Chapter 6

Summations and Transformations

for $3\psi_3$

Reference [14] is based on this chapter
6.1 Introduction

In the category of summations formulae, the following Ramanujan’s $\psi_1$ sum is one of the earliest known result.

$$1\psi_1(a; b; q, z) = \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_{\infty} (q/az)_{\infty} (b/a)_{\infty}}{(z)_{\infty} (b/az)_{\infty} (q/a)_{\infty}}, \quad (6.1.1)$$

where $|b/a| < |z| < 1$. The sum (6.1.1) may be regarded as the bilateral extension of well known $q$-binomial theorem [85]. This formula has a unique place in the theory of $q$-series because of its wide range of applications.

The proofs of many well known basic bilateral hypergeometric series identities rely on it. For example, the proof of the following $\psi_2$ sum of Bhargava and Adiga [48].

$$2\psi_2 \left( \begin{array}{c} q/a, b \\ d, bq \end{array} : q, a \right) = \sum_{n=-\infty}^{\infty} \frac{(q/a)_n (b)_n}{(d)_n (bq)_n} a^n = \frac{(d/b)_{\infty} (ab)_{\infty} (q^2)_{\infty}}{(q/b)_{\infty} (d)_{\infty} (a)_{\infty} (bq)_{\infty}}, \quad (6.1.2)$$

where $|a| < 1$, $|d| < 1$. Vasuki and Rajanna [144] have given an alternative proof of (6.1.2) and deduce certain Rogers Ramanujan type identities.

D. D. Somashekara et.al. [139] have derived a new summation formula for $\psi_2$ basic bilateral hypergeometric series using the $\psi_1$ summation formula (6.1.1) by the method of parameter augmentation. In particular, they have proved
\(2\psi_2\left(\begin{array}{c} a, \ bc/azq \ ; q, \ z \\ b, \ c \end{array}\right)\)
\[= \frac{(az)_\infty(q/az)_\infty(b/a)_\infty(c/a)_\infty(bc/azq)_\infty}{(z)_\infty(b)_\infty(c)_\infty(b/az)_\infty(c/az)_\infty(q/a)_\infty}. \quad (6.1.3)\]

The proof of (6.1.3) is based on the application of an operator \(E(b\theta)\) [107, 161] in (6.1.1). Somashekara et. al. [140] also derived the following \(2\psi_2\) transformation formula of Bailey from the \(q\)-Gauss summation formula

\[2\psi_2\left(\begin{array}{c} a, \ b \ ; q, \ z \\ c, \ d \end{array}\right)\]
\[= \frac{(az,c/b,d/a,dq/abz)_\infty}{(z,d,q/a,cd/abz)_\infty} 2\psi_2\left(\begin{array}{c} a, \ abz/d \ ; q, \ d/a \\ c, \ az \end{array}\right). \quad (6.1.4)\]

In chapter 4 (see [12]), we also have used (6.1.1) to derived a new summation formula for \(2\psi_2\) basic bilateral hypergeometric series, which is stated as

\[2\psi_2\left(\begin{array}{c} a, \ c \ ; q, \ z \\ b, \ acz \end{array}\right)\]
\[= \frac{(q)_\infty(b/a)_\infty(az)_\infty(q/az)_\infty(cz)_\infty}{(b)_\infty(q/a)_\infty(z)_\infty(b/az)_\infty(acz)_\infty}. \quad (6.1.5)\]

The origin of the developments in the present chapter is also (6.1.1). In particular, in the present chapter, we work on the class of summation formulae of basic bilateral hypergeometric series and have been able to give certain summation formulae for \(3\psi_3\) basic bilateral hypergeometric series. In Section 6.2 of this chapter, we give some new transformation formulae for \(3\psi_3\) basic bilateral hypergeometric series by the method of parameter augmentation. In
Section 6.3, we give some interesting special cases. In Section 6.4, we derive many identities involving $q$-gamma and $q$-beta function identities and in Section 6.5, we establish some eta-function identities.

To make the following sections of this chapter readable, we recall the following definitions of the $q$-differential operator and $q$-shifted operator

$$D_q f(a) = \frac{f(a) - f(aq)}{a}.$$ 

and

$$\zeta(f(a)) = f(aq).$$

Chen and Liu [56, 57] constructed operator

$$\theta = \zeta^{-1} D_q.$$ 

Based on this, they introduced the following two operators:

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q; q)_n}. \quad (6.1.6)$$

and

$$E(b\theta) = \sum_{k=0}^{\infty} \frac{(b\theta)^k q^{k(k-1)/2}}{(q)_k}. \quad (6.1.7)$$
The following identities hold for these operators

\[
T(bD_q) \frac{1}{(at)_\infty} = \frac{1}{(at, bt)_\infty}. \tag{6.1.8}
\]

\[
T(bD_q) \frac{1}{(as, at)_\infty} = \frac{(abst)_\infty}{(as, at, bs, bt)_\infty}. \tag{6.1.9}
\]

\[
E(b\theta)(at; q)_\infty = (at, bt)_\infty. \tag{6.1.10}
\]

\[
E(b\theta)(as, at; q)_\infty = \frac{(as, at, bs, bt; q)_\infty}{(abst/q; q)_\infty}, \tag{6.1.11}
\]

where \(|abst/q| < 1|.

### 6.2 Main Results

**Theorem 6.2.1.** We have

\[
3\psi_3 \begin{pmatrix}
q/a, \ b, \ c; q, \ a \\
\ d, \ bq, \ abc
\end{pmatrix} = \frac{(q)_\infty^2 (d/b)_\infty (ab)_\infty (ac)_\infty (abc)_\infty}{(a)_\infty (q/b)_\infty (d/bq)_\infty (c)_\infty (eq)_\infty}. \tag{6.2.1}
\]

**Proof:** Equation (6.1.2) can be written as

\[
\sum_{n=0}^{\infty} \frac{(q/a)_n (b)_n}{(d)_n (bq)_n} a^n + \sum_{n=1}^{\infty} \frac{(q/d)_n (1/b)_n}{(q/b)_n (a)_n} d^n = \frac{(d/b)_\infty (ab)_\infty (q^2)_\infty}{(q/b)_\infty (d)_\infty (a)_\infty (bq)_\infty}.
\]

which on using (1.2.7) and (1.2.8) becomes

\[
\sum_{n=0}^{\infty} \frac{(q/a)_n (b)_n}{(d)_n (bq)_n} a^n \left\{ \frac{1}{(bq^n)_\infty (a)_\infty} \right\} + \sum_{n=1}^{\infty} \frac{(q/d)_n (1/b)_n (b)_n}{(a)_n (a)_n} a^n \left\{ \frac{1}{(bq^{-n})_\infty (a)_\infty} \right\}
\]
Applying the operator $T(cD_q)$ on both sides and using the identities (6.1.8) and (6.1.9), we obtain

\[
\sum_{n=0}^{\infty} \frac{(q/a)_n(b)_\infty a^n}{(d/b)_n} \cdot \frac{(bc^n a)_{\infty}}{(bq^n)_{\infty}(ca)_\infty(cq)_\infty}
+ \sum_{n=1}^{\infty} \frac{(q/d)_n(1/b)_n(b)_{\infty}(bd)_n}{(a)_n(-1)^n q^{n(n+1)/2}} \cdot \frac{(bc^{-n} a)_{\infty}}{(bq^{-n})_{\infty}(ca)_{\infty}(cq)_{\infty}}

= \frac{(q)_\infty(d/b)_{\infty}}{(q/b)_{\infty}(a)_{\infty}(bq)_{\infty}(cq)_{\infty}}.
\]

After some simplification, we get

\[
\sum_{n=0}^{\infty} \frac{(q/a)_n(b)_n(c)_n a^n}{(d/b)_{\infty}(bc)_n} \cdot \frac{1}{(ab)_\infty(ac)_\infty}
+ \sum_{n=1}^{\infty} \frac{(q/a)_{-n}(b)_{-n}(c)_{-n}}{(d)_{-n}(bc)_{-n}(abc)_{-n}} a^{-n} \cdot \frac{1}{(ab)_\infty(ac)_\infty}

= \frac{(q)_\infty(d/b)_{\infty}}{(q/b)_{\infty}(a)_{\infty}(bq)_{\infty}(cq)_{\infty}}.
\]

which gives (6.2.1).

**Theorem 6.2.2.** We have

\[
\_{3}q_{3}^{3} \left( a, \frac{bc}{azq}, \frac{bd}{azq}; q, z \right)

= \frac{(az)_\infty(q/az)_\infty(b/a)_\infty(c/a)_\infty(bc/azq)_\infty(d/azq)_\infty(d/a)_\infty(bd/azq)_\infty}{(z)_\infty(b/az)_\infty(c/az)_\infty(q/a)_\infty(bd/azq)_\infty}.
\]

(6.2.2)
Proof: Equation (6.1.3) can be written as

\[
\sum_{n=0}^{\infty} \frac{(a)_n(bc/aqz)_n}{(b)_n(c)_n} z^n + \sum_{n=1}^{\infty} \frac{(q/b)_n(q/c)_n}{(q/a)_n(q^2az/bc)_n} q^n = \frac{(az)_\infty(q)_\infty(q/az)_\infty(b/a)_\infty(c/a)_\infty(bc/azq)_\infty}{(z)_\infty(b)_\infty(c)_\infty(b/az)_\infty(c/az)_\infty(q/a)_\infty}.
\]

which on using (1.2.7) and (1.2.8) becomes

\[
\sum_{n=0}^{\infty} \frac{(a)_n(bc/azq)_n z^n}{(c)_n} \{ (bq^n)_\infty(b/az)_\infty \} + \sum_{n=1}^{\infty} \frac{(q/c)_n(b)_\infty(-1)^n q^n(n+1)/2(q/b)_n}{(q/a)_n(q^2az/bc)_n} \{ (bq^{-n})_\infty(b/az)_\infty \} = \frac{(az)_\infty(q)_\infty(q/az)_\infty(b/a)_\infty(c/a)_\infty(bc/azq)_\infty}{(z)_\infty(c)_\infty(c/az)_\infty(q/a)_\infty}.
\]

Applying the operator \( E(d\theta) \) on both sides and using the identities (6.1.10) and (6.1.11), we obtain

\[
\sum_{n=0}^{\infty} \frac{(a)_n(bc/azq)_n z^n}{(c)_n} \{ (bdq^n/qaz)_\infty \} + \sum_{n=1}^{\infty} \frac{(q/c)_n(b)_\infty(-1)^n q^n(n+1)/2(q/b)_n}{(q/a)_n(q^2az/bc)_n} \{ (bdq^{-n}/aqz)_\infty \} = \frac{(az)_\infty(q)_\infty(q/az)_\infty(b/a)_\infty(c/a)_\infty(bc/azq)_\infty}{(z)_\infty(c)_\infty(c/az)_\infty(q/a)_\infty(bdc/a^2zq^2)_\infty}.
\]

Multiplying throughout by \( \frac{(bd/azq)_\infty}{(b)_\infty(d)_\infty} \), we obtain

\[
\sum_{n=0}^{\infty} \frac{(a)_n(bc/azq)_n(bd/azq)_n}{(b)_n(c)_n(d)_n} z^n
\]
\[
\sum_{n=1}^{\infty} \frac{(a)_{-n}(bd/aqz)_{-n}(bc/aqz)_{-n}}{(b)_{-n}(c)_{-n}(d)_{-n}} z^{-n} \]
\[= (az)_{\infty}(q)_{\infty}(q/az)_{\infty}(b/a)_{\infty}(c/a)_{\infty}(bc/aqz)_{\infty}(d/a)_{\infty}(bd/aqz)_{\infty} \]
\[\cdot \frac{(z)_{\infty}(b)_{\infty}(c)_{\infty}(d)_{\infty}(b/az)_{\infty}(c/az)_{\infty}(q/a)_{\infty}(bcd/azq)_{\infty}(d/azq)_{\infty}}{z_{\infty}(b)_{\infty}(c)_{\infty}(b/az)_{\infty}(c/az)_{\infty}(d/azq)_{\infty}}.
\]

which gives (6.2.2).

**Theorem 6.2.3.** We have

\[
\psi_{3}^{3} \left( \begin{array}{cccc} a, & bc/azq, & d & ; q, z \\ b, & c, & adz \end{array} \right) = \frac{(q)_{\infty}(q/az)_{\infty}(b/a)_{\infty}(c/a)_{\infty}(bc/aqz)_{\infty}(az)_{\infty}(dz)_{\infty}}{(z)_{\infty}(b)_{\infty}(c)_{\infty}(b/az)_{\infty}(c/az)_{\infty}(q/a)_{\infty}(adz)_{\infty}}. \tag{6.2.3}
\]

**Proof:** Equation (6.1.3) can be written as

\[
\sum_{n=0}^{\infty} \frac{(a)_{n}(bc/azq)_{n}}{(b)_{n}(c)_{n}} z^{n} + \sum_{n=1}^{\infty} \frac{(q/b)_{n}(q/c)_{n}}{(q/a)_{n}(q^{2}az/bc)_{n}} q^{n} = \frac{(az)_{\infty}(q/az)_{\infty}(b/a)_{\infty}(c/a)_{\infty}(bc/aqz)_{\infty}}{(z)_{\infty}(b)_{\infty}(c)_{\infty}(b/az)_{\infty}(c/az)_{\infty}(q/a)_{\infty}}.
\]

which on using (1.2.7) and (1.2.8) becomes

\[
\sum_{n=0}^{\infty} \frac{(bc/azq)_{n} z^{n}}{(b)_{n}(c)_{n}} \left\{ \frac{1}{(aq^{n})_{\infty}(az)_{\infty}} \right\} + \sum_{n=1}^{\infty} \frac{(q/b)_{n}(q/c)_{n}(aq)_{n}}{(q^{2}az/bc)_{n}(-1)^{n} q^{n(n+1)/2} \{ (aq)_{n}^{-1}(az)_{\infty} \}} = \frac{(az)_{\infty}(q/az)_{\infty}(b/a)_{\infty}(c/a)_{\infty}(bc/aqz)_{\infty}}{(z)_{\infty}(b)_{\infty}(c)_{\infty}(b/az)_{\infty}(c/az)_{\infty}(q/a)_{\infty}(a)_{\infty}}.
\]

Applying the operator \(T(dD_{d})\) on both sides and using the identities (6.1.8) and (6.1.9), we obtain
\[
\sum_{n=0}^{\infty} \frac{(bc/aqz)_n z^n}{(b)_n(c)_n} \left\{ \frac{(adzq^n)_\infty}{(aq^n)_\infty (az)_\infty (dq^n)_\infty (dz)_\infty} \right\} \\
+ \sum_{n=1}^{\infty} \frac{(q/b)_n(q/c)_n(aq)_n}{(q^2az/bc)_n(-1)^nq^{n(n+1)/2}} \left\{ \frac{(adzq^{n-1})_\infty}{(aq^{n-1})_\infty (az)_\infty (dq^{n-1})_\infty (dz)_\infty} \right\} \\
= \frac{(q)_\infty(q/az)_\infty(b/az)_\infty(c/a)_\infty(bc/azq)_\infty}{(z)_\infty(b)_\infty(c)_\infty(b/az)_\infty(c/az)_\infty(q/a)_\infty(a)_\infty(d)_\infty}. 
\]

Multiplying throughout by \( \frac{(a)_\infty(d)_\infty}{(adz)_\infty} \), we obtain

\[
\sum_{n=0}^{\infty} \frac{(a)_n(bc/azq)_n(d)_n}{(b)_n(c)_n(adz)_n} z^n \\
+ \sum_{n=1}^{\infty} \frac{(a)_{-n}(bc/azq)_{-n}(d)_{-n}}{(b)_{-n}(c)_{-n}(adz)_{-n}} z^{-n} \\
= \frac{(az)_\infty(q)_\infty(q/az)_\infty(b/az)_\infty(c/a)_\infty(bc/azq)_\infty(dz)_\infty}{(z)_\infty(b)_\infty(c)_\infty(b/az)_\infty(c/az)_\infty(q/a)_\infty(adz)_\infty}, 
\]

which yields (6.2.3).

**Theorem 6.2.4.** We have

\[
\psi_3 \left( \begin{array}{ccc}
  a, & c, & d \\
  b, & acz, & adz
  \end{array} ; q, z \right) = \frac{(q)_\infty(q/az)_\infty(b/az)_\infty(cz)_\infty(acdz)_\infty(az)_\infty(dz)_\infty}{(z)_\infty(b)_\infty(q/a)_\infty(b/az)_\infty(acz)_\infty(cdz)_\infty(adz)_\infty}. 
\]  
(6.2.4)

**Proof:** Equation (6.1.5) can be written as

\[
\sum_{n=0}^{\infty} \frac{(a)_n(c)_n}{(b)_n(acz)_n} z^n + \sum_{n=1}^{\infty} \frac{(q/b)_n(q/acz)_n b^n}{(q/a)_n(q/c)_n} = \frac{(q)_\infty(b)_\infty(q/az)_\infty(cz)_\infty}{(z)_\infty(b)_\infty(q/a)_\infty(b/az)_\infty(acz)_\infty}. 
\]
which on using (1.2.7) and (1.2.8) becomes

\[
\sum_{n=0}^{\infty} \frac{(c)_n z^n}{(b)_n (acz)_n} \left\{ \frac{1}{(aq^n)_\infty (az)_\infty} \right\} + \sum_{n=1}^{\infty} \frac{(q/b)_n (q/ac)_n (ab)_n}{(q/c)_n (-1)^n q^n (n+1)/2} \left\{ \frac{1}{(aq)_\infty (az)_\infty} \right\} \\
= \frac{(q)_\infty (q/az)_\infty (b/a)_\infty (cz)_\infty}{(z)_\infty (b)_\infty (q/a)_\infty (b/az)_\infty (acz)_\infty (a)_\infty}.
\]

Applying the operator \(T(dD_q)\) on both sides and using the identities (6.1.8) and (6.1.9), we obtain

\[
\sum_{n=0}^{\infty} \frac{(c)_n z^n}{(b)_n (acz)_n} \left\{ \frac{(adzq^n)_\infty}{(aq^n)_\infty (az)_\infty (dq^n)_\infty (dz)_\infty} \right\} + \sum_{n=1}^{\infty} \frac{(q/b)_n (q/ac)_n (ab)_n}{(q/c)_n (-1)^n q^n (n+1)/2} \left\{ \frac{(adzq^{-n})_\infty}{(aq^{-n})_\infty (az)_\infty (dq^{-n})_\infty (dz)_\infty} \right\} \\
= \frac{(q)_\infty (q/az)_\infty (b/a)_\infty (cz)_\infty (acd)_\infty}{(z)_\infty (b)_\infty (q/a)_\infty (b/az)_\infty (acz)_\infty (a)_\infty (dcz)_\infty (d)_\infty}.
\]

Multiplying throughout by \(\frac{(a)_\infty (d)_\infty}{(adz)_\infty}\), we obtain

\[
\sum_{n=0}^{\infty} \frac{(a)_n (c)_n (d)_n}{(b)_n (acz)_n (adz)_n} z^n + \sum_{n=1}^{\infty} \frac{(a)_{-n} (c)_{-n} (d)_{-n}}{(b)_{-n} (acz)_{-n} (adz)_{-n}} z^{-n} \\
= \frac{(q)_\infty (q/az)_\infty (b/a)_\infty (cz)_\infty (acd)_\infty (az)_\infty (dz)_\infty}{(z)_\infty (b)_\infty (q/a)_\infty (b/az)_\infty (acz)_\infty (cdz)_\infty (adz)_\infty}.
\]

which gives (6.2.4).

**Theorem 6.2.5.** We have
\[ \begin{pmatrix} a, b, d & ; q, q/a \\ c, bq, bdq/a & \end{pmatrix} = (q)_{\infty}(q)_{\infty}(c/b)_{\infty}(bq/a)_{\infty}(dq/a)_{\infty} \]  
\[ \frac{(q/a)_{\infty}(q/b)_{\infty}(c)_{\infty}(bq)_{\infty}(dq)_{\infty}}{(q/a)_{\infty}(q/b)_{\infty}(c)_{\infty}(bq)_{\infty}(dq)_{\infty}}. \quad (6.2.5) \]

Proof: On taking \( d = bq \) and \( z = q/a \) in Bailey’s \( 2\psi_2 \) transformation formula \((6.1.4)\), we can easily obtained

\[ 2\psi_2 \begin{pmatrix} a, b & ; q, q/a \\ c, bq & \end{pmatrix} = \frac{(q,q,bq/a,c/b)_{\infty}}{(q/a,bq,q/b,c)_{\infty}}. \quad (6.2.6) \]

Equation (6.2.6) can be written as

\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(bq)_n} (q/a)^n + \sum_{n=1}^{\infty} \frac{(1/b)_n(q/c)_n}{(q/a)_n(q/b)_n} c^n = \frac{(q,q,bq/a,c/b)_{\infty}}{(q/a,bq,q/b,c)_{\infty}}.
\]

which on using \((1.2.7)\) and \((1.2.8)\) becomes

\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(bq)_n} (q/a)^n \frac{1}{(bq^n)_{\infty}(bq/a)_{\infty}} + \sum_{n=1}^{\infty} \frac{(q/c)_n(1/b)_n(b)_n}{(q/a)_n} \frac{1}{(q/a)_n(1)^n q^{n+1}/2 (bq)_{\infty}^{-n}(bq/a)_{\infty}} 
\]

\[
= \frac{(q,q,c/b)_{\infty}}{(q/a,q/b,c,bq)_{\infty}}.
\]

Applying the operator \( T(dD_q) \) with respect to \( b \) and using the identities \((6.1.8)\) and \((6.1.9)\), we obtain

\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(bq)_n} (q/a)^n \frac{1}{(bq^n)_{\infty}(bq/a)_{\infty}} = \frac{(bdq^n+1/a)_{\infty}}{(bq^n)_{\infty}(bq/a)_{\infty}(dq^n)_{\infty}(dq/a)_{\infty}}.
\]
which on simplification gives (6.2.5).

**Theorem 6.2.6.** We have

\[
3ψ_3\left( \begin{array}{c} a, b, \frac{cde}{abz}q ; q, z \\ c, d, e \end{array} \right) = \frac{(az)∞(d/a)∞(dq/abz)∞(e/b)∞(cde/abzq)∞}{(z)∞(d)∞(q/a)∞(ce/bq)∞(cd/abz)∞(ed/abz)∞} \]

\[
3ψ_3\left( \begin{array}{c} a, \frac{abz}{d}, \frac{ce/bq}{q/a} ; q, \frac{d}{a} \\ c, az, e \end{array} \right) \quad (6.2.7)
\]

**Proof:** Equation (6.1.4) can be written as

\[
\sum_{n=0}^∞ \frac{(a)n(b)n}{(c)n(d)n}z^n + \sum_{n=1}^∞ \frac{(q/c)n(q/d)n}{(q/a)n(q/b)}(cd/abz)^n
\]

\[
= \frac{(az)∞(c/b)∞(d/a)∞(dq/abz)∞}{(z)∞(d)∞(q/a)∞(cd/abz)∞} \left\{ \sum_{n=0}^∞ \frac{(a)n(abz/d)n}{(c)n(az)n}(d/a)^n \right. \\
+ \left. \sum_{n=1}^∞ \frac{(q/c)n(q/az)n}{(q/a)n(qd/abz)n}(c/b)^n \right\}.
\]

which on using (1.2.7) and (1.2.8) becomes

\[
\sum_{n=0}^∞ \frac{(a)n(b)n}{(d)n(c)∞}z^n \left\{ (cq^n)∞(cd/abz)∞ \right\}
\]

\[
+ \sum_{n=1}^∞ \frac{(q/d)n(-1)^nq^{n(n+1)/2}(d/abz)^n}{(q/a)n(q/b)n(c)∞} \left\{ (cq^{-n})∞(cd/abz)∞ \right\}
\]

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which after simplification gives identities (6.1.10) and (6.1.11), we obtain

\[
\sum_{n=0}^{\infty} \frac{(a)_{n} (b)_{n} z^{n}}{(d)_{n} (q/a)_{\infty}} \left\{ \frac{(cq^{n})_{\infty} (cd/abz)_{\infty}}{(cdeq^{n}/qabz)_{\infty}} \right\} (d/abz)_{\infty} \sum_{n=0}^{\infty} \frac{(a)_{n} (az/d)_{n} (d/a)_{n}^{n} (cq^{n})_{\infty} (c/b)_{\infty}}{(c)_{\infty} (az)_{n} (c)_{n}^{n}} (e^{\theta})_{\infty} (eq^{n})_{\infty} (e/\theta)_{\infty} \\
+ \sum_{n=1}^{\infty} \frac{(q/az)_{n} (-1)^{n} q^{n(n+1)/2} (1/b)^{n}}{(q/a)_{n} (q/b)_{n} (c)_{\infty}} \left\{ \frac{(cq^{n})_{\infty} (cd/abz)_{\infty}}{(cdeq^{n}/qabz)_{\infty}} \right\} (c/d)_{\infty} (d/abz)_{\infty} (e/\theta)_{\infty} \\
= \frac{(az)_{\infty} (d/a)_{\infty}}{(z)_{\infty} (d)_{\infty} (q/a)_{\infty}} \left\{ \sum_{n=0}^{\infty} \frac{(a)_{n} (az/d)_{n} (d/a)_{n}^{n} (cq^{n})_{\infty} (c/b)_{\infty}}{(az)_{n} (c)_{\infty} (c/a)_{\infty} (eq^{n})_{\infty} (e/\theta)_{\infty}} \\
+ \sum_{n=1}^{\infty} \frac{(q/az)_{n} (-1)^{n} q^{n(n+1)/2} (1/b)^{n}}{(q/a)_{n} (q/b)_{n} (c)_{\infty}} \left\{ \frac{(cq^{n})_{\infty} (c/b)_{\infty}}{(ceq^{n}/bq)_{\infty}} \right\} (ceq^{n}/bq)_{\infty} \right\}.
\]

which after simplification gives

\[
\frac{(e)_{\infty} (cd/abz)_{\infty} (ed/abz)_{\infty}}{(cde/abqz)_{\infty}} \left\{ \sum_{n=0}^{\infty} \frac{(a)_{n} (b)_{n} (ced/abqz)_{n} z^{n}}{(e)_{n} (d)_{n} (e)_{n}} \\
+ \sum_{n=1}^{\infty} \frac{(a)_{n} (b)_{n} (ced/abqz)_{n} z^{n}}{(e)_{n} (d)_{n} (e)_{n}} \right\} \\
= \frac{(az)_{\infty} (d/a)_{\infty}}{(z)_{\infty} (d)_{\infty} (q/a)_{\infty}} \left\{ \sum_{n=0}^{\infty} \frac{(a)_{n} (az/d)_{n} (ce/bq)_{n} (d/a)_{n}}{(c)_{n} (az)_{n} (e)_{n} (c/a)_{\infty} (eq^{n})_{\infty} (e/\theta)_{\infty}} \\
+ \sum_{n=1}^{\infty} \frac{(a)_{n} (az/d)_{n} (ce/bq)_{n} (d/a)_{n}}{(c)_{n} (az)_{n} (e)_{n} (c/a)_{\infty} (eq^{n})_{\infty} (e/\theta)_{\infty}} \right\}.
\]

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which yields (6.2.7).

**Theorem 6.2.7.** We have

\[
\begin{align*}
3\psi_3 \left( \begin{array}{cccc}
a, & b, & de/q ; q, & z \\
c, & d, & e \\
\end{array} \right) \\
= \frac{(az)_\infty (d/a)_\infty (dq/abz)_\infty (c/b)_\infty (e/a)_\infty (eq/abz)_\infty (de/q)_\infty}{(z)_\infty (q/a)_\infty (cd/abz)_\infty (de/abz)_\infty (de/2abz)_\infty (e)_\infty (c)_\infty (d)_\infty} \cdot 2\psi_2 \left( \begin{array}{cccc}
a, & abz/d ; q, & d/a \\
c, & az \\
\end{array} \right).
\end{align*}
\]

(6.2.8)

**Proof:** Equation (6.1.4) can be written as

\[
\sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n(d)_n} z^n + \sum_{n=1}^\infty \frac{(q/c)_n(q/d)_n}{(q/a)_n(q/b)} (cd/abz)^n (d/a)^n (dq/abz)_\infty
\]

\[
= \frac{(az)_\infty (c/b)_\infty (d/a)_\infty (dq/abz)_\infty}{(z)_\infty (d)_\infty (q/a)_\infty (cd/abz)_\infty} \left\{ \sum_{n=0}^\infty \frac{(a)_n(abz/d)_n}{(c)_n(az)_n} (d/a)^n + \sum_{n=1}^\infty \frac{(q/c)_n(q/az)_n}{(q/a)_n(qd/abz)} (c/b)^n \right\}.
\]

which on using (1.2.7) and (1.2.8) becomes

\[
\sum_{n=0}^\infty \frac{(a)_n(b)_nz^n}{(c)_n(d)_n} (dq^n)_\infty (d)_\infty \cdot \sum_{n=1}^\infty \frac{(q/c)_n(-1)^nq^{n+1/2}(c/abz)_n}{(q/a)_n(q/b)_n(d)_n} (dq^{-n})_\infty (d)_\infty
\]

\[
= \frac{(az)_\infty (c/b)_\infty}{(z)_\infty (cd/abz)_\infty (q/a)_\infty} \left\{ \sum_{n=0}^\infty \frac{(a)_n(abz/d)_n(d/a)_n}{(c)_n(az)_n} (d/a)_\infty (dq/abz)_\infty + \sum_{n=1}^\infty \frac{(q/c)_n(q/az)_n(c/b)_n}{(q/a)_n(qd/abz)} (d/a)_\infty (dq/abz)_\infty \right\}.
\]

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Applying the operator $E(e\theta)$ on both sides with respect to $d$ and using the identities (6.1.10) and (6.1.11), we obtain

\[\sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n(d)_n} \left\{ \frac{(dq^n)_\infty(d)_\infty(eq^n)_\infty(e)_\infty}{(deq^n/q)_\infty} \right\} \]

\[+ \sum_{n=1}^{\infty} \frac{(q/c)_n(-1)^n q^{n(n+1)/2}(c/abz)_n}{(q/a)_n(q/b)_n(d)_n} \left\{ \frac{(dq^{-n})_\infty(d)_\infty(eq^{-n})_\infty(e)_\infty}{(deq^{-n}/q)_\infty} \right\} \]

\[= \frac{(az)_\infty(c/b)_\infty}{(z)_\infty(q/a)_\infty(cd/abz)_\infty} \left\{ \sum_{n=0}^{\infty} \frac{(a)_n(abz/d)_n(d/a)_n}{(az)_n(c)_n} \frac{(dq/abz)_\infty(e/a)_\infty(eq/abz)_\infty}{(de/a^2bz)_\infty} \right\} \]

\[+ \sum_{n=1}^{\infty} \frac{(q/c)_n(q/az)_n(c/b)_n}{(q/a)_n(qd/abz)_n} \frac{(d/a)_\infty(dq/abz)_\infty(e/a)_\infty(eq/abz)_\infty}{(de/a^2bz)_\infty} \}

which after simplification gives

\[\frac{(e)_\infty(e)_\infty(d)_\infty}{(de/q)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(d/e)_n z^n}{(c)_n(d)_n(e)_n} \]

\[+ \sum_{n=1}^{\infty} \frac{(a)^{-n}(b)^{-n}(d/e)_n}{(c)^{-n}(d)_n(e)_n} \]

\[= \frac{(az)_\infty(d/a)_\infty(dq/abz)_\infty(c/b)_\infty(e/a)_\infty(eq/abz)_\infty}{(z)_\infty(q/a)_\infty(cd/abz)_\infty(de/a^2bz)_\infty} \left\{ \sum_{n=0}^{\infty} \frac{(a)_n(abz/d)_n}{(c)_n(az)_n} \frac{(d/a)^n}{(d/e)_n} \right\} \]

\[+ \sum_{n=1}^{\infty} \frac{(a)^{-n}(abz/d)_n}{(c)^{-n}(az)_n} \frac{(d/a)^{-n}}{d/a}_\infty \}

which yields (6.2.8).

**Theorem 6.2.8.** We have

\[3\psi_3 \left( \begin{array}{ccc} a, & b, & e; & q, & z \\ c, & d, & aez \end{array} \right) \]
\[
\sum_{n=0}^{\infty} \frac{(a)_{\infty}(b)_{\infty}}{(c)_{n}(d)_{n}} z^n \left\{ \frac{1}{(aq^n)_{\infty}}(az)_{\infty} \right\} + \sum_{n=1}^{\infty} \frac{(a)_{\infty}(q/c)_{n}(q/d)_{n}(cd/bz)_{n}}{(q/b)_{n}(-1)^nq^n(n+1)/2} \left\{ \frac{1}{(aq^{-n})_{\infty}}(az)_{\infty} \right\}
\]

\[
\sum_{n=0}^{\infty} \frac{(a)_{\infty}(b)_{\infty}}{(c)_{n}(d)_{n}} z^n \left\{ \frac{1}{(aq^n)_{\infty}}(azq^n)_{\infty}(d/a)_{n} \right\} + \sum_{n=1}^{\infty} \frac{(q/c)_{n}(q/az)_{n}(c/b)_{n}}{(q/a)_{n}(qd/abz)_{n}(az)_{\infty}} \left\{ \frac{1}{(aq^n)_{\infty}}(azq^n)_{\infty}(d/a)_{n} \right\}
\]

which on simplification gives (6.2.9).
6.3 Special Cases

If we take \( c = q \) in (6.2.7), then we obtain the following transformation of \( 3\phi_2 \) series

\[
3\phi_2 \left( \begin{array}{ccc}
  a, & b, & de/abz ; q, & z \\
  d, & e
\end{array} \right) = \frac{(az)_\infty(d/a)_\infty(q/b)_\infty}{(z)_\infty(d)_\infty(q/a)_\infty} \quad 3\phi_2 \left( \begin{array}{ccc}
  a, & abz/d, & e/b ; q, & d/a \\
  az, & e
\end{array} \right) .
\] (6.3.1)

where \(|z| < 1, |d/a| < 1\).

If we let \( a \to \infty \) in (6.2.5), we have

\[
\sum_{n=-\infty}^{\infty} \frac{(b)_n(d)_n(-1)^nq^{n(n+1)/2}}{(c)_n(bq)_n} = \frac{(q,q,c/b)_\infty}{(q/b,c,bq,dq)_\infty}.
\]

On taking \( c = q \) and \( d = 0 \), we obtain

\[
\sum_{n=0}^{\infty} \frac{(b)_n(-1)^nq^{n(n-1)/2}}{(q)_n(bq)_n}q^n = \frac{(q)_\infty}{(bq)_\infty}.
\] (6.3.2)

which is the well-known \( 1\phi_1 \) sum.
6.4 Some $q$-Gamma and $q$-Beta Function Identities

1. If $0 < q < 1$, $0 < x, y < 1$, then

$$B_q(x, y) = \frac{\Gamma_q(2x)(1 - q)^x}{\Gamma_q(x + 1)\Gamma_q(2 - x)\Gamma_q(x)} \sum_{n=-\infty}^{\infty} \frac{(q^{-x})_n(q^{y-1})_n(q^{2-1})_n}{(q^x)^2_n(q^{2x+y-1})_n} (q^1+x)_n.$$

(6.4.1)

**Proof.** Putting $a = q^{-x}$, $b = q^{x-1}$, $c = q^x$ and $d = q^{y-1}$ in (6.2.5), we have

$$\sum_{n=-\infty}^{\infty} \frac{(q^{-x})_n(q^{y-1})_n(q^{2-1})_n}{(q^x)^2_n(q^{2x+y-1})_n} (q^1+x)_n = \frac{(q^{2x})_\infty(q)_{\infty}(q)_{\infty}(q)_{\infty}}{(q^1)_{\infty}(q^{2-x})_{\infty}(q^x)_{\infty}(q^y)_{\infty}}.$$

Using (1.2.19), (1.2.20) and (1.2.21), we obtain (6.4.1).

2. If $0 < q < 1$, $0 < x, y < 1$, then

$$B_q^2(x, y)$$

$$= \frac{(1 - q)^{1+2x+y}\Gamma_q(x + 2y)\Gamma_q(1 + x - y)}{(1 - q^x)(1 + y)\Gamma_q(1 + y)} \sum_{n=-\infty}^{\infty} \frac{(q^{1-x})_n(q^y)_n^2}{(q^1+x)_n(q^1+y)_n(q^{x+2y})_n} q^{xn}.$$

(6.4.2)

**Proof.** Putting $a = q^x$, $d = q^{1+x}$, $b = q^y$ and $c = q^y$ in (6.2.1), we get

$$\sum_{n=-\infty}^{\infty} \frac{(q^{1-x})_n(q^y)_n^2}{(q^1+x)_n(q^1+y)_n(q^{x+2y})_n} q^{xn} = \frac{(q^{1+y})_\infty(q^y)_\infty(q^{x+2y})_\infty(q^{x+y})_\infty(q^{y+x})_\infty}{(q^{1-y})_\infty(q^{1+x})_\infty(q^x)_\infty(q^y)_\infty(q^{1+y})_\infty(q^{1+y})_\infty}.$$
Using (1.2.19), (1.2.20) and (1.2.21), we obtain (6.4.2).

3. If $0 < q < 1$, $0 < x, y < 1$, then

$$B_q^2(x, y) = \frac{(1 - q)^{1+x}}{\Gamma_q(2 - x)(1 - q^y)} \sum_{n=-\infty}^{\infty} \frac{(q^{1-x})_n^2(q^y)_n}{(q^{1+y})_n^2(q^{2-x})_n} q^{xn}. \quad (6.4.3)$$

**Proof.** Putting $a = q^x$, $d = q^{y+1}$, $b = q^{1-x}$ and $c = q^y$ in (6.2.1), we get

$$\sum_{n=-\infty}^{\infty} \frac{(q^{1-x})_n^2(q^y)_n}{(q^{1+y})_n^2(q^{2-x})_n} q^{xn} = \frac{(q^{x+y})_{\infty}(q^y)^2(q^x)^{\infty}(q^{x+y})_{\infty}}{(q^{x+y})_{\infty}(q^{y+2})_{\infty}(q^{x+y+2})_{\infty}(q^y)^{\infty}}.$$

Using (1.2.19), (1.2.20) and (1.2.21), we obtain (6.4.3).

### 6.5 Eta-function Identities

In this section, we derive some new interesting eta-function identities.

Choosing $a = q^{1/2}$, $b = q^{1/2}$, $c = q^2$ and $d = q^{3/2}$ in (6.2.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(q^{1/2})_n(q^{1/2})_n(q^2)_n}{(q^{3/2})_n(q^{3/2})_n(q^3)_n} q^{n/2} = \frac{(q^{1/2})_{\infty}(q^{5/2})_{\infty}}{(q^{1/2})_{\infty}(q^{3/2})_{\infty}(q^2)_{\infty}}.$$

A change of base $q$ to $q^2$ gives

$$\sum_{n=-\infty}^{\infty} \frac{(q; q^2)_n^2(q^4; q^2)_n}{(q^3; q^2)_n^2(q^6; q^2)_n} q^{n} = \frac{(1 - q^2)(q^2)^3}{(1 - q^3)(q^2)^{\infty}}.$$

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After some simplification and using (1.2.22), we get (6.5.1).

$$\frac{\eta^3(2\tau)}{\eta(\tau)} = \frac{q^{5/24}(1 - q^3)(1 - q)(1 - q^4)(q^3; q^2)\infty(q; q^2)\infty}{(1 + q)(q; q)\infty} \sum_{n=-\infty}^{\infty} q^n. \quad (6.5.1)$$

Similarly, putting $a = q^{1/2}$, $b = q^{1/2}$, $c = q$ and $d = q$ in (6.2.5), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(q^{1/2})_n(q^{1/2})_n}{(q^{3/2})_n(q^2)_n} q^{n/2} = \frac{(q)\infty(q)\infty}{(q^{1/2})\infty(q^2)\infty}. \quad (6.5.2)$$

then changing $q$ to $q^2$ gives

$$\sum_{n=-\infty}^{\infty} \frac{(q; q^2)_n(q; q^2)_n}{(q^3; q^2)_n(q^4; q^2)_n} q^n = \frac{(q^2; q^2)\infty(q^2; q^2)\infty}{(q; q^2)\infty(q^4; q^2)\infty}. \quad (6.5.2)$$

which after simplification with use of (1.2.22) yields (6.5.2).

$$\frac{\eta^2(2\tau)}{\eta(\tau)} = \frac{q^{1/8}}{(1 + q)} \sum_{n=-\infty}^{\infty} \frac{(q; q^2)_n}{(q^4; q^2)_n} q^n. \quad (6.5.2)$$