ON THE DISTRIBUTION MODULO 1 OF AN ARITHMETIC FUNCTION

5.1. Introduction. To every real number \( r \), there corresponds a uniquely determined non-negative real number \( r_{\text{fr}} \), called its fractional part, given by \( r - [r] \), where \([r]\) denotes the largest integer which does not exceed \( r \). We can extend the idea of the fractional part of a real number to cover real-valued arithmetic functions as follows:

Let \( \alpha \) be any real-valued arithmetic function and let \( \alpha_{\text{fr}}(n) \) denote the fractional part of \( \alpha(n) \) for every positive integer \( n \). This defines the arithmetic function \( \alpha_{\text{fr}} \), which is uniquely determined by \( \alpha \). We call \( \alpha_{\text{fr}} \) the fractional part function of the arithmetic function \( \alpha \).

In this chapter we study the distribution modulo 1 of real-valued arithmetic functions \( \alpha \) such that \( \alpha(n) - \alpha(n-1) \to 0 \); that is, we study the distribution of the values \( \alpha_{\text{fr}}(n) \) in relation to the convergent or divergent or the oscillatory behaviour of the function \( \alpha \). In fact, we establish connections between the behaviour of the function \( \alpha \) and the derived set of the range of the fractional part function \( \alpha_{\text{fr}} \).
We denote the range of $\alpha_f$ by $F$ and its derived set by $F'$. Also we denote the integral part of $\alpha(n)$, $[\alpha(n)]$, by $I(n)$, so that we have always

$$\alpha(n) = \alpha_f(n) + I(n),$$

for every positive integer $n$.

Since $0 \leq \alpha_f(n) < 1$ for every $n$, the set $F$ is bounded both below and above. So $F$ is finite if and only if $F'$ is empty.

5.2. *Main results.* We now state our main results. In all the theorems that follow it is assumed that the arithmetic function $\alpha$ is such that $\alpha(n) - \alpha(n-1) \to 0$.

(5.2.1) **Theorem.** The arithmetic function $\alpha$ converges to a limit if and only if $F$ has at most one limit point in $[0,1]$ or $F' \subseteq \{0,1\}$.

This theorem is closely related to

(5.2.2) **Theorem.** The arithmetic function $\alpha$ converges to a limit if and only if $\alpha_f$ converges to a limit or the sequence $(\alpha_f(n))_{n=1}^{\infty}$ has the only subsequential limits $0$ and $1$.

(5.2.3) **Theorem.** (a) Let the arithmetic function $\alpha$ be non-oscillatory. Then the function diverges to $+\infty$ or $-\infty$ if and only if $F' = [0,1]$.

(b) If the function is of infinite oscillation then also $F' = [0,1]$. 
(5.2.4) **Theorem.** If the arithmetic function $\alpha$ is oscillatory, then $F'$ is a closed sub-interval of $[0,1]$ or $[0,1] = (a,b)$, where $(a,b) \subseteq (0,1)$.

(5.2.5) **Note.** $F'$ may be $[0,1]$ or may take any of the following forms:

$$[a,b], [0,a] \cup [b,1], [0,a] \cup \{1\}, \{0\} \cup [b,1].$$

where $0 \leq a < b \leq 1$.

### 5.3. **Proof of Theorems 5.2.1 and 5.2.2.**

For convenience we split into two lemmas the first half of the theorem, namely: if $\alpha$ converges to a limit, then $F$ has at most one limit point in $[0,1]$ or $F' \subseteq \{0,1\}$.

(5.3.1) **Lemma.** If the function $\alpha$ converges to a non-integral value, $s$, then $\alpha_f$ converges to $s_f$, the fractional part of $s$.

**Proof.** Let $[s]$ be the integral part of $s$. Since $s$ is not an integer, $0 < s_f < 1$ and we can choose a positive number $\varepsilon < \min(s_f, 1-s_f)$, so that

$$0 < s_f - \varepsilon < s_f + \varepsilon < 1.$$

Since $\alpha(n) \rightarrow s$ as $n \rightarrow \infty$, we have, corresponding to $\varepsilon$, a positive integer $N$ such that $n > N$ implies

$$s - \varepsilon < \alpha(n) < s + \varepsilon,$$

that is,
Hence, from (5.3.2) we have

$$[s] < \alpha(n) < [s] + 1.$$ 

Therefore, the integral part of \(\alpha(n)\) is \([s]\). So, from (5.3.3) we have

$$s^\ast - \varepsilon < \alpha^\ast(n) < s^\ast + \varepsilon$$

showing that \(\alpha^\ast(n) \to s^\ast\) as \(n \to \infty\). This proves the lemma.

(5.3.4) **Remark.** Since \(\alpha^\ast\) converges to a limit, its range, \(F^\ast\), can have at most one limit point. So \(F^\ast\) is either null or a single-element set. But if \(F\) is an infinite set, then \(F^\ast\) must be a single-element set.

(5.3.5) **Lemma.** If the function \(\alpha\) converges to an integer, then \(F\) has no limit point in \((0,1)\); that is \(F^\ast \subseteq \{0,1\}\).

**Proof.** Let the function converge to an integer \(J\). If \(F\) is finite \(F^\ast\) is null. Assume then, that \(F\) is infinite, so that it has a limit point. Let, if possible, \(k\) be a limit point of \(F\) in the interval \((0,1)\). We can choose a positive

$$\varepsilon < \min (k,1 - k)$$

and also a positive

$$\delta < \min (k - \varepsilon, 1 - k - \varepsilon),$$

so that

$$0 < \delta < k - \varepsilon < k < k + \varepsilon < 1 - \delta < 1.$$
Since $\alpha(n) \to J$, there corresponds to this $\delta$ a positive integer $N$ such that, for all $n > N$,

$$J - \delta < \alpha(n) < J + \delta.$$ 

Also, since $k$ is a limit point of $F$, $\alpha_f(n)$ will lie in the interval $(k-\varepsilon, k+\varepsilon)$ for infinitely many $n$ and so we can find a positive integer $m > N$ such that $\alpha_f(m)$ belongs to this interval. Now $m > N$ implies

$$J - \delta < \alpha(m) < J + \delta,$$

that is,

$$J - \delta < I(m) + \alpha_f(m) < J + \delta,$$

and so

$$c - \delta < \alpha_f(m) < c + \delta,$$

where $c = J - I(m)$ is an integer. But

$$0 \leq \alpha_f(m) < 1$$

and $0 < \delta < 1$

and so we have

$$-1 < c < 2,$$

so that $c = 0$ or $1$. If $c = 0$, we have

$$0 \leq \alpha_f(m) < \delta < k - \varepsilon,$$

and if $c = 1$,

$$k + \varepsilon < 1 - \delta < \alpha_f(m) < 1.$$ 

In either case we have the contradiction that $\alpha_f(m)$ is not a member of $(k - \varepsilon, k + \varepsilon)$. Hence $F$ has no limit point in $(0,1)$; and this means that its limit point (or limit points) lie outside $(0,1)$. But since any limit point of $F$ must lie in $[0,1]$, we
conclude that \( F' \subseteq \{0, 1\} \). If \( F \) is finite \( F' = \emptyset \subseteq \{0, 1\} \).

So, if the function \( \alpha \) converges to an integer, then \( F' \subseteq \{0, 1\} \).

We have now completed the proof of the first half of Theorem 5.2.1.

Following exactly the same arguments we can prove that the sequence \((\alpha_F(n))\) has no subsequential limits in \((0, 1)\), for we could have taken \( k \) to be a possible subsequential limit of \((\alpha_F(n))\) and arrived at the same contradiction. Therefore, if \((\alpha(n))\) tends to an integer, then \((\alpha_F(n))\) either tends to 0 or 1, or has the only subsequential limits 0 and 1. Combining this with Lemma 5.3.1 we conclude that if the arithmetic function \( \alpha \) converges to a limit, then \( \alpha_F \) tends to a limit or the sequence \((\alpha_F(n))\) has the only subsequential limits 0 and 1, which is exactly the necessity part of Theorem 5.2.2.

Thus we have incidentally proved the first part of Theorem 5.2.2 also.

We now prove the other half of Theorem 5.2.1; namely:

if \( F' \) is null or a single-element set or a (non-empty) subset of \( \{0, 1\} \), then the function \( \alpha \) converges to a limit.

Case (i). Let \( F' = \emptyset \). Then \( F \) is a finite set.

If \( F \) contains only one element \( g \), then \( \alpha_F(n) = g \) for every \( n \).
and \( \alpha(n) - \alpha(n-1) \to 0 \) implies that \( I(n) - I(n-1) \to 0 \).

Therefore, the sequence of integers \((I(n))\) converges to an integer, say \( J \). Since \( \alpha_x(n) = g \) for all \( n \), it follows that \( \alpha(n) \to J + g \) as \( n \to \infty \).

If \( F \) contains the elements \( g_1, g_2, \ldots, g_k \), let

\[ \delta_1 = \min(|g_i - g_j|), \quad i \neq j, \text{ and } \delta_2 = \max(|g_i - g_j|). \]

Since \( 0 \leq g_i < 1 \) for \( 1 \leq i \leq k \), \( 0 < |g_i - g_j| < 1 \) for unequal \( i \) and \( j \) so that \( 0 < \delta_1, \delta_2 < 1 \). Choose now any positive number \( \epsilon < \min(\delta_1, 1 - \delta_2) \).

Since \( \alpha(n) = \alpha(n-1) \to 0 \), we have, corresponding to \( \epsilon \), a positive integer \( N \) such that \( n > N \) implies

\[ |\alpha(n) - \alpha(n-1)| < \epsilon, \]

that is,

\[ |I(n) + \alpha_x(n) - (I(n-1) + \alpha_x(n-1))| < \epsilon, \]

that is,

\[ (5.3.6) \quad \alpha_x(n-1) - \alpha_x(n) - \epsilon < I(n) - I(n-1) < \alpha_x(n-1) - \alpha_x(n) + \epsilon. \]

Now, since \( \alpha_x(n) \) for every \( n \) is a \( g_i \) for some \( i \), we have

\[ s(\alpha_x(n) - \alpha_x(n-1)) \leq |\alpha_x(n) - \alpha_x(n-1)| \]

\[ = |g_i - g_j| \]

for some \( i \) and \( j \)

\[ \leq \delta_2. \]

and so from (5.3.6) we must have

\[ -\delta_2 - \epsilon < I(n) - I(n-1) < \delta_2 + \epsilon. \]
Since $\epsilon < 1 - \delta_2$, the above inequality gives

$$-1 < I(n) - I(n-1) < 1.$$ 

Therefore, $I(n) = I(n-1)$ and hence

$$|\alpha_f(n) - \alpha_f(n-1)| < \epsilon < \delta_1$$

for all $n > N$. This inequality, by definition of $\delta_1$, must imply that $\alpha_f(n) = \alpha_f(n-1)$ for $n > N$. Hence, for $n > N$,

$$\alpha(n) = I(n) + \alpha_f(n) = I(n) + \alpha_f(n) = \alpha(n),$$

and the function $\alpha$ converges to $\alpha(N)$.

**Case (ii).** Let $F'$ contain the lone element $\xi$, $0 < \xi < 1$. (In case (iii) we take $\xi$ to be 0 or 1.) Since $F' \neq \emptyset$, $F$ is an infinite set. We first prove that $\alpha_f(n) \to \xi$ by proving that every subsequence of $(\alpha_f(n))$ converges to $\xi$ only. Later we prove that the convergence of $\alpha_f$, under the condition $\alpha(n) - \alpha(n-1) \to 0$, implies the convergence of the function $\alpha$.

Since $\xi$ is a limit point of the range of $(\alpha_f(n))$, some subsequence of it must tend to $\xi$. If $\alpha_f(n)$ does not tend to $\xi$, then some subsequence of $(\alpha_f(n))$ must tend to a limit, say $g_1$, different from $\xi$. Now $\xi$, $1 - \xi$ and $|\xi - g_1|$ are all positive numbers and so we can choose a positive number

$$\delta < \min \left(\xi, 1 - \xi, |\xi - g_1|\right),$$

Clearly, the interval $(\xi - \delta, \xi + \delta)$ lies completely within the
interval $[0,1]$ and does not contain $g_1$. Since $\xi$ is the only limit point of $F$, there can be only finitely many members of it outside the interval $(\xi - \delta, \xi + \delta)$; and it is possible that some or all of these are the limits of some subsequences of $(\alpha_x(n))$. Suppose that the subsequential limits lying outside the interval $[\xi - \delta, \xi + \delta]$ are $g_1, \ldots, g_k$. (Note: The possibility that one of $\xi - \delta$ and $\xi + \delta$ or both of them are subsequential limits of $(\alpha_x(n))$ cannot be ignored.) Since $g_1, \ldots, g_k$ are not the limit points of $F$, but are subsequential limits of $(\alpha_x(n))$, it is clear that each of $g_1, \ldots, g_k$ must repeat itself infinitely many times in the sequence $(\alpha_x(n))$.

Again, outside the interval $(\xi - \delta, \xi + \delta)$ there are at most finitely many members of the sequence $(\alpha_x(n))$ that are not subsequential limits of it. Therefore, beyond a certain stage, every member of $(\alpha_x(n))$ must be either one of $g_1, \ldots, g_k$ or a member of $[\xi - \delta, \xi + \delta]$. Thus we can find a positive integer $N_1$ such that $n > N_1$ implies

$$\alpha_x(n) \in \{g_1, \ldots, g_k\} \cup [\xi - \delta, \xi + \delta].$$

Let $\delta$ be the least of the positive differences between any two members of the set

$$\{g_1, \ldots, g_k, \xi - \delta, \xi + \delta\}$$
and \( \delta_2 \) the greatest of the same (that is, greatest of the positive differences). Since \( \delta_1 \) and \( \delta_2 \) are positive numbers each less than 1, we can choose a positive number

\[ \varepsilon < \min(\delta_1, 1 - \delta_2). \]

Corresponding to this \( \varepsilon \), we have a positive integer \( N_2 \) such that for all \( n > N_2 \), the inequality

\[ (5.3.8) \alpha_x(n-1) - \alpha_x(n) - \varepsilon < I(n) - I(n-1) < \alpha_x(n-1) - \alpha_x(n) + \varepsilon, \]

holds as in case (1). (The above entry and (5.3.6) are same.)

If \( N > \max(N_1, N_2) \), then \( n > N \) must imply both (5.3.7) and (5.3.8). But (5.3.7) clearly implies, when \( n > N \),

\[ |\alpha_x(n) - \alpha_x(n-1)| < \delta_2 \]

and since \( \varepsilon < 1 - \delta_2 \), we must have, from inequality (5.3.8),

\[ -1 < I(n) - I(n-1) < 1. \]

Hence the integer \( I(n) - I(n-1) = 0 \), that is, \( I(n) = I(N) \) for all \( n > N \). Also from (5.3.8) we have

\[ |\alpha_x(n) - \alpha_x(n-1)| < \varepsilon \]

for all \( n > N \).

Since \( g_1 \) occurs infinitely many times in the sequence \( (\alpha_x(n)) \), we can chose a positive integer \( m > N \) such that \( \alpha_x(m) = g_1 \). Hence

\[ |g_1 - \alpha_x(m-1)| = |\alpha_x(m) - \alpha_x(m-1)| < \varepsilon < \delta_1. \]
If \( g_1 \neq \alpha_f(m-1) \), then, in view of (5.3.7),
\[
\alpha_f(m-1) \in \left[ \xi - \delta, \xi + \delta \right]
\]
or \( \alpha_f(m-1) \) is one of \( g_2, \ldots, g_k \), since \( m-1 \geq N \geq N_1 \).
In either case, by definition of \( S_1 \), we have
\[
| g_1 - \alpha_f(m-1) | \geq \delta_1,
\]
which is not the case. Therefore \( g_1 = \alpha_f(m) = \alpha_f(m-1) \).
In the same way, we have also, \( g_1 = \alpha_f(m) = \alpha_f(m+1) \)
and so we have
\[
g_1 = \alpha_f(N) = \alpha_f(N+1) = \alpha_f(N+2) = \ldots.
\]
But then \( F \) is a finite set, that is \( F' = \emptyset \), which is a contradiction. Hence \( \alpha_f(n) \to \xi \).

We have shown already that \( I(n) = I(N) \) for \( n > N_0 \)
that is, \((I(n))\) converges to \( I(N) \). Since \( \alpha_f(n) \to \xi \), we conclude
that \( \alpha(n) \to I(N) + \xi \). Thus we have proved that if \( F' = \{ \xi \} \)
\( 0 < \xi < 1 \), then the function \( \alpha \) converges to a limit.

**Case (iii).** Let \( F' \) be a non-empty subset of \( \{0, 1\} \).
Clearly \( F \) is an infinite set.

Let \( 0 < k < 1 \). Define an arithmetic function \( \beta \) by
the relation: \( \beta(n) = \alpha(n) + k \) for every \( n \). We denote the range
of \((G_f(n))\) \( G_f(n) = \text{fractional part of } \alpha(n) + k \), that is of
\( \alpha_f(n) + k \) by \( G \) and its derived set by \( G' \).
We prove that if \( P' = \{0\} \) or \( \{1\} \) or \( \{0, 1\} \), then 
\[ G' = \{k\}, \]
which by case (ii) implies the convergence of the function \( \beta \) and hence of \( \alpha \).

(a) First, let \( P' = \{0\} \). Choose a positive number 
\( \varepsilon < k \) so that 
\[ 0 < \varepsilon, k - \varepsilon < k - k + \varepsilon < 1. \]

Now, if \( \beta^1(i) \notin (k, k+\varepsilon) \), then \( \beta^1(i) < k \) or \( \beta^1(i) > k + \varepsilon \); and since one of \( \alpha^1(i) + k \) and \( \alpha^1(i) + k - \varepsilon \) that lies in 
\[ [0, 1) \] is \( \beta^1(i) \) we have \( \alpha^1(i) \in (\varepsilon, 1) \). But since 0 is the only limit point of \( F \), at most finitely many members of \( F \) lie in 
\( (\varepsilon, 1) \). Therefore, at most finitely many members of \( G \) lie 
outside \( (k, k+\varepsilon) \), that is \( k \) is the only limit point of \( G \).

So \( G' = \{k\} \).

(b) If \( P' = \{1\} \), only finitely many members of \( F \) lie 
in the interval \( (0, 1-\varepsilon) \) and \( \beta^1(i) \notin (k-\varepsilon, k) \) implies 
\( \alpha^1(i) \in (0, 1-\varepsilon) \). Hence at most finitely many members of \( G \) lie 
outside \( (k-\varepsilon, k) \), so that \( k \) is the only limit point of \( G \), 
that is, \( G' = \{k\} \).

(c) If \( P' = \{0, 1\} \), the interval \( (\varepsilon, 1-\varepsilon) \) will contain 
only a finite number of members of \( F \). Also, \( \beta^1(i) \notin (k-\varepsilon, k+\varepsilon) \) 
implies \( \alpha^1(i) \notin (\varepsilon, 1-\varepsilon) \). So, again, \( k \) is the only limit point 
of \( G \) and \( G' = \{k\} \).
Thus we have shown that if \( \emptyset \neq F' \subseteq \{0, 1\} \), then the function \( \alpha \) converges to a limit. With this we have completed the proof of Theorem 5.2.1.

We now prove the other half of Theorem 5.2.2.

If \( \alpha_f(n) \) tends to a limit, then \( \alpha_f(n) - \alpha_f(n-1) \to 0 \) and this together with the condition \( \alpha(n) - \alpha(n-1) \to 0 \) readily implies \( I(n) - I(n-1) \to 0 \). Therefore, the sequence of integers \( (I(n)) \) converges. Since the sequence \( (\alpha_f(n)) \) also is convergent, we conclude that the function \( \alpha \) converges to a limit.

If \( (\alpha_f(n)) \) has the only subsequential limits 0 and 1, we can prove practically on the same lines as in subcase (c) above, that only finitely many members of the sequence \( (\beta_f(n)) \) lie outside the interval \( (k-\epsilon, k+\epsilon) \). So the function \( \beta_f \) and hence \( \alpha_f \) converges to a limit. As already proved, the convergence of \( \alpha_f \) implies that of \( \alpha \) itself, hence \( \alpha \).

The proof of Theorem 5.2.2 is now complete.

(5.3.9) Remark. If \( \emptyset \neq F' \subseteq \{0, 1\} \), then the function \( \alpha \) must converge to an integer. For if \( \alpha \) converges to a non-integral value \( s \), then, by Lemma 5.3.1, \( \alpha_f \) converges to \( s_f \), the fractional part of \( s \), \( 0 < s_f < 1 \). Therefore, either \( F' \) is empty or \( s_f \in F' \), which is contrary to our assumption. Hence \( \alpha \) converges to an integer.
In the following remarks we indicate the manner of convergence of the function \( \alpha \) to an integer in each of the cases \( F' = \{0\}, F' = \{1\} \) and \( F' = \{0,1\} \). (We need these remarks in course of the proof of Theorem 5.2.4.)

**Remarks.** Let \( \alpha(n) \) tend to an integer \( I \) and \( \alpha(n) \neq I \) for infinitely many \( n \). Clearly, there will be subsequences of \( (\alpha(n)) \) tending to \( I \) through values greater than or less than \( I \). If \( (\alpha(m_i))_{i=1}^\infty \) is a subsequence of \( (\alpha(n)) \) tending to \( I \) through values \( > I \), then given a positive \( \varepsilon < 1 \) we must have \( 0 < \alpha(m_i) - I < \varepsilon \) for all sufficiently large \( i \), so that the integral part of \( \alpha(m_i) = I \). So \( 0 < \alpha_{\lfloor}(m_i) < \varepsilon \) for all sufficiently large \( i \), that is \( 0 \in F' \).

If \( (\alpha(m_i)) \) tends to \( I \) through values less than \( I \), then we have \( 0 < I - \alpha(m_i) < \varepsilon \) for all sufficiently large \( i \).

Here the integral part of \( \alpha(m_i) \) is \( I = 1 \), so that

\[
1 - \varepsilon < \alpha_{\lfloor}(m_i) < 1.
\]

Hence \( 1 \in F' \).

**(5.3.10a)** Let \( F' = \{0\} \). Then clearly \( \alpha(n) \neq I \) for infinitely many \( n \). If \( \alpha(n) < I \) for infinitely many \( n \), then a subsequence of \( (\alpha(n)) \) tends to \( I \) through values \( < I \) and so \( 1 \in F' \), which is not the case. So \( \alpha(n) \geq I \) for all sufficiently large \( n \); and
since $\alpha(n) \neq I$ for infinitely many $n$, it follows that a subsequence of $(\alpha(n))$ tends to $I$ through values greater than $I$.

Conversely, if $\alpha(n) \to I$, $\alpha(n) \geq I$ for all sufficiently large $n$, and a subsequence of $(\alpha(n))$ tends to $I$ through values $> I$, then we must have $F' = \{0\}$. For given $\varepsilon > 0$ we can find an $N$ such that $n > N$ implies $0 \leq \alpha(n) - I < \varepsilon$, that is, $0 \leq \alpha_x(n) < \varepsilon$; and in view of a subsequence of $(\alpha(n))$ tending to $I$ through values $> I$, the strict inequality $0 < \alpha_x(n) < \varepsilon$ must hold for infinitely many $n$. Hence $F' = \{0\}$.

(5.3.10b) Suppose $F' = \{1\}$. No subsequence of $(\alpha(n))$ can converge to $I$ through values $> I$, for then $0 \in F'$. So for sufficiently large $n$, $\alpha(n) \leq I$; and since $\alpha(n) \neq I$ for infinitely many $n$ we conclude that a subsequence of $(\alpha(n))$ tends to $I$ through values $< I$.

Conversely, if $\alpha(n) \to I$, $\alpha(n) \leq I$ for all sufficiently large $n$, and a subsequence of $(\alpha(n))$ tends to $I$ through values $< I$, then $F' = \{1\}$. This is so because $F'$ is non-empty and is a subset of $\{0, 1\}$, and $0 \notin F'$ since, in this case, there is no subsequence of $(\alpha(n))$ tending to $I$ through values $> I$.

(5.3.10c) If $F' = \{0, 1\}$, then $\alpha(n) \neq I$ for infinitely many $n$ as in other cases. Also, if $\alpha(n) \geq I$ for all sufficiently
large $n$, then $F' = \{0\}$; if $I > \alpha(n)$ for all sufficiently large $n$, then $F' = \{1\}$. Both of these, therefore, are impossible.

Hence $\alpha(n) > I$ and $I > \alpha(n)$ for infinitely many $n$, so that $(\alpha(n))$ has subsequences tending to $I$ from either side of it.

Again, if subsequences of $(\alpha(n))$ tending to $I$ from either side exist, then $0$ and $1$ are members of $F'$, that is, $F' = \{0, 1\}$.

5.4. Proof of Theorem 5.2.3. Proof of part (a).

We suppose first that the function $\alpha$ diverges to $+\infty$ and prove that $F$ is everywhere dense in the closed unit interval $[0, 1]$, that is, $F' = [0, 1]$.

Let $k \in (0, 1)$ and let $m$ be any positive integer.

Since $\alpha(n) \to \infty$, there exists a positive integer $N$ such that $m + k < \alpha(n)$ for all $n > N$. Let $m'$ be the least positive integer such that $m + k < \alpha(m')$. Then

(5.4.1) $\alpha(m'-1) \leq m + k < \alpha(m').$

(We may take $\alpha(0)$ to be $0$ here.) Clearly, $m'$ is uniquely determined by $m$ and $m \to \infty$ implies $m' \to \infty$. Since $0 < k < 1$, we can choose a positive number $\epsilon < \min(k, 1-k)$, so that

(5.4.2) $0 < k - \epsilon < k < k + \epsilon < 1.$

Since $\alpha(n) - \alpha(n-1) \to 0$ we can make

$\alpha(m') - \alpha(m'-1) < \epsilon$.
by taking \( m \) sufficiently large; and in view of (5.4.1) we have

\[(5.4.3) \quad m + k - \epsilon < \alpha(m - 1) \leq m + k < \alpha(m) < m + k + \epsilon.\]

Using (5.4.2) we get

\[m < \alpha(m - 1) \leq m + k < \alpha(m) < m + 1,\]

which implies that the integral parts of \( \alpha(m - 1) \) and \( \alpha(m) \) are each equal to \( m \). Hence from (5.4.3) we have

\[k - \epsilon < \alpha(m - 1) \leq k < \alpha(m) < k + \epsilon,\]

showing that \( k \) is a limit point of \( F \). So every member of the open interval \((0,1)\) is a member of \( F' \). But since \( 0 \) and \( 1 \) are limit points of \((0,1)\), they also belong to \( F' \). Therefore \( F' = [0,1] \).

Suppose now that \( \alpha \) diverges to \(-\infty\), so that \( -\alpha(n) \to -\infty \).

By the result just proved the fractional parts of numbers \(-\alpha(n)\) must be everywhere dense in \([0,1]\). The fractional part of \(-\alpha(n)\) is 1 - \( \alpha(n) \) if \( \alpha(n) \neq 0 \), and 0 if \( \alpha(n) = 0 \). So, if the fractional parts of numbers \(-\alpha(n)\) are everywhere dense in \([0,1]\), then the fractional parts of numbers \( \alpha(n) \) also must be everywhere dense in \([0,1]\), that is \( F' = [0,1] \). Thus we have proved that if the function \( \alpha \) diverges to \(+\infty\) or \(-\infty\), then \( F' = [0,1] \). This proves the first half of part (a) of Theorem 5.2.3.
Note. It is known (See Pólya and Szegő [8], Part I, Chapter 3, Problem 101, p.23) that the fractional parts of the partial sums of an infinite series of positive terms in which the general term tends to zero are everywhere dense in the closed unit interval. This is equivalent to the statement: if $\alpha(n) \to \infty$ and $\alpha(n) - \alpha(n-1)$ tends to zero through positive values, then $F' = [0,1]$. So Theorem 5.2.3 is more general than this result. Further the proof given is on independent lines.

We now prove the second half of the theorem. Suppose that the function $\alpha$ is non-oscillatory and then $F' = [0,1]$. Clearly $\alpha$ cannot be convergent, and since the function is not oscillatory, it must diverge to $+\infty$ or $-\infty$. This completes the proof of part (a) of Theorem 5.2.3.

Before proceeding to prove part (b) of the theorem we require to prove the following lemma.

(5.4.4) **Lemma.** Let $\alpha$ be an arithmetic function such that $\alpha(n) - \alpha(n-1) \to 0$. If $\alpha$ is of infinite oscillation, that is, if $\alpha$ oscillates between finite and infinite limits or between $+\infty$ and $-\infty$, then there exists a subsequence $\alpha(n_i)$ of the sequence $\alpha(n)$ which diverges to $+\infty$ or $-\infty$ and is such that $\alpha(m_i) - \alpha(m_{i-1}) \to 0$. 
Proof. Case (i). We first suppose that the function oscillates between a finite number $a$ and $+\infty$ or between $-\infty$ and $+\infty$. Let $b$ be any positive number $> a$. Since $\omega(n) = \omega(n-1) \to 0$, the range of $\omega$ is everywhere dense between $a$ and $+\infty$ or between $-\infty$ and $+\infty$ according as the function oscillates between $a$ and $+\infty$ or between $-\infty$ and $+\infty$ by Lemma 2.5.1. In either case, since $b > a$, every number $\geq b$ must be a limit point of the range of the function $\omega$. So if $\xi \geq b$, then any neighbourhood of $\xi$ will contain $\xi$ infinitely many members of the sequence $(\omega(n))$.

Consider now a sequence $(\beta(n))$, where

$$\beta(n) = b + \frac{b}{2} + \frac{b}{3} + \ldots + \frac{b}{n}.$$ 

This sequence diverges to $+\infty$ and is such that $\beta(n) - \beta(n-1) \to 0$.

We can determine a sequence of non-overlapping intervals $(\Delta_n)$ such that $\beta(n) \in \Delta_n$ for every $n$. We may, for example, take $\Delta_n$ to be the interval $(\beta(n) - \epsilon_n, \beta(n) + \epsilon_n)$, where $\epsilon_n = b/3(n+1)$. Since $\beta(n) \geq b$ for each $n$, each of the intervals $\Delta_n$ must contain an infinite number of the members of $(\omega(n))$.

Let $\omega(m_1)$ be any member of $(\omega(n))$ lying in the interval $\Delta_1$. Since infinitely many members of $(\omega(n))$ lie in $\Delta_2$, we can choose a member $\omega(m_2)$ of the sequence with $m_2 > m_1$. Clearly $\omega(m_1) < \omega(m_2)$ since $\beta(1) < \beta(2)$ and the intervals $\Delta_1$ and $\Delta_2$
are non-overlapping. In this way we can construct an increasing subsequence \( (\alpha(m_i))_{i=1}^{\infty} \) of \( (\alpha(n)) \) such that \( \alpha(m_i) \in A_i \) for each \( i \).

Since \( \alpha(m_i) > \beta(i-1) \) for each \( i \) and \( \beta(i) \to \infty \) as \( i \to \infty \), the sequence \( (\alpha(m_i)) \) diverges to \( +\infty \). Also

\[
0 < \alpha(m_i) - \alpha(m_{i-1}) < \beta(i+1) - \beta(i-2),
\]

and this implies that \( \alpha(m_i) - \alpha(m_{i-1}) \to 0 \) since \( \beta(i) - \beta(i-2) \to 0 \) as \( i \to \infty \). This disposes of case (i).

**Case (ii).** If \( \alpha \) oscillates between a finite number and \(-\infty\), the sequence \( (-\alpha(n)) \) oscillates between a finite number and \(+\infty\); and by case (i) we have a subsequence \( (\alpha(m_i))_{i=1}^{\infty} \) that diverges to \(+\infty\) and is such that \( \alpha(m_i) - \alpha(m_{i-1}) \to 0 \).

Therefore the subsequence \( (\alpha(m_i)) \) of \( (\alpha(n)) \) diverges to \(-\infty\) and is such that \( \alpha(m_i) - \alpha(m_{i-1}) \to 0 \). With this we have completed the proof of the lemma.

We shall now prove part (b) of Theorem 5.2.3. If the arithmetic function \( \alpha \) satisfying the condition \( \alpha(n) = \alpha(n-1) \) tends to 0 is of infinite oscillation, then, by Lemma 5.4.4 there exists a subsequence \( (\alpha(m_i))_{i=1}^{\infty} \) of \( (\alpha(n)) \) diverging to \(+\infty\) or \(-\infty\) such that \( \alpha(m_i) - \alpha(m_{i-1}) \to 0 \). Applying Theorem 5.2.3 to \( (\alpha(m_i))_{i=1}^{\infty} \) we see that the range of \( (\frac{\alpha(m_i)}{m_i})_{i=1}^{\infty} \), which is a subset of the range of \( (\frac{\alpha(n)}{n}) \), is everywhere dense in \([0,1]\).
Therefore the range of \( \alpha \) is everywhere dense in \([0,1]\), that is,
\[ F' = [0,1]. \] This proves part (b).

Theorem 5.2.3 is now completely proved.

5.5. Proof of Theorem 5.2.4.

We first prove a lemma that is required in the proof of the theorem.

(5.5.1) Lemma. Let \( \alpha \) be an arithmetic function of bounded oscillation such that \( \alpha(n) - \alpha(n-1) \to 0 \), and \( A \) and \( \lambda \) the limit superior and limit inferior of \( \alpha(n) \). Also, let \( A \) denote the closed interval \([\lambda, A]\) and \( F_A \), the set of the fractional parts of the members of \( A \). Then \( F_A = F' \) if no integer belongs to \( A \); if an integer belongs to \( A \), then

\[
(5.5.2) \quad F_A \cup \{1\} = F' \cup \{0,1\}.
\]

Proof. Let \( S \) be the set of the subsequential limits of \( \alpha(n) \). Then \( A \) and \( \lambda \) are the supremum and the infimum of \( S \), so that \( S \subset [\lambda, A] = A \). Since \( \alpha(n) - \alpha(n-1) \to 0 \), the members of the sequence \( \alpha(n) \) are everywhere dense between \( \lambda \) and \( A \) by Lemma 2.5.1. Hence if \( s \in [\lambda, A] \), then a subsequence of \( \alpha(n) \) tends to \( s \). Hence \( A \subset S \). Thus \( S = A \); that is, \( A \) is the set of the subsequential limits of \( \alpha(n) \).

Suppose first that there is no integer in \( A \). If \( k \in F_A \),
then \( k \neq 0 \), and there exists an integer \( I \), such that \( I + k \in A \).

Hence a subsequence of (distinct members of) \( (\alpha(n)) \) converges to \( I + k \). If \( F_1 \) is the set of the fractional parts of the members of this subsequence, then, by Lemma 5.3.1 and Remark 5.3.6, \( F_1' = \{ k \} \). Clearly \( F_1 \subset F \) and so \( F_1' \subset F' \), which implies \( k \in F' \).

Hence \( F_A \subset F' \).

Now suppose that \( k \in F' \). Then there exists a subsequence \( (\alpha_f(m_i))_{i=1}^{\infty} \) of \( (\alpha_f(n)) \) that tends to \( k \). Let the corresponding subsequence of \( (\alpha(n)) \) be \( (\alpha(m_i))_{i=1}^{\infty} \). Since \( \alpha \) is bounded, \( (\alpha(m_i)) \) is either convergent itself or has a convergent subsequence. Suppose it converges to \( b \). Clearly \( b \in A \); and since \( A \) does not contain any integer, \( b \) is not an integer. So, by Lemma 5.3.1, the sequence \( (\alpha_f(m_i)) \) converges to \( b_f \), the fractional part of \( b \). Therefore \( k = b_f \). Since \( b \in A \), \( b_f \in F_A \); that is, \( k \in F_A \). If \( (\alpha(m_i)) \) does not converge, then it has a convergent subsequence, say \( (\alpha(p_j))_{j=1}^{\infty} \). Then the corresponding subsequence of \( (\alpha_f(m_i)) \) is \( (\alpha_f(p_j))_{j=1}^{\infty} \). Since \( (\alpha(m_i)) \) converges to \( k \), \( (\alpha_f(p_j)) \) also converges to \( k \). If \( (\alpha(p_j)) \) converges to \( c \), then, arguing as before, we have \( c_f = k \) and \( c \in A \), so that \( c_f = k \in F_A \).

Hence \( F' \subset F_A \). Already, \( F_A \subset F' \). So \( F' = F_A \).

If \( A \) contains an integer, then \( 0 \in F_A \); and if \( 0 \not\in k \in F_A \).
we can prove, as before, that \( k \in F' \). Therefore

\[ F_A \cup \{1\} \subseteq F' \cup \{0,1\}. \]

If \( 0 \neq k \in F' \), then again as before, \( k \in F_A \) and since \( 0 \in F_A \) we have \( F' \cup \{0,1\} \subseteq F_A \cup \{1\} \). Hence \( F_A \cup \{1\} = F' \cup \{0,1\} \) and the lemma is proved.

Now we can prove Theorem 5.2.4. We first assume that the function \( \alpha \) is of bounded oscillation and that \( \lambda \) and \( \lambda \) are the limit superior and limit inferior of \( (\alpha(n)) \). Then two cases arise: either \( \lambda - \lambda < 1 \) or \( \lambda - \lambda > 1 \).

**Case (i).** Let \( \lambda - \lambda < 1 \).

(a) If \( \lambda \) and \( \lambda \) lie strictly between two consecutive integers, then \( s \in \lambda = [\lambda, \lambda] \implies \lambda \leq s \leq \lambda \). (\( \lambda \) and \( \lambda \) are fractional parts of \( \lambda \) and \( s \) respectively.) Hence \( F_A = [\lambda, \lambda] \). Since \( \lambda \) does not contain any integer, we must have, by Lemma 5.5.1, \( F' = [\lambda, \lambda] \).

(b) If \( \lambda \) is an integer, then \( \lambda \) is not an integer

and \( F_A = [0, \lambda] \) since \( \lambda = 0 \). Clearly a subsequence of \( (\alpha(n)) \) tends to \( \lambda \), an integer, through values greater than it and so

by Remark 5.3.10, \( 0 \in F' \). If there is no subsequence of \( (\alpha(n)) \) tending to \( \lambda \) through values less than it, then, by Remark 5.3.10b

\( 1 \not\in F' \) and so (5.5.2) implies \( F_A \cup \{1\} = F' \cup \{1\} \). Every member
of $F_A$ (being a fractional part) is $< 1$ and so $1 \not\in F_A$. Also

$1 \not\in F^1$. Therefore,

$$F^1 = F_A = [0, \frac{1}{F}].$$

If there is a subsequence of $(\alpha(n))$ tending to the integer $\lambda$ through values less than it, then $1 \in F^0$ by Remark 5.3.10b. Already $0 \in F^0$. So $F^0 \cup \{0, 1\} = F^0$ and (5.5.2) becomes

$$F^0 = \left[0, \frac{1}{F}\right] \cup \{1\} = [0, 1] - (\frac{1}{F}, 1).$$

(c) If $\lambda$ is an integer, then $\lambda$ is not an integer; and $s \in \left[\lambda, \frac{1}{F}\right]$ implies either $\lambda_s \leq s < 1$ or $s = 0$ (if $s = \lambda$). So $F_A = \left[\frac{1}{F}, 1\right] \cup \{0\}$. Since there is a subsequence of $(\alpha(n))$ tending to the integer $\lambda$ through values less than it, $1 \in F^0$. There may or may not be a subsequence of $(\alpha(n))$ tending to $\lambda$ through values greater than it. In the first case, $0 \in F^0$, and in the second $0 \not\in F^1$. If $0 \in F^0$, then, since already $1 \in F^0, F^0 \cup \{0, 1\} = F^0$. Now

$$F_A \cup \{1\} = \left[\frac{1}{F}, 1\right] \cup \{0\} \cup \{1\} = \left[\frac{1}{F}, 1\right] \cup \{0\} \cup \{1\} = [0, 1] - (0, \frac{1}{F}).$$

Thus, in this case,

$$F^0 = [0, 1] - (0, \frac{1}{F}).$$

If $0 \not\in F^0, F^0 \cup \{0, 1\} = F^0 \cup \{0\}$ since $1 \in F^0$. Also,
\( F_A \cup \{1\} = [\lambda_{\frac{1}{2}}, 1] \cup \{0\}. \) Therefore, \( F' \cup \{0\} = [\lambda_{\frac{1}{2}}, 1] \cup \{0\}, \)

which implies

\[ F' = [\lambda_{\frac{1}{2}}, 1] \]

since \( 0 \not\in F' \) and \( 0 \not\in [\lambda_{\frac{1}{2}}, 1]. \)

(d) If none of \( \lambda \) and \( \Lambda \) is an integer, and the integral part of \( \Lambda \), \( \lfloor \Lambda \rfloor \), lies (strictly) between \( \lambda \) and \( \Lambda \), then there exist subsequences of \( (\alpha(n)) \) tending to \( \lfloor \Lambda \rfloor \) from either side of it.

Hence by Remark 5.3.10(c), both 0 and 1 are members of \( F' \), so that

\( F' \cup \{0, 1\} = F'. \) Now if \( \lambda \leq s < \lfloor \Lambda \rfloor \), then \( s_{\frac{1}{2}} \in [\lambda_{\frac{1}{2}}, 1] \) and if \( \lfloor \Lambda \rfloor \leq s \leq \Lambda \), then \( s_{\frac{1}{2}} \in [0, \lambda_{\frac{1}{2}}] \). Hence

\[
F_A = [\lambda_{\frac{1}{2}}, 1] \cup [0, \lambda_{\frac{1}{2}}] = [0, 1] = (\lambda_{\frac{1}{2}}, \lambda_{\frac{1}{2}}).
\]

so, in this case

\[ F' = [0, 1] = (\lambda_{\frac{1}{2}}, \lambda_{\frac{1}{2}}). \]

\textbf{Case (ii).} Let \( \Lambda - \lambda \geq 1 \). Clearly \( F_A = [0, 1] \),

so that \( F_A \cup \{1\} = [0, 1] \). Now either both \( \lambda \) and \( \Lambda \) are

(consecutive) integers or there is an integer lying between them. If both of them are integers, then a subsequence of

\( (\alpha(n)) \) tends to \( \lambda \) through values greater than it while there

is another tending to \( \Lambda \) through values less than it so that

0 and 1 are both members of \( F' \). If there is an integer \( I \) lying

between \( \lambda \) and \( \Lambda \) then also there are subsequences tending to \( I \)
from either side of it. So in either case 0 and 1 ∈ F'.

Hence, F' ∪ {0, 1} = F' and (5.5.2) becomes

\[ F' = [0, 1]. \]

If the function \( \alpha \) is of infinite oscillation, then

by part (b) of Theorem 5.2.3, \( F' = [0, 1] \).

The proof of Theorem 5.2.4 is now complete.

5.6. Concluding remark. In conclusion we observe that

if \( \alpha \) is any real-valued arithmetic function with \( \alpha(n) - \alpha(n-1) \to 0 \),

then \( F' \) can take any of the following five forms:

\[ \emptyset; \{ \frac{a}{b} \}; 0 \leq \frac{a}{b} \leq 1; \{ 0, 1 \}; \]

\[ [a, b], 0 < a < b \leq 1; [0, 1] \setminus (a, b), (a, b) \notin (0, 1). \]

If \( F' \) takes any of the first three forms, then \( \alpha \) converges;

if it takes any of the last two forms, then \( \alpha \) will not converge.