo}(\frac{k-r-1}{m}) \right|^2 \right] = P \cdot \frac{1+(m+1)\rho}{1+m\rho} \]  
(B.10)

\[ \mathbb{E}\left[ \left| e_{fo}(\frac{k-r}{m}) \right|^2 \right] = P \cdot \frac{1+(m+1)\rho}{1+m\rho} \]  
(B.11)

It is seen that the forward and backward error energies are independent of time for an optimally tracking filter. This is in agreement with the case of the transversal filter, where it is shown that the optimal wiener-Hopf transversal filter tracks a chirp signal perfectly and that the Wiener-Hopf minimum error is independent of time [25].
APPENDIX C

DERIVATION OF RLS PT EQUATIONS

In this section, the recursive relationships that constitute the RLS PT modeling approach are derived. A recursive expression for the PT transition matrix \( P_{t+1} R_{t+1} \) (more specifically, the PT transition matrix augmented by the Lagrange multipliers associated with the zero mean constraint) is obtained in terms of \( P_{t+1} R_{t} \). As with conventional RLS, this recursion involves an inversion of a measurement sample covariance matrix (augmented covariance matrix in the case of the PT RLS) that can also be shown to satisfy a recursive equation.

Eqn. (3.112), are repeated here for convenience:

\[
\begin{bmatrix}
    P_{t} R_{t+1} \\
    \mu_{t} R_{t+1}
\end{bmatrix} = \begin{bmatrix}
    \mathbb{E}_t \{ a_t^i z(k) z'(k) \} & \mathbb{E}_t \{ \beta_t^i z'(k) \} \\
    \mathbb{E}_t \{ \beta_t^i z'(k) \} & 0
\end{bmatrix}^{-1} \begin{bmatrix}
    \mathbb{E}_t \{ a_t^i z(k) x'(k + 1) \} \\
    \mathbb{E}_t \{ \beta_t^i x'(k + 1) \}
\end{bmatrix} \tag{C.1}
\]

where

\[
\begin{align*}
\mathbb{E}_t \{ a_t^i z(k) z'(k) \} &= \sum_{i=0}^{t} \alpha^{t-i} z(i) z'(i) \\
\mathbb{E}_t \{ a_t^i z(k) x'(k + 1) \} &= \sum_{i=0}^{t} \alpha^{t-i} z(i) x'(i + 1) \\
\mathbb{E}_t \{ \beta_t^i z(k) \} &= \sum_{i=0}^{t} \beta^{t-i} z(i) \\
\mathbb{E}_t \{ \beta_t^i x'(k + 1) \} &= \sum_{i=0}^{t} \gamma^{t-i} x'(i + 1)
\end{align*}
\tag{C.2}
\]

where \( \alpha, \beta \) and \( \gamma \in [0,1] \), and in general \( \beta \neq \alpha \neq \gamma \).

Let \( \Pi_t^{-1} \) denote the matrix to be inverted in equation (B.1) i.e.

\[
\Pi_t^{-1} = \begin{bmatrix}
    \mathbb{E}_t \{ a_t^i z(k) z'(k) \} & \mathbb{E}_t \{ \beta_t^i z'(k) \} \\
    \mathbb{E}_t \{ \beta_t^i z'(k) \} & 0
\end{bmatrix}
\tag{C.6}
\]

where \( \Pi_t^{-1} \in \mathbb{R}^{(m+1) \times (m+1)} \), and introducing the notation
equation (3.11) can be written as

\[ \Phi_t = \begin{bmatrix} P_t R_{t+1} \\ \mu_t R_{t+1} \end{bmatrix} \]  \hspace{1cm} (C.7)

\( \Phi_t \) obeys the following recursive equation

\[
\Pi_t^{-1} \Phi_t = \begin{bmatrix} \widetilde{E}_t \{ \alpha z(k) x'(k+1) \} \\
\widetilde{V}_t x'(k+1) \end{bmatrix} \]  \hspace{1cm} (C.8)

Proof:

From Eqn. (C.3) and (C.5), Eqn. (C.8) can be expanded as

\[
\Pi_t^{-1} \Phi_t = \begin{bmatrix} I_{m \times m} 0_{m \times 1} \\
0_{1 \times m} \frac{\alpha}{\beta} \end{bmatrix} \Pi_t^{-1} \begin{bmatrix} I_{m \times m} 0_{m \times 1} \\
0_{1 \times m} \frac{\alpha}{\beta} \end{bmatrix} \Phi_{t-1} + \Pi_t^{-1} \begin{bmatrix} z(k) x'(k+1) \\
x'(k+1) \end{bmatrix} - \begin{bmatrix} I_{m \times m} 0_{m \times 1} \\
0_{1 \times m} \frac{\alpha}{\beta} \end{bmatrix} = : \Phi_{t-1} + \Pi_t^{-1} \begin{bmatrix} z(k) x'(k+1) \\
x'(k+1) \end{bmatrix}
\]  \hspace{1cm} (C.9)

Where \( I_{m \times m} \) denotes the \( m \times m \) identity matrix, \( 0_{i \times j} \) denotes the \( i \times j \) matrix (vector) of zeros, and where the use is made of recursive expression.

\[
\widetilde{E}_t \{ \alpha z(k) x'(k+1) \} = \alpha \widetilde{E}_t \{ \alpha z(k-1) x'(k) \} + z(k) x'(k+1)
\]  \hspace{1cm} (C.11)

\[
\widetilde{V}_t x'(k+1) = \gamma \widetilde{V}_t \{ \alpha z(k) \} + x(k+1)
\]  \hspace{1cm} (C.12)

Which are easily derived from the basic defining Eqn. (C.3) and (C.5) respectively.

By replacing \( k \) with \( k - 1 \) everywhere in Eqn. (C.8) we have,

\[
\Pi_{t-1}^{-1} \Phi_{t-1} = \begin{bmatrix} \widetilde{E}_{t-1} \{ \alpha z(k-1) x'(k) \} \\
\widetilde{V}_{t-1} x'(k) \end{bmatrix} \]  \hspace{1cm} (C.13)

Substituting Eqn. (C.13) into Eqn. (C.10), we have
From Eqn. (C.6), (C.6) and (C.2), we have

\[
\Pi_{k-1}^{-1} = \alpha \left[ \begin{array}{cc} I_{m \times m} & 0_{m \times 1} \\ 0_{1 \times m} & \frac{\beta}{\alpha} \end{array} \right] \Pi_{k-1}^{-1} \left[ \begin{array}{cc} I_{m \times m} & 0_{m \times 1} \\ 0_{1 \times m} & \frac{\beta}{\alpha} \end{array} \right] + \left[ \begin{array}{c} z(k) \chi'(k+1) \\ \chi'(k+1) \end{array} \right] (C.15)
\]

Solving for \( \Pi_{k-1}^{-1} \) from Eqn. (C.15) and substituting the result into Eqn. (C.14), desired recursive equation for \( \Phi_i \) is obtained.

Eqn. (C.9) involves the inversion of the augmented measurement sample covariance matrix \( \Pi_i \). To complete the derivation of the RLS PT formulation, it is now shown here that inversion can be computed recursively in a manner which is analogous to but is more general than the conventional RLS method.

The inverse of the augmented sample covariance matrix \( \Pi_i^{-1} \) satisfies the following recursion.

\[
\alpha \Pi_i = I_{a,\beta} \Pi_{k-1}^{-1} I_{a,\beta} - I_{a,\beta} \Pi_{k-1}^{-1} I_{a,\beta} Z_1(k) \cdot (Z_1(k) I_{a,\beta} \Pi_{k-1}^{-1} I_{a,\beta} Z_1(k) + \alpha I)^{-1} 
\]

\[
\cdot Z_2(k) \left( I_{a,\beta} \Pi_i I_{a,\beta} \right) (C.16)
\]

Where

\[
I_{a,\beta} = \left[ \begin{array}{cc} I_{m \times m} & 0_{m \times 1} \\ 0_{1 \times m} & \frac{\alpha}{\beta} \end{array} \right] (C.17)
\]

And where \( Z_1(k) \in \mathcal{H}^{(m+1)^2} \) and \( Z_2(k) \in \mathcal{H}^{(m+1)^2} \) are defined by relations

\[
Z_1(k) = \left[ \begin{array}{c} \frac{1}{\sqrt{2}} z(k) \\ \frac{1}{\sqrt{2}} z(k) \\ \sqrt{2} \end{array} \right] (C.18)
\]

\[
Z_2(k) = \left[ \begin{array}{c} \frac{1}{\sqrt{2}} \chi'(k) \\ \frac{1}{\sqrt{2}} \chi'(k) \\ \sqrt{2} \end{array} \right] (C.19)
\]

Proof:

From Eqn. (C.15), we have
Employing the generalization of matrix inversion, we have

\[(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}\]  \hspace{1cm} (C.21)

Where \(A \in \mathbb{R}^{n \times n}\) and \(C \in \mathbb{R}^{m \times m}\) are nonsingular, and \(B\) and \(D\) are matrices of appropriate dimensions. The inverse of \(\Pi_i^{-1}\) can then be expressed recursively in terms of \(\Pi_{i-1}\) with above parameters defined as under.

\[
\Pi_i^{-1} = \begin{bmatrix} I_{m \times m} & 0_{m \times 1} \\ 0_{1 \times m} & \beta/a \end{bmatrix} \Pi_{i-1}^{-1} \begin{bmatrix} I_{m \times m} & 0_{m \times 1} \\ 0_{1 \times m} & \beta/a \end{bmatrix} + z_1(k)z_2(k) \hspace{1cm} (C.20)
\]

This involves the inversion of the augmented measurement sample covariance matrix. To complete the derivation of the RLS PT formulation, it can be shown that the inversion can be computed recursively in a manner analogous to but more general than the conventional RLS method.
APPENDIX 'G'

SIMILITUDE PROBLEM

The condition for two electromagnetic boundary-value problems to be equivalent has been obtained, and it is stated that a scale model cannot make the measurement equivalent to the real target (the non-scale model) exactly because of the limitation of the conductivities are very high, the scale model can realize the equivalent measurement within a negligible error.

The condition of similitude requires that the two characteristic parameters $c_1$ and $c_2$ in

$$\mu_0 \varepsilon_0 \left(\frac{l_0}{f_0}\right)^2 = c_{1,0} \mu_0 \varepsilon_0 \left(\frac{l_0^2}{f_0}\right) = c_2$$

be invariant to a change of scale. Since it is difficult or impossible to change $\varepsilon$ and/or $\mu$ practically, it should be assumed that they are left unchanged. The three parameters $l, l_0$ and $\sigma$ should be adjusted so that $c_1$ and $c_2$ are constant. To perform the field measurement in a scale model equivalent to the real measurement in a radar system 'if the length in the scale model is $1/n$ times as long as that in the real target, the frequency and the conductivity in the scale model must be $n$ times as high as those in the real radar and target system' where $n$ is a scale factor.

In an anechoic chamber, i.e., the measured field of a scale model, the choice of the frequency is made by the size of the scale model to keep the product of the frequency and the size constant. Namely the $c_1$ is kept constant. With regards to the second parameter in Eq(5.1), it is impossible to keep the parameter identical because the range of conductivities of the metals is limited. But since the conductivities are very high, the second parameter in both models can be considered infinite. Therefore it might be expected that both the models have equivalent measurement. Here two things are investigated:

(a) The influence of the conductivities in the electromagnetic field
(b) The influence of the thickness of the metallic coating on the scale of the model in electromagnetic field

The second option is examined because usually scale model of the aircraft, metallic coating is done on the plastic model. In this appendix, both influences are examined using the reflection of the electromagnetic wave incident on the metallic plate at normal incidence.

D.1 REFLECTION OF THE ELECTROMAGNETIC WAVE INCIDENT ON THREE LAYER MEDIA AT NORMAL ANGLE

It is assumed that plane waves with frequency $\omega$ are incident from the medium with $\mu_1$ and $\varepsilon_1$ to the second medium with $\mu_2$ and $\varepsilon_2$ at normal angle as shown in Fig.(1.1)

$$\frac{E_r}{E_i}$$

The reflection coefficient $R = \frac{E_r}{E_i}$ is given by

$$R = \frac{(1 - \eta_3/\eta_1)\cos k_z l + j/(\eta_3/\eta_1 - \eta_2/\eta_1)\sin k_z l}{(1 + \eta_3/\eta_1)\cos k_z l + j/(\eta_3/\eta_1 + \eta_2/\eta_1)\sin k_z l}$$

Consider a real target, i.e. an aircraft. Since usually the thickness of the aircraft is so large that the reflection from the third layer in Fig. A.1 can be neglected, the thickness can be taken to infinite. Using Equn. (A.2) and physical constants and conditions in Table A.1, the reflection coefficient $R$ is

Table D.1: Physical constants and measurement condition

<table>
<thead>
<tr>
<th>metal</th>
<th>$\sigma [\Omega^{-1}m^{-1}]$</th>
<th>Freq [GHz]</th>
<th>$\delta_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>duralumin</td>
<td>$2.94 \times 10^7$</td>
<td>0.1</td>
<td>$9.28 \times 10^{-4}$</td>
</tr>
<tr>
<td>Al</td>
<td>$3.64 \times 10^7$</td>
<td>10.0</td>
<td>$8.34 \times 10^{-7}$</td>
</tr>
<tr>
<td>Cu</td>
<td>$5.65 \times 10^7$</td>
<td>10.0</td>
<td>$6.69 \times 10^{-7}$</td>
</tr>
<tr>
<td>Ag</td>
<td>$6.17 \times 10^7$</td>
<td>10.0</td>
<td>$6.41 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

$R$ can be considered as "-1".
D.2. SCALE MODEL

Usually silver leaf is used for coating of a metallic scale model. This situation corresponds to that of Fig.D.1. Assume that the scale factor $n$ is 100. Using Eqn.(A.1) and Physical constants and conditions Table D.1, reflection coefficient is given. Here the thickness is greater than 0.01 skin depth, the error in the amplitude of the reflection coefficient is negligible when the thickness is greater than 0.01 skin depth. Similar results are obtained for copper and aluminium.

D.3 CONCLUSIONS

In the scale model measurements, it is required that the thickness of the metallic coating be greater than 0.01 skin depth in order to perform electromagnetic measurements equivalent to non-scale model. Usually the thickness of the metallic coating is thicker than 0.01 skin depth, so it can be said equivalent measurements are performed if the first condition of Eqn. (D.1) is satisfied.
APPENDIX 'E'
OUTER PRODUCT MODEL

E.1 DEFINITIONS

In order to analyze the outer product model, some definitions of terms are introduced and summarized for convenience.

\( V^{(m)} \): Unipolar binary \((0,1)\) stored feature vectors. \( m = 1,2,\ldots,M \)

\( V_i^{(m)} \): \( i-th \) component of \( V^{(m)} \), taking the values 0 or 1. \( i = 1,2,\ldots,N \).

\( U \): Unipolar binary \((0,1)\) input vectors, having a of size of \( N \).

\( M \) and \( N \): The number of feature vectors and the size of the vectors.

Usually for the purpose of analysis some statistical assumptions are introduced to the feature vectors \( V^{(m)} \). Namely assume that \( V_i^{(m)} \) is a random variable with a probability

\[
\text{Prob}[V_i^{(m)} = 1] = \text{Prob}[V_i^{(m)} = 0] = \frac{1}{2}
\]  \hspace{1cm} (E.1)

Where \( \text{Prob}[\ ] \) expresses the probability of the quantity in the brackets.

\( V_i^{(m)} \) is independent of \( i \) and \( m \). Namely the components \( V_i^{(m)} \) \( i = 1,2,\ldots,N \) and \( m = 1,2,\ldots,M \) are un-correlated. Therefore

\[
E[V_i^{(m)} V_j^{(n)}] = E[V_i^{(m)}] E[V_j^{(n)}] \text{ for } i \neq j \text{ or } m \neq n
\]  \hspace{1cm} (E.2)

Where \( E[\ ] \) expresses the expectation of the quantity in brackets.

For the outer product model, a \( N \times N \) 2-D matrix \( T'_{i,j} \), is called a "synaptic connection net", is calculated according to the relation

\[
T'_{i,j} = \begin{cases} 
\sum_{m=1}^{M} (2V_i^{(m)} - 1)(2V_j^{(m)} - 1) & \text{if } i \neq j \\
0 & \text{otherwise}
\end{cases}
\]  \hspace{1cm} (E.3)

\( T'_{i,j} \) is seen to be multi-valued, ranging between \( +M \) and \( -M \) in steps of two.
E.2 THE OUTER PRODUCT MODEL ANALYSIS

Consider a partial version (input vector) \( U \) of the stored vector \( V^{(m0)} \) as follows

\[
U_i = \begin{cases} 
    V_i^{(m0)} & i \leq \eta N \\
    X_i & \text{otherwise} 
\end{cases}
\]  

(E.4)

Where the \( \eta \) is the ratio of known bits to total bits in the input vector, and \( \eta N \) is an integer.

\( X_i \) is now random variable with probability

\[
P[X_i = 0] = P[X_i = 1] = \frac{1}{2}.
\]  

(E.5)

\( X_i \) is independent of \( i \) and the \( V_i^{(m)} \)s.

E.2.1 CONDITIONAL MEAN AND VARIANCE OF THE \( i \)-th BIT IN THE ESTIMATE FOR \( i \leq \eta N \).

By choosing \( i = 1 \), which does not sacrifice the generality of the results below, the estimated \( \hat{V}_1^{(m0)} \) is obtained:

\[
\hat{V}_1^{(m0)} = \sum_{j=1}^{N} T'_{i,j} U_j
\]

\[
= \sum_{j \neq 1}^{N} \sum_{m=1}^{K} (eV_1^{(m)} - 1) eV_j^{(m)} - 1 + \sum_{j \neq 1}^{N} \sum_{m=1}^{K} (2V_1^{(m)} - 1) V_j^{(m)} - 1 X_j
\]

(E.6)

\[
= \sum_{j \neq 1}^{N} \sum_{m=0}^{K} (eV_1^{(m)} - 1)eV_j^{(m)} - 1 V_j^{(m0)} + \sum_{j \neq 1}^{N} (2V_1^{(m0)} - 1) eV_j^{(m0)} - 1 V_j^{(m0)}
\]

\[+ \sum_{j \neq 1}^{N} \sum_{m=1}^{K} (2V_1^{(m)} - 1) eV_j^{(m)} - 1 X_j \]
\[
\begin{align*}
&= \sum_{j=2}^{N} \sum_{m=0}^{M} (2V_j^{(m)} - 1)(2V_j^{(m)} - 1) + \sum_{j=2}^{N} (2V_j^{(m)} - 1)V_j^{(m)} \\
&\quad + \sum_{j=2}^{N} \sum_{m=0}^{M} (2V_j^{(m)} - 1)(2V_j^{(m)} - 1)X_j
\end{align*}
\]  
(E.7)

Here, the relation \((2V_j^{(m)} - 1)V_j^{(m)} = V_j^{(m)}\) has been used. The first and the third terms are noise terms that have zero mean, while the second term contribute the desired signal.

We obtain the conditional mean and variance for the estimate \(\hat{V}_1^{(m)}\) given the component \(\hat{V}_1^{(m)}\). Since the first and the third terms have zero means in Eqn (E.7),

We have by denoting the conditional expectation by \(E_1\),

\[
m_1 = E_1[\hat{V}_1^{(m)}] = E_1\left[\sum_{j=2}^{N} (2V_j^{(m)} - 1)V_j^{(m)}\right]
\]

\[
= \sum_{j=2}^{N} (2V_j^{(m)} - 1)E[V_j^{(m)}]
\]

since \(E[V_j^{(m)}] = \frac{1}{2}\),

\[
m_1 = \frac{nN - 1}{2}(2V_1^{(m)} - 1)
\]  
(E.8)

the variance \(\sigma_1^2\) of the estimate is given by

\[
\sigma_1^2 = E_1\left[(\hat{V}_1^{(m)} - m_1)^2\right]
\]  
(E.9)

Substituting Eqn. (E.6) in to the above Eqn. we obtain,

\[
\sigma_1^2 = E_1\left[\sum_{j=2}^{N} \sum_{m=0}^{M} (2V_j^{(m)} - 1)(2V_j^{(m)} - 1)V_j^{(m)}\right] + \sum_{j=2}^{N} \sum_{m=0}^{M} (2V_j^{(m)} - 1)(2V_j^{(m)} - 1)X_j - m_1^2
\]  
(E.10)
Since the first term is noise term and is un-correlated with the second term (signal term) and third term. The expectation of the first term zero, the following equation is obtained,

\[ \sigma_i^2 = E[\left( \sum_{m=1}^{M} (2V_{i}^{(m)} - 1)(2V_{j}^{(m)} - 1)V_{j}^{(m_0)} + \sum_{j=2}^{M} (2V_{i}^{(m_0)} - 1)V_{j}^{(m_0)} \right)^2] \]

\[ + E[\left( \sum_{j=2}^{M} (2V_{i}^{(m_0)} - 1)V_{j}^{(m_0)} - m_1 \right)^2] \]


