CHAPTER-3
CHAPTER 3
COMMON FIXED POINTS, COMMON COUPLED FIXED POINTS
IN PARTIAL METRIC SPACES USING ADMISSIBLE FUNCTIONS

The aim of this Chapter 3 is to obtain common fixed point theorems and common coupled fixed point theorems for maps using admissible functions in partial metric spaces. We divide this chapter into three sections.

SECTION 3.1: COMMON FIXED POINTS FOR FOUR MAPS USING \((\alpha, \eta)\)-ADMISSIBLE FUNCTIONS IN PARTIAL METRIC SPACES

Recently Samet et al.[8] defined partial compatible pair of maps in partial metric spaces as follows:

**Definition 3.1.1.** Let \((X, p)\) be a partial metric space and \(F, g : X \rightarrow X\). Then the pair \((F, g)\) is said to be partial compatible if the following conditions hold:

(i) \(p(x, x) = 0 \Rightarrow p(gx, gx) = 0\) whenever \(x \in X\),

(ii) \(\lim_{n \to \infty} p(Fgx_n, gFx_n) = 0\) whenever there exists a sequence \(\{x_n\}\) in \(X\) such that \(Fx_n \to t\) and \(gx_n \to t\) for some \(t \in X\).

We observe that the Definition 3.1.1 seems to be insufficient. Hence we modify it as

**Definition 3.1.2.** Let \((X, p)\) be a partial metric space and \(F, g : X \rightarrow X\). Then the pair \((F, g)\) is said to be partial\(^{(\ast)}\) compatible if the following conditions hold:

(i) \(p(x, x) = 0 \Rightarrow p(gx, gx) = 0\) whenever \(x \in X\),

(ii) \(\lim_{n \to \infty} p(Fgx_n, gFx_n) = 0\) whenever there exists a sequence \(\{x_n\}\) in \(X\) such that \(Fx_n \to t\) and \(gx_n \to t\) for some \(t \in X\) with \(p(t, t) = 0\).

Now we give an example in which the pair \((F, g)\) is partial\(^{(\ast)}\) compatible, but not partial compatible.
Example 3.1.3. Let $X = [0,1]$ and $p(x, y) = \max\{x, y\}$. Let $F, g : X \to X$ as $Fx = \frac{x^2}{2}$ and $gx = x^2$.

Clearly $p(x, x) = 0 \Rightarrow x = 0$. Hence $p(gx, gx) = 0$. Let $\{x_n\}$ be any sequence in $X$ such that $Fx_n \to t$ and $gx_n \to t$ as $n \to \infty$ for some $t \in X$ with $p(t, t) = 0$. Then clearly $t = 0$ and $x_n \to 0$ as $n \to \infty$. Hence $\lim_{n \to \infty} p(Fgx_n, gFx_n) = 0$.

Thus the pair $(F, g)$ is partial compatible.

If we take $\{x_n\} = \{1\}$ and $t = 2$, then $Fx_n \to t$ and $gx_n \to t$ as $n \to \infty$. But $\lim_{n \to \infty} p(Fgx_n, gFx_n) = \frac{1}{2} \neq 0$. Hence the pair $(F, g)$ is not partial compatible.

The aim of this section is to prove two unique common fixed point theorems for four maps using $(\alpha, \eta)$-admissible function in partial metric spaces. We also give an example to illustrate our main theorem.

**MAIN RESULTS**:

In this section $\Psi$ denote the class of all functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, where $\psi$ is continuous, monotonically increasing and $\sum \psi^n(t) < \infty$ for each $t > 0$. It is clear that $\psi(t) < t$ for every $t > 0$.

**Theorem 3.1.4.** Let $(X, p)$ be a complete partial metric space and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. Let $f, g, S$ and $T$ be self mappings on $X$ satisfying

(3.1.4.1) $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$,

(3.1.4.2) $\alpha(Sx, fx) \alpha(Ty, gy) \geq \eta(Sx, fx) \eta(Ty, gy)$, $\forall \ x, y \in X$ implies $p(fx, gy) \leq \psi(M(x, y))$, $\forall \ x, y \in X$ where $\psi \in \Psi$ and

$$M(x, y) = \max \left\{ \frac{p(Sx, Ty), p(Sx, fx), p(Ty, gy)}{2}, \frac{1}{2} [p(Sx, gy) + p(Ty, fx)] \right\},$$

(3.1.4.3) $\alpha(Sx_1, fx_1) \geq \eta(Sx_1, fx_1)$ for some $x_1 \in X$,
(3.1.4.4) pair \((f, g)\) satisfies \((\alpha, \eta)\)-admissible condition with respect to the pair 
\((S, T)\),

(3.1.4.5) the pairs \((f, S)\) and \((g, T)\) are partial\(^{(s)}\) compatible and \(S\) and \(T\) are 
continuous on \(X\),

(3.1.4.6) if there exists a sequence \(\{y_n\}\) in \(X\) such hat \(\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})\),
\(\forall n \in \mathbb{N}\) and \(y_n \to z\) for some \(z \in X\), then we have
\[\alpha(Sy_{2n}, fy_{2n}) \geq \eta(Sy_{2n}, fy_{2n}), \ \alpha(Ty_{2n+1}, gy_{2n+1}) \geq \eta(Ty_{2n+1}, gy_{2n+1}),\]
\(\forall n \in \mathbb{N}\), \(\alpha(z, fz) \geq \eta(z, fz)\) and \(\alpha(z, gz) \geq \eta(z, gz)\).

Then \(f, g, S\) and \(T\) have a common fixed point.

(3.1.4.7) Further if we assume that \(\alpha(u, u) \geq \eta(u, u)\) whenever \(u\) is a common
fixed point of \(f, g, S\) and \(T\) then \(f, g, S\) and \(T\) have a unique common
fixed point in \(X\).

**Proof.** From (3.1.4.3), we have \(\alpha(Sx_1, fx_1) \geq \eta(Sx_1, fx_1)\) for some \(x_1 \in X\).

From (3.1.4.1), define the sequences \(\{x_n\}\) and \(\{y_n\}\) as follows:

\[y_1 = fx_1 = Tx_2, y_2 = gx_2 = Sx_3, y_3 = fx_3 = Tx_4, y_4 = gx_4 = Sx_5, \cdots\]

\[y_{2n+1} = fx_{2n+1} = Tx_{2n+2}, y_{2n+2} = gx_{2n+2} = Sx_{2n+3}, n = 0, 1, 2, \cdots\]

Now

\[\alpha(Sx_1, Tx_2) \geq \eta(Sx_1, Tx_2), \ \text{from (3.1.4.3)}\]

\[\Rightarrow \alpha(fx_1, gx_2) \geq \eta(fx_1, gx_2), \ \text{from (3.1.4.4), i.e.} \ \alpha(y_1, y_2) \geq \eta(y_1, y_2)\]

\[\Rightarrow \alpha(Tx_2, Sx_3) \geq \eta(Tx_2, Sx_3), \ \text{from definition of } \{y_n\}\]

\[\Rightarrow \alpha(gx_2, fx_3) \geq \eta(gx_2, fx_3), \ \text{from (3.1.4.4), i.e.} \ \alpha(y_2, y_3) \geq \eta(y_2, y_3)\]

\[\Rightarrow \alpha(Sx_3, Tx_4) \geq \eta(Sx_3, Tx_4), \ \text{from definition of } \{y_n\}\]

\[\Rightarrow \alpha(fx_3, gx_4) \geq \eta(fx_3, gx_4), \ \text{from (3.1.4.4), i.e.} \ \alpha(y_3, y_4) \geq \eta(y_3, y_4)\]

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Continuing in this way, we have
\[ \alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1}) \quad \text{for} \quad n = 1, 2, 3, \ldots . \] (1)

**Case (a):** Suppose \( y_{2m} = y_{2m+1} \) for some \( m \).

\[
\alpha(Sx_{2m+1}, fx_{2m+1}) \alpha(Tx_{2m+2}, gx_{2m+2}) = \alpha(y_{2m}, y_{2m+1}) \alpha(y_{2m+1}, y_{2m+2}) \\
\geq \eta(y_{2m}, y_{2m+1}) \eta(y_{2m+1}, y_{2m+2}), \text{from (1)} \\
= \eta(Sx_{2m+1}, fx_{2m+1}) \eta(Tx_{2m+2}, gx_{2m+2}).
\]

Hence from (3.1.4.2), we have
\[
p(y_{2m+1}, y_{2m+2}) = p(fx_{2m+1}, gx_{2m+2}) \\
\leq \psi(M(x_{2m+1}, x_{2m+2})),
\]
where
\[
M(x_{2m+1}, x_{2m+2}) = \max \left\{ p(y_{2m}, y_{2m+1}), p(y_{2m}, y_{2m+1}), p(y_{2m+1}, y_{2m+2}), \right. \\
\left. \frac{1}{2} [p(y_{2m}, y_{2m+2}) + p(y_{2m+1}, y_{2m+1})] \right\}.
\]

But \( p(y_{2m}, y_{2m+1}) = p(y_{2m+1}, y_{2m+2}) \leq p(y_{2m+1}, y_{2m+2}), \text{from } (p_2). \)

\[
\frac{1}{2} [p(y_{2m}, y_{2m+2}) + p(y_{2m+1}, y_{2m+1})] \leq \frac{1}{2} [p(y_{2m}, y_{2m+1}) + p(y_{2m+1}, y_{2m+2})], \text{from } (p_1) \\
\leq p(y_{2m+1}, y_{2m+2}).
\]

Hence \( M(x_{2m+1}, x_{2m+2}) = p(y_{2m+1}, y_{2m+2}). \)

Thus \( p(y_{2m+1}, y_{2m+2}) \leq \psi(p(y_{2m+1}, y_{2m+2})) \)

which in turn yields that \( p(y_{2m+1}, y_{2m+2}) = 0 \) so that \( y_{2m+1} = y_{2m+2}. \)

Continuing in this way we get \( y_{2m} = y_{2m+1} = y_{2m+2} = \cdots . \)

Hence \( \{y_n\} \) is Cauchy.

**Case (b):** Suppose that \( y_n \neq y_{n+1}, \forall n. \)

Then
\[
\alpha(Sx_{2n+1}, fx_{2n+1}) \alpha(Tx_{2n+2}, gx_{2n+2}) = \alpha(y_{2n}, y_{2n+1}) \alpha(y_{2n+1}, y_{2n+2}) \\
\geq \eta(y_{2n}, y_{2n+1}) \eta(y_{2n+1}, y_{2n+2}), \text{from (1)} \\
= \eta(Sx_{2n+1}, fx_{2n+1}) \eta(Tx_{2n+2}, gx_{2n+2}).
\]
As in Case (a), we have
\[ p(y_{2n+1}, y_{2n+2}) \leq \psi(M(x_{2n+1}, x_{2n+2})) \] (2)
where \( M(x_{2n+1}, x_{2n+2}) = \max \{ p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2}) \} \).

If \( M(x_{2n+1}, x_{2n+2}) = p(y_{2n+1}, y_{2n+2}) \) then we get
\[ p(y_{2n+1}, y_{2n+2}) \leq \psi(p(y_{2n+1}, y_{2n+2})) \]
\[ < p(y_{2n+1}, y_{2n+2}). \]

It is a contradiction. Hence
\[ p(y_{2n+1}, y_{2n+2}) \leq \psi(p(y_{2n}, y_{2n+1})). \]

Similarly we can show that \( p(y_{2n}, y_{2n+1}) \leq \psi(p(y_{2n-1}, y_{2n})). \)

Thus \( p(y_{n+1}, y_{n+2}) \leq \psi(p(y_n, y_{n+1})) \) for \( n = 1, 2, 3, \ldots \)
\[ p(y_{n+1}, y_{n+2}) \leq \psi(p(y_n, y_{n+1})) \]
\[ \leq \psi^2(p(y_{n-1}, y_n)) \]
\[ \leq \psi^3(p(y_{n-2}, y_{n-1})) \]
\[ \vdots \]
\[ \leq \psi^n(p(y_1, y_2)) \] (3)
\[ \to 0 \text{ as } n \to \infty. \]

For \( n > m \), consider
\[ p(y_m, y_n) \leq p(y_m, y_{m+1}) + p(y_{m+1}, y_{m+2}) + \ldots + p(y_{n-1}, y_n) \]
\[ \leq \psi^{m-1}(p(y_1, y_2)) + \psi^m(p(y_1, y_2)) + \ldots + \psi^{n-2}(p(y_1, y_2)) \]
\[ \leq \sum_{k=m-1}^{\infty} \psi^k(p(y_1, y_2)) \to 0 \] (4)

Thus \( \{y_n\} \) is a Cauchy sequence in \((X, p)\). Since \((X, p)\) is a complete partial metric space, there exists \( z \in X \) such that \( p(z, z) = \lim_{n \to \infty} p(y_n, y_m). \)
From (4), we have $p(z, z) = 0$ \hspace{1cm} (5)

Hence

$$p(z, z) = \lim_{n \to \infty} p(fx_{2n+1}, z) = \lim_{n \to \infty} p(gx_{2n+2}, z)$$

$$= \lim_{n \to \infty} p(Sx_{2n+1}, z) = \lim_{n \to \infty} p(Tx_{2n+2}, z) = 0 \hspace{1cm} (6)$$

Since the pair $(f, S)$ is partial(*) compatible, from (5), we have $p(Sz, Sz) = 0$.

and $\lim_{n \to \infty} p(fSx_{2n+1}, fSx_{2n+1}) = 0 \hspace{1cm} (7)$

Since $S$ is continuous at $z$, we have $\lim_{n \to \infty} p(SSx_{2n+1}, Sz) = p(Sz, Sz) = 0 \hspace{1cm} (8)$

and $\lim_{n \to \infty} p(Sfx_{2n+1}, Sz) = p(Sz, Sz) = 0 \hspace{1cm} (9)$

Also $p(fSx_{2n+1}, Sz) \leq p(fSx_{2n+1}, Sfx_{2n+1}) + p(Sfx_{2n+1}, Sz)$.

Now by using (7) and (9), we have $\lim_{n \to \infty} p(fSx_{2n+1}, Sz) \leq 0$. 

Hence $\lim_{n \to \infty} p(fSx_{2n+1}, Sz) = 0 \hspace{1cm} (10)$

Now $p(fSx_{2n+1}, SSx_{2n+1}) \leq p(fSx_{2n+1}, Sz) + p(Sz, SSx_{2n+1})$.

$$\lim_{n \to \infty} p(fSx_{2n+1}, SSx_{2n+1}) \leq 0 \hspace{1cm} from \hspace{1cm} (10) \hspace{1cm} and \hspace{1cm} (8).$$

Hence

$$\lim_{n \to \infty} p(fSx_{2n+1}, SSx_{2n+1}) = 0 \hspace{1cm} (11)$$

Letting $n \to \infty$ and using (10) and (6) in

$|p(fSx_{2n+1}, gx_{2n}) - p(z, Sz)| \leq p(fSx_{2n+1}, Sz) + p(z, gx_{2n})$, we get

$$\lim_{n \to \infty} p(fSx_{2n+1}, gx_{2n}) = p(Sz, z) \hspace{1cm} (12)$$

Letting $n \to \infty$ and using (8)and(6) in

$|p(SSx_{2n+1}, Tx_{2n}) - p(Sz, z)| \leq p(SSx_{2n+1}, Sz) + p(z, Tx_{2n})$, we get

$$\lim_{n \to \infty} p(SSx_{2n+1}, Tx_{2n}) = p(Sz, z) \hspace{1cm} (13)$$

Letting $n \to \infty$ and using (8)and(6) in

$|p(SSx_{2n+1}, gx_{2n}) - p(Sz, z)| \leq p(SSx_{2n+1}, Sz) + p(z, gx_{2n})$, we get

$$\lim_{n \to \infty} p(SSx_{2n+1}, gx_{2n}) = p(Sz, z) \hspace{1cm} (14)$$

Letting $n \to \infty$ and using (10)and(6) in

$$\lim_{n \to \infty} p(SSx_{2n+1}, gx_{2n}) = p(Sz, z) \hspace{1cm} (15)$$
\[ |p(Tx_{2n}, fSx_{2n+1}) - p(z, Sz)| \leq p(fSx_{2n+1}, Sz) + p(z, Tx_{2n}), \] we get

\[
\lim_{n \to \infty} p(Tx_{2n}, fSx_{2n}) = p(Sz, z) \quad (15)
\]

Now Consider

\[
\alpha(SSx_{2n+1}, fSx_{2n+1}) \alpha(Tx_{2n}, gx_{2n}) = \alpha(Sy_{2n}, fy_{2n}) \alpha(y_{2n-1}, y_{2n}) \\
\geq \eta(Sy_{2n}, fy_{2n}) \eta(y_{2n-1}, y_{2n}), \text{ from (3.1.4.6)} \\
= \eta(SSx_{2n+1}, fSx_{2n+1}) \eta(Tx_{2n}, gx_{2n}).
\]

From (12), we have

\[
p(Sz, z) = \lim_{n \to \infty} p(fSx_{2n+1}, gx_{2n}) \\
\leq \lim_{n \to \infty} \psi(M(Sx_{2n+1}, x_{2n})), \text{ from (2.1.2)}
\]

where

\[
M(Sx_{2n+1}, x_{2n}) = \max \left\{ \begin{array}{l}
p(SSx_{2n+1}, Tx_{2n}), p(SSx_{2n+1}, fSx_{2n+1}), \\
p(Tx_{2n}, gx_{2n}), \\
\frac{1}{2} [p(SSx_{2n+1}, gx_{2n}) + p(Tx_{2n}, fSx_{2n+1})]
\end{array} \right\}
\]

\[ \rightarrow p(Sz, z), \text{ from (13), (11), (3), (14) and (15)}. \]

Thus

\[ p(Sz, z) \leq \phi(p(Sz, z)) \]

which in turn yields that \(Sz = z\).

Similarly using the continuity of \(T\) and partial\(^{(s)}\) compatibility of \((g, T)\) and \(\alpha(Ty_{2n+1}, gy_{2n+1}) \geq \eta(Ty_{2n+1}, gy_{2n+1})\) \text{ from (3.1.4.6)}, we can show that \(Tz = z\).

We have

\[
\alpha(Sz, fz) \alpha(Tx_{2n}, gx_{2n}) = \alpha(z, fz) \alpha(y_{2n-1}, y_{2n}) \\
\geq \eta(z, fz) \eta(y_{2n-1}, y_{2n}), \text{ from (1) and (2.1.6)} \\
= \eta(Sz, fz) \eta(Tx_{2n}, gx_{2n}).
\]
Now from (3.1.4.2),

\[ p(fz, z) = \lim_{n \to \infty} p(fz, gx_{2n}) \leq \lim_{n \to \infty} \psi(M(z, x_{2n})) , \]

where

\[ M(z, x_{2n}) = \max \left\{ p(z, Tx_{2n}), p(z, fz), p(Tx_{2n}, gx_{2n}), \frac{1}{2} [p(z, gx_{2n}) + p(Tx_{2n}, fz)] \right\} \]

\[ \to p(z, fz) \text{ from (6), (3), Lemma 1.2.5}. \]

Hence \( p(fz, z) \leq \psi(p(fz, z)) \) which in turn yields that \( fz = z \).

Similarly we can show that \( gz = z \) by using \( \alpha(z, gz) \geq \eta(z, gz) \) from (3.1.4.6).

Thus \( z \) is a common fixed point of \( f, g, S \) and \( T \).

Suppose \( z' \) is another common common fixed point of \( f, g, S \) and \( T \). We have

\[ \alpha(Sz, fz) \alpha(Tz', gz') = \alpha(z, z) \alpha(z', z') \geq \eta(z, z) \eta(z', z'), \text{ from (3.1.4.7)} \]

\[ = \eta(Sz, fz) \eta(Tz', gz'). \]

Hence from (3.1.4.2), we have \( p(z, z') = p(fz, gz') \leq \psi(M(z, z')) \), where

\[ M(z, z') = \max \left\{ p(z, z'), p(z, z), p(z', z'), \frac{1}{2} [p(z, z') + p(z, z')] \right\} \]

\[ = p(z, z'), \text{ from (p2)}. \]

Thus

\[ p(z, z') \leq \psi(p(z, z')) < p(z, z') \]

which is a contradiction. Hence \( z = z' \). Thus \( f, g, S \) and \( T \) have a unique common fixed point.

Now we give an example to support Theorem 3.1.4.

**Example 3.1.5** Let \( X = [0, 1] \) be endowed with metric \( p(x, y) = \max\{x, y\}, \)

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∀ x, y ∈ X and f, g, S, T : X → X be defined by

\( fx = \left(\frac{x}{2}\right)^8 \),  \( gx = \left(\frac{x}{2}\right)^4 \),

\( Sx = \left(\frac{x}{2}\right)^4 \) and \( Tx = \left(\frac{x}{2}\right)^2 \), ∀ x ∈ X.

Define \( \alpha, \eta : X \times X \rightarrow \mathbb{R}^+ \) by

\[
\begin{align*}
\alpha(x, y) &= \begin{cases} 
2, & \text{if } x, y \in [0, \frac{1}{4}] \\
0, & \text{otherwise}
\end{cases}, \\
\eta(x, y) &= \begin{cases} 
1, & \text{if } x, y \in [0, \frac{1}{4}] \\
0, & \text{otherwise}
\end{cases}.
\end{align*}
\]

Define \( \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) by \( \psi(t) = \frac{1}{4}t \), ∀ t ∈ \( \mathbb{R}^+ \).

Clearly \( fx, gx, Sx, Tx \in [0, \frac{1}{4}], \forall x \in X \).

Clearly, \( \forall x, y \in X \), we have \( \alpha(Sx, fx) \alpha(Ty, gy) \geq \eta(Sx, fx) \eta(Ty, gy) \).

Also, \( \forall x, y \in X \), we have

\[
p(fx, gy) = \max\{\left(\frac{x}{2}\right)^8, \left(\frac{y}{2}\right)^4\}
\leq \max\{\frac{1}{16} \left(\frac{x}{2}\right)^4, \frac{1}{4} \left(\frac{y}{2}\right)^2\}
\leq \frac{1}{4}p(Sx, Ty)
\leq \frac{1}{4}M(x, y) = \psi(M(x, y))
\]

One can easily verify all the other conditions of Theorem 3.1.4. Here "0" is the unique common fixed point of \( f, g, S \) and \( T \).

Finally we can prove a theorem similar to Theorem 3.1.4 by changing some conditions on the function \( \alpha \) and \( \eta \) and relaxing some conditions on the maps \( f, g, S \) and \( T \). Actually we give the following theorem without proof.

**Theorem 3.1.6.** Let \((X, p)\) be a complete partial metric space and

\( \alpha, \eta : X \times X \rightarrow \mathbb{R}^+ \) be two functions. Let \( f, g, S \) and \( T \) be self mappings on \( X \) satisfying (3.1.4.1),(3.1.4.2),(3.1.4.3) and (3.1.4.4).

Further assume that

(3.1.6.1)(a) \( S \) is continuous, the pair \((f, S)\) is partial\(^*\) compatible and the pair \((g, T)\) is weakly compatible and if there exists a sequence \( \{y_n\} \) in \( X \) such that

\( \alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1}) \) and \( \alpha(y_{n+1}, y_n) \geq \eta(y_{n+1}, y_n) \), ∀\( n \in \mathbb{N} \) and
\[ y_n \to z \text{ for some } z \in X, \text{ then we have} \]
\[ \alpha(Sy_{2n}, fy_{2n}) \geq \eta(Sy_{2n}, fy_{2n}), \forall n \in \mathbb{N}, \alpha(Tz, gz) \geq \eta(Tz, gz), \]
\[ \alpha(z, fz) \geq \eta(z, fz), \alpha(z, gw) \geq \eta(z, gw) \text{ and } \alpha(z, z) \geq \eta(z, z), \]
where \( Tw = z. \)

(OR)

(3.1.6.1)(b) \( T \) is continuous, the pair \((g, T)\) is partial(*) compatible and the pair \((f, S)\) is weakly compatible and if there exists a sequence \(\{y_n\}\) in \(X\) such that
\[ \alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1}) \text{ and } \alpha(y_{n+1}, y_n) \geq \eta(y_{n+1}, y_n), \forall n \in \mathbb{N} \]
y \( y_n \to z \text{ for some } z \in X, \) then we have
\[ \alpha(Ty_{2n+1}, gy_{2n+1}) \geq \eta(Ty_{2n+1}, gy_{2n+1}), \forall n \in \mathbb{N}, \alpha(z, gz) \geq \eta(z, gz), \]
\[ \alpha(z, fw) \geq \eta(z, fw), \alpha(Sz, fz) \geq \eta(Sz, fz) \text{ and } \alpha(z, z) \geq \eta(z, z), \]
where \( Sw = z. \)

Then \( f, g, S \) and \( T \) have a common fixed point.

(3.1.6.2) Further if we assume that \( \alpha(u, u) \geq \eta(u, u) \) whenever \( u \) is a common fixed point of \( f, g, S \) and \( T \) then \( f, g, S \) and \( T \) have a unique common fixed point in \( X. \)
SECTION 3.2 : SUZUKI TYPE UNIQUE COMMON FIXED POINT
THEOREM FOR FOUR MAPS USING $\alpha$-ADMISSIBLE
FUNCTIONS IN ORDERED PARTIAL METRIC SPACES

The aim of this section is to obtain a Suzuki type unique common
fixed point theorem for four maps of which one pair is partial($\ast$) compatible
in ordered partial metric spaces using $\alpha$-admissible functions. We also present
an example to illustrate our main theorem.

MAIN RESULT:

Theorem 3.2.1. Let $(X, p, \preceq)$ be a partially ordered complete partial metric
space and $\alpha : X \times X \to \mathbb{R}^+$ be a function. Let $f, g, S$ and $T$ be self mappings
on $X$ satisfying

(3.2.1) $f$ and $g$ are dominating maps and $f$ and $g$ are weak annihilators of $S$
and $T$ respectively,

(3.2.2) $f(X) \subseteq T(X), g(X) \subseteq S(X),$

(3.2.3) $\frac{1}{2} \min \{p(fx, Sx), p(gy, Ty)\} \leq \max \{p(Sx, Ty), p(fx, gy)\}$ implies
$\alpha(Sx, Ty) \psi(p(fx, gy)) \leq \phi(M(x, y)) - \varphi(M(x, y)), \forall$ comparable ele-
ments $x, y \in X$, where

$$M(x, y) = \max \left\{ p(Sx, Ty), p(Sx, fx), p(Ty, gy), \frac{1}{2} [p(Sx, gy) + p(Ty, fx)] \right\}$$

and $\psi, \phi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are such that $\psi$ is monotonically increasing and
continuous and $\phi$ and $\varphi$ are upper and lower semi continuous respectively
and satisfying the following condition:

$\psi(t) - \phi(t) + \varphi(t) > 0$ for $t > 0$ \hspace{1cm} (A)
(3.2.1.4) the pair \((f, g)\) is triangular \(\alpha\)-admissible w.r.to the pair \((S, T)\),

(3.2.1.5) \(\alpha(Sx_1, fx_1) \geq 1\) and \(\alpha(fx_1, Sx_1) \geq 1\) for some \(x_1 \in X\),

(3.2.1.6)(a) \(S\) is continuous, the pair \((f, S)\) is partial\(^*(\ast)\) compatible and the pair \((g, T)\) is weakly compatible and if there exists a sequence \(\{y_n\}\) in \(X\) such that 
\[\alpha(y_n, y_{n+1}) \geq 1, \quad \alpha(y_{n+1}, y_n) \geq 1, \quad \forall n \in \mathbb{N}\] and \(y_n \to z\) for some \(z \in X\), then we have 
\[\alpha(Sy_{2n}, y_{2n-1}) \geq 1, \quad \alpha(y_{2n}, y_{2n-1}) \geq 1, \quad \forall n \in \mathbb{N}\] and 
\[\alpha(z, Tz) \geq 1, \quad \forall n \in \mathbb{N}\]

(OR)

(3.2.1.6)(b) \(T\) is continuous, the pair \((g, T)\) is partial\(^*(\ast)\) compatible and the pair \((f, S)\) is weakly compatible and if there exists a sequence \(\{y_n\}\) in \(X\) such that 
\[\alpha(y_n, y_{n+1}) \geq 1, \quad \alpha(y_{n+1}, y_n) \geq 1, \quad \forall n \in \mathbb{N}\] and \(y_n \to z\) for some \(z \in X\), then we have 
\[\alpha(y_{2n}, Ty_{2n-1}) \geq 1, \quad \alpha(y_{2n}, z) \geq 1, \quad \forall n \in \mathbb{N}\] and 
\[\alpha(Sz, z) \geq 1, \quad \forall n \in \mathbb{N}\]

(3.2.1.7) if for a non-decreasing sequence \(\{x_n\}\) in \(X\) with \(x_n \preceq y_n, \forall n \in \mathbb{N}\) and \(y_n \to u\) implies \(x_n \preceq u, \forall n \in \mathbb{N}\).

Then \(f, g, S\) and \(T\) have a unique common fixed point in \(X\).

(3.2.1.8) Further if we assume that \(\alpha(u, v) \geq 1\) whenever \(u\) and \(v\) are common fixed points of \(f, g, S\) and \(T\) and the set of common fixed points of \(f, g, S\) and \(T\) is well ordered then \(f, g, S\) and \(T\) have unique common fixed point in \(X\).

**proof.** From (3.2.1.5), we have \(\alpha(Sx_1, fx_1) \geq 1\) for some \(x_1 \in X\).

From (3.2.1.2), there exist sequences \(\{x_n\}\) and \(\{y_n\}\) as follows:
$y_{2n+1} = f x_{2n+1} = T x_{2n+2}, \ y_{2n+2} = g x_{2n+2} = S x_{2n+3}, \ n = 0, 1, 2, \ldots.$

Now

\[ \alpha(S x_1, f x_1) \geq 1 \Rightarrow \alpha(S x_1, T x_2) \geq 1, \ \text{from definition of } \{y_n\} \]
\[ \Rightarrow \alpha(f x_1, g x_2) \geq 1, \ \text{from (3.2.1.4), i.e } \alpha(y_1, y_2) \geq 1 \]
\[ \Rightarrow \alpha(T x_2, S x_3) \geq 1, \ \text{from definition of } \{y_n\} \]
\[ \Rightarrow \alpha(g x_2, f x_3) \geq 1, \ \text{from (3.2.1.4), i.e } \alpha(y_2, y_3) \geq 1 \]
\[ \Rightarrow \alpha(S x_3, T x_4) \geq 1, \ \text{from definition of } \{y_n\} \]
\[ \Rightarrow \alpha(f x_3, g x_4) \geq 1, \ \text{from (3.2.1.4), i.e } \alpha(y_3, y_4) \geq 1 \]

Continuing in this way, we have

\[ \alpha(y_n, y_{n+1}) \geq 1, \ \forall \ n \in \mathbb{N} \]  \hspace{1cm} (1)

Similarly by using $\alpha(f x_1, S x_1) \geq 1$, we can show that

\[ \alpha(y_{n+1}, y_n) \geq 1, \ \forall \ n \in \mathbb{N} \]  \hspace{1cm} (2)

From (3.2.1.4), using triangular property, we have

\[ \alpha(y_m, y_n) \geq 1 \text{ for } m < n. \]  \hspace{1cm} (3)

From (3.2.1.1), we have

\[ x_{2n+1} \preceq f x_{2n+1} = T x_{2n+2} \preceq f T x_{2n+2} \preceq x_{2n+2}, \]
\[ x_{2n+2} \preceq g x_{2n+2} = S x_{2n+3} \preceq g S x_{2n+3} \preceq x_{2n+3}. \text{ Thus} \]
\[ x_n \preceq x_{n+1}, \forall n \in \mathbb{N} \]  \hspace{1cm} (4)

**Case (i):** Suppose $y_{2m} = y_{2m+1}$ for some $m$.

Assume that $y_{2m+1} \neq y_{2m+2}$. i.e. $p(y_{2m+1}, y_{2m+2}) > 0$.

Now $\alpha(S x_{2m+1}, T x_{2m+2}) = \alpha(y_{2m}, y_{2m+1}) \geq 1$, from (1).
Also we have

\[ \frac{1}{2} \min \{ p(f_{2m+1}, S_{2m+1}), p(g_{2m+2}, T_{2m+2}) \} \]

\[ \leq \max \{ p(S_{2m+1}, T_{2m+2}), p(f_{2m+1}, g_{2m+2}) \}, \text{from def. of } \{ y_n \}. \]

From (3.2.1.3) and (4), we have

\[ \psi(p(y_{2m+1}, y_{2m+2})) = \psi(p(f_{2m+1}, g_{2m+2})), \]

\[ \leq \alpha(S_{2m+1}, T_{2m+2})\psi(p(f_{2m+1}, g_{2m+2})), \]

\[ \leq \phi(M(x_{2m+1}, x_{2m+2})) - \varphi(M(x_{2m+1}, x_{2m+2})), \text{from (3.2.1.2)} \]

(5)

where

\[ M(x_{2m+1}, x_{2m+2}) = \max \{ p(y_{2m}, y_{2m+1}), p(y_{2m}, y_{2m+1}), p(y_{2m+1}, y_{2m+2}), \]

\[ \frac{1}{2} [p(y_{2m}, y_{2m+2}) + p(y_{2m+1}, y_{2m+1})] \}. \]

But \( p(y_{2m}, y_{2m+1}) = p(y_{2m+1}, y_{2m+1}) \leq p(y_{2m+1}, y_{2m+2}) \), from \( (p_2) \).

\[ \frac{1}{2} [p(y_{2m}, y_{2m+2}) + p(y_{2m+1}, y_{2m+1})] \leq \frac{1}{2} [p(y_{2m}, y_{2m+1}) + p(y_{2m+1}, y_{2m+2})] \]

\[ \leq p(y_{2m+1}, y_{2m+2}) \text{ from } (p_4). \]

Hence \( M(x_{2m+1}, x_{2m+2}) = p(y_{2m+1}, y_{2m+2}) \).

Now (5) becomes

\[ \psi(p(y_{2m+1}, y_{2m+2})) \leq \phi(p(y_{2m+1}, y_{2m+2})) - \varphi(p(y_{2m+1}, y_{2m+2})). \]

It is a contradiction to \( (A) \). Hence \( y_{2m+1} = y_{2m+2} \).

Continuing in this way we can conclude that \( y_n = y_{n+k}, \forall \text{ positive integers } k \). Thus \( \{ y_n \} \) is a Cauchy sequence in \( X \).

**Case (ii):** Suppose that \( y_n \neq y_{n+1}, \forall n \in \mathbb{N} \).

Now \( \alpha(S_{2n+1}, T_{2n+2}) = \alpha(y_{2n}, y_{2n+1}) \geq 1 \), from (1).

As in Case (i), we have

\[ \psi(p(y_{2n+1}, y_{2n+2})) \leq \phi(M(x_{2n+1}, x_{2n+2})) - \varphi(M(x_{2n+1}, x_{2n+2})) \]
where \( M(x_{2n+1}, x_{2n+2}) = \max \{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2})\} \).

If \( M(x_{2n+1}, x_{2n+2}) = p(y_{2n+1}, y_{2n+2}) \) then

\[
\psi(p(y_{2n+1}, y_{2n+2})) \leq \phi(p(y_{2n+1}, y_{2n+2})) - \varphi(p(y_{2n+1}, y_{2n+2})).
\]

It is a contradiction to (A). Hence

\[
\psi(p(y_{2n+1}, y_{2n+2})) \leq \phi(p(y_{2n}, y_{2n+1})) - \varphi(p(y_{2n}, y_{2n+1})) \tag{6}
\]

\[
< \psi(p(y_{2n}, y_{2n+1})), \text{ from (A)}.
\]

Since \( \psi \) is increasing, we have \( p(y_{2n+1}, y_{2n+2}) \leq p(y_{2n}, y_{2n+1}) \).

Similarly using (2), we can show that \( p(y_{2n}, y_{2n+1}) \leq p(y_{2n-1}, y_{2n}) \).

Thus \( \{p(y_n, y_{n+1})\} \) is a decreasing sequence of non-negative real numbers and hence converges to some real number \( r \geq 0 \). Hence

\[
\lim_{n \to \infty} p(y_n, y_{n+1}) = r.
\]

Suppose \( r > 0 \).

Letting \( n \to \infty \) in (6), we get \( \psi(r) \leq \phi(r) - \varphi(r) \).

It is a contradiction to (A). Hence \( r = 0 \). Thus

\[
\lim_{n \to \infty} p(y_n, y_{n+1}) = 0 \tag{7}
\]

From (\( p_2 \)), we have

\[
\lim_{n \to \infty} p(y_n, y_n) = 0 \tag{8}
\]

By the definition of \( p^s \), (7) and (8), we have

\[
\lim_{n \to \infty} p^s(y_n, y_{n+1}) = 0 \tag{9}
\]

Now we prove that \( \{y_{2n}\} \) is a Cauchy sequence in \( (X, p^s) \).

On contrary, suppose that \( \{y_{2n}\} \) is not a Cauchy sequence. Then there exist
$\epsilon > 0$ and monotone increasing sequences of natural numbers $\{y_{2m_k}\}$ and $\{y_{2n_k}\}$ such that $n_k > m_k$,

\[ p^s(y_{2m_k}, y_{2n_k}) \geq \epsilon \quad (10) \]

\[ p^s(y_{2m_k}, y_{2n_k-2}) < \epsilon \quad (11) \]

Now from (10) and (11), we obtain

\[
\epsilon \leq p^s(y_{2m_k}, y_{2n_k}) \leq p^s(y_{2m_k}, y_{2n_k-2}) + p^s(y_{2n_k-2}, y_{2n_k-1}) + p^s(y_{2n_k-1}, y_{2n_k}) 
< \epsilon + p^s(y_{2n_k-2}, y_{2n_k-1}) + p^s(y_{2n_k-1}, y_{2n_k}).
\]

Letting $k \to \infty$ and using (9), we get

\[
\lim_{k \to \infty} p^s(y_{2m_k}, y_{2n_k}) = \epsilon \quad (12)
\]

Hence from the definition of $p^s$ and (8), we have

\[
\lim_{k \to \infty} p(y_{2m_k}, y_{2n_k}) = \frac{\epsilon}{2} \quad (13)
\]

Letting $k \to \infty$ and then using (12) and (9), in

\[
|p^s(y_{2m_k+1}, y_{2n_k}) - p^s(y_{2m_k}, y_{2n_k})| = p^s(y_{2m_k}, y_{2m_k+1}),
\]

\[
|p^s(y_{2m_k}, y_{2n_k-1}) - p^s(y_{2m_k}, y_{2n_k})| \leq p^s(y_{2n_k-1}, y_{2n_k}) \quad \text{and}
\]

\[
|p^s(y_{2n_k-1}, y_{2m_k-1}) - p^s(y_{2m_k}, y_{2n_k})| \leq p^s(y_{2n_k-1}, y_{2n_k}) + p^s(y_{2m_k}, y_{2m_k+1}),
\]

we obtain upon using definition of $p^s$ and (8) that

\[
\lim_{k \to \infty} p(y_{2m_k+1}, y_{2n_k}) = \frac{\epsilon}{2} \quad (14)
\]

\[
\lim_{k \to \infty} p(y_{2m_k}, y_{2n_k-1}) = \frac{\epsilon}{2} \quad (15)
\]

and

\[
\lim_{k \to \infty} p(y_{2n_k-1}, y_{2m_k+1}) = \frac{\epsilon}{2} \quad (16)
\]
If \( \frac{1}{2} \min \{ p(y_{2m_k+1}, y_{2m_k}), p(y_{2n_k}, y_{2n_k-1}) \} \)
\[ > \max \{ p(y_{2m_k}, y_{2n_k-1}), p(y_{2m_k+1}, y_{2n_k}) \} \]
then letting \( k \to \infty \) and using (7), (15) and (14), we obtain \( 0 \geq \epsilon \). It is a contradiction. Hence

\[ \frac{1}{2} \min \{ p(y_{2m_k+1}, y_{2m_k}), p(y_{2n_k}, y_{2n_k-1}) \} \leq \max \{ p(y_{2m_k}, y_{2n_k-1}), p(y_{2m_k+1}, y_{2n_k}) \} \]

Also \( \alpha(Sx_{2m_k+1}, Tx_{2n_k}) = \alpha(y_{2m_k}, y_{2n_k-1}) \geq 1 \), from (3).

Hence from (3.2.1.3) and (4), we have
\[
\psi(p(y_{2m_k+1}, y_{2n_k})) = \psi(p(fx_{2m_k+1}, gx_{2n_k})) \\
\leq \alpha(Sx_{2m_k+1}, Tx_{2n_k}) \psi(p(fx_{2m_k+1}, gx_{2n_k})) \\
\leq \phi(M(x_{2m_k+1}, x_{2n_k})) - \varphi(M(x_{2m_k+1}, x_{2n_k}))
\]

where
\[
M(x_{2m_k+1}, x_{2n_k}) = \max \left\{ p(y_{2m_k}, y_{2n_k-1}), p(y_{2n_k}, y_{2m_k-1}), p(y_{2n_k-1}, y_{2m_k}), \right. \\
\left. \frac{1}{2} [p(y_{2m_k}, y_{2n_k}) + p(y_{2n_k}, y_{2m_k})] \right\}
\]

\[ \to \frac{\epsilon}{2} \text{ as } k \to \infty, \text{ from (15), (7), (13), (16)} \]

Letting \( k \to \infty \) in (17) and using (14) we obtain
\[
\psi(\frac{\epsilon}{2}) \leq \phi(\frac{\epsilon}{2}) - \varphi(\frac{\epsilon}{2}).
\]

It is a contradiction to (A).

Hence \( \{y_{2n}\} \) is a Cauchy sequence in \( (X, p^*) \).

Letting \( n \to \infty, m \to \infty \) in
\[
|p^*(y_{2n+1}, y_{2m+1}) - p^*(y_{2m}, y_{2n})| \leq p^*(y_{2n+1}, y_{2n}) + p^*(y_{2m}, y_{2m+1}),
\]
we obtain \( \lim_{n \to \infty} p^*(y_{2n+1}, y_{2m+1}) = 0. \)

Hence \( \{y_{2n+1}\} \) is a Cauchy sequence in \( (X, p^*) \). Thus \( \{y_n\} \) is a Cauchy sequence.
in \((X,p^s)\).

Hence we have \(\lim_{n \to \infty} p^s(y_n, y_m) = 0\) and hence from def.of \(p^s\) and (8), we have

\[
\lim_{n \to \infty} p(y_n, y_m) = 0
\]  
(18)

Thus \(\{y_n\}\) is a Cauchy sequence in \((X,p)\). Since \((X,p)\) is a complete partial metric space, there exists \(z \in X\) such that \(p(z,z) = \lim_{n \to \infty} p(y_n, y_m)\).

From (18),

\[
p(z,z) = 0
\]  
(19)

Hence

\[
p(z,z) = \lim_{n \to \infty} p(fx_{2n+1}, z) = \lim_{n \to \infty} p(gx_{2n+2}, z) = \lim_{n \to \infty} p(Sx_{2n+1}, z) = \lim_{n \to \infty} p(Tx_{2n+2}, z) = 0.
\]  
(20)

Suppose (3.2.1.6)(a) holds.

Suppose \(Sz \neq z\).

Since the pair \((f,S)\) is partial\(^*(-)\) compatible, from (19), we have \(p(Sz,Sz) = 0\) and

\[
\lim_{n \to \infty} p(fSx_{2n+1}, Sfx_{2n+1}) = 0
\]  
(21)

Since \(S\) is continuous at \(z\), we have

\[
\lim_{n \to \infty} p(SSx_{2n+1}, Sz) = p(Sz, Sz) = 0
\]  
(22)

and

\[
\lim_{n \to \infty} p(Sfx_{2n+1}, Sz) = p(Sz, Sz) = 0
\]  
(23)

Also \(p(fSx_{2n+1}, Sz) \leq p(fSx_{2n+1}, Sfx_{2n+1}) + p(Sfx_{2n+1}, Sz)\).

Now by using (21) and (23), we have \(\lim_{n \to \infty} p(fSx_{2n+1}, Sz) \leq 0\). Hence

\[
\lim_{n \to \infty} p(fSx_{2n+1}, Sz) = 0
\]  
(24)
Now \( p(fSx_{2n+1}, SSx_{2n+1}) \leq p(fSx_{2n+1}, Sz) + p(Sz, SSx_{2n+1}) \).

\[
\lim_{n \to \infty} p(fSx_{2n+1}, SSx_{2n+1}) \leq 0 \quad \text{from (24) and (22).}
\]

Hence

\[
\lim_{n \to \infty} p(fSx_{2n+1}, SSx_{2n+1}) = 0 \quad (25)
\]

Letting \( n \to \infty \) and using (24),(19) and (20) in

\[|p(fSx_{2n+1}, gx_{2n}) - p(z, Sz)| \leq p(fSx_{2n+1}, Sz) + p(z, gx_{2n}),\] we get

\[
\lim_{n \to \infty} p(fSx_{2n+1}, gx_{2n}) = p(Sz, z) \quad (26)
\]

Letting \( n \to \infty \) and using (22),(19) and (20) in

\[|p(SSx_{2n+1}, Tx_{2n}) - p(Sz, z)| \leq p(SSx_{2n+1}, Sz) + p(z, Tx_{2n}),\] we get

\[
\lim_{n \to \infty} p(SSx_{2n+1}, Tx_{2n}) = p(Sz, z) \quad (27)
\]

Letting \( n \to \infty \) and using (22),(19) and (20) in

\[|p(SSx_{2n+1}, gx_{2n}) - p(Sz, z)| \leq p(SSx_{2n+1}, Sz) + p(z, gx_{2n}),\] we get

\[
\lim_{n \to \infty} p(SSx_{2n+1}, gx_{2n}) = p(Sz, z) \quad (28)
\]

Letting \( n \to \infty \) and using (24),(19) and (20) in

\[|p(Tx_{2n}, fSx_{2n+1}) - p(z, Sz)| \leq p(fSx_{2n+1}, Sz) + p(z, Tx_{2n}),\] we get

\[
\lim_{n \to \infty} p(Tx_{2n}, fSx_{2n}) = p(Sz, z) \quad (29)
\]

If \( \frac{1}{2} \min \{p(fSx_{2n+1}, SSx_{2n+1}), p(gx_{2n}, Tx_{2n})\} > \max\{p(SSx_{2n+1}, Tx_{2n}), p(fSx_{2n+1}, gx_{2n})\}\) then letting \( n \to \infty \) and using (25),(7),(27) and (26), we get \( 0 \geq p(Sz, z) \) which is a contradiction. Hence

\[
\frac{1}{2} \min \{p(fSx_{2n+1}, SSx_{2n+1}), p(gx_{2n}, Tx_{2n})\} \\
\leq \max\{p(SSx_{2n+1}, Tx_{2n}), p(fSx_{2n+1}, gx_{2n})\}.
\]
Clearly $\alpha(SSx_{2n+1}, Tx_{2n}) = \alpha(Sy_{2n}, y_{2n-1}) \geq 1$, from (3.2.1.6)(a).

From (3.2.1.1), we have $x_{2n} \leq gx_{2n} = Sx_{2n+1}$.

Using the continuity of $\psi$ and (26), we get

$$
\psi(p(Sz, z)) = \lim_{n \to \infty} \psi(p(fSx_{2n+1}, gx_{2n})) \\
\leq \lim_{n \to \infty} \alpha(SSx_{2n+1}, Tx_{2n}) \psi(p(fSx_{2n+1}, gx_{2n})) \\
\leq \lim_{n \to \infty} \left[ \phi(M(Sx_{2n+1}, x_{2n})) - \varphi(M(Sx_{2n+1}, x_{2n})) \right] \text{ from (3.2.1.3)}
$$

where

$$
M(Sx_{2n+1}, x_{2n}) = \max \left\{ \frac{1}{2} \left[ p(SSx_{2n+1}, gx_{2n}) + p(Tx_{2n}, fSx_{2n+1}) \right], p(SSx_{2n+1}, Tx_{2n}), p(SSx_{2n+1}, fSx_{2n+1}), p(Tx_{2n}, gx_{2n}) \right\}
$$

$$
\to p(Sz, z), \text{ from (27), (25), (7), (28) and (29)}.
$$

Hence $\psi(p(Sz, z)) \leq \phi(p(Sz, z)) - \varphi(p(Sz, z))$.

It is a contradiction to (A). Hence $Sz = z$.

Suppose $fz \neq z$.

If $\frac{1}{2} \min \{p(Sz, fz), p(gx_{2n}, Tx_{2n})\} > \max \{p(Sz, Tx_{2n}), p(fz, gx_{2n})\}$

then letting $n \to \infty$, we get $0 \geq p(fz, z)$, from (7), (19) and Lemma 1.2.5.

It is a contradiction. Hence

$$
\frac{1}{2} \min \{p(Sz, fz), p(gx_{2n}, Tx_{2n})\} \leq \max \{p(Sz, Tx_{2n}), p(fz, gx_{2n})\}
$$

Also $\alpha(Sz, Tx_{2n}) = \alpha(z, y_{2n-1}) \geq 1$, from (3.2.1.6)(a).

Since $x_{2n} \leq gx_{2n}$ and $gx_{2n} \to z$, by (3.2.1.7), we have $x_{2n} \leq z$.

Using the continuity of $\psi$, Lemma 1.2.5 and (20), we get

$$
\psi(p(fz, z)) = \lim_{n \to \infty} \psi(p(fz, gx_{2n})) \\
\leq \lim_{n \to \infty} \alpha(Sz, Tx_{2n}) \psi(p(fz, gx_{2n})) \\
\leq \lim_{n \to \infty} \left[ \phi(M(z, x_{2n})) - \varphi(M(z, x_{2n})) \right] \text{ from (3.2.1.3)}
$$

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where

\[
M(z, x_{2n}) = \max \left\{ p(z, Tx_{2n}), p(z, fz), p(Tx_{2n}, gx_{2n}), \frac{1}{2} [p(z, gx_{2n}) + p(Tx_{2n}, fz)] \right\}
\]

\[
\rightarrow p(z, fz) \quad \text{from (20), (7), Lemma 1.2.5.}
\]

Hence \( \psi(p(fz, z)) \leq \phi(p(fz, z)) - \varphi(p(fz, z)). \)

It is a contradiction to (A). Hence \( fz = z. \)

Since \( f(X) \subseteq T(X), \) there exists \( w \in X \) such that \( z = fz = Tw. \) Also we have \( z = fz = Tw \leq Tw \leq w, \) from (3.2.1.1).

From (3.2.1.6)(a), we have \( \alpha(Sz, Tw) = \alpha(z, z) \geq 1. \)

Suppose \( z \neq gw. \) Now from (19),

\[
\frac{1}{2} \min \{ p(Sz, fz), p(gw, Tw) \} = \min \{ p(z, z), p(gw, z) \}
\]

\[
= 0 < \max \{ p(Sz, Tw), p(fz, gw) \}.
\]

From (3.2.1.3), we have

\[
\psi(p(z, gw)) = \psi(p(fz, gw)) \\
\leq \alpha(Sz, Tw) \psi(p(fz, gw)) \\
\leq \phi(M(z, w)) - \varphi(M(z, w)),
\]

where

\[
M(z, w) = \max \left\{ p(z, z), p(z, z), p(z, gw), \frac{1}{2} [p(z, gw) + p(z, z)] \right\}
\]

\[
= p(z, gw).
\]

Thus

\[
\psi(p(z, gw)) \leq \phi(p(z, gw)) - \varphi(p(z, gw)) < \psi(p(z, gw)), \text{ from (A)}
\]

which is a contradiction. Hence \( z = gw. \)

Since the pair \((g, T)\) is weakly compatible, we have \( gz = gTw = Tgw = Tz. \)
From (3.2.1.6)(a), we have \( \alpha(Sz, Tz) = \alpha(z, Tz) \geq 1 \).

Suppose \( z \neq gz \). Now from (19),

\[
\frac{1}{2} \min \{ p(Sz, fz), p(gz, Tz) \} = \min \{ p(z, z), p(gz, Tz) \} = 0 < \max \{ p(Sz, Tz), p(fz, gz) \}.
\]

From (3.2.1.3), we have

\[
\psi(p(z, gz)) = \psi(p(fz, gz)) \\
\leq \alpha(Sz, Tz) \psi(p(fz, gz)) \\
\leq \phi(M(z, z)) - \varphi(M(z, z)),
\]

where

\[
M(z, z) = \max \left\{ p(z, Tz), p(z, z), p(Tz, gz), \frac{1}{2} [p(z, gz) + p(gz, z)] \right\} \\
= p(z, gz), \text{from (p2)}
\]

Thus

\[
\psi(p(z, gz)) \leq \phi(p(z, gz)) - \varphi(p(z, gz)) < \psi(p(z, gz)), \text{from (A)}
\]

which is a contradiction. Hence \( z = gz = Tz \). Thus \( z \) is a common fixed point of \( f, g, S \) and \( T \).

Suppose \( z' \) is another common common fixed point of \( f, g, S \) and \( T \).

From (3.2.1.8), we have \( \alpha(Sz, Tz') = \alpha(z, z') \geq 1 \) and \( z \preceq z' \).

Now

\[
\frac{1}{2} \min \{ p(fz, Sz), p(gz', Tz') \} = \frac{1}{2} \min \{ 0, p(gz', Tz') \} = 0 < p(z, z') = \max \{ p(Sz, Tz'), p(fz, gz') \}.
\]

Hence

\[
\psi(p(z, z')) = \psi(p(fz, gz')) \\
\leq \alpha(Sz, Tz') \psi(p(fz, gz')) \\
\leq \phi(M(z, z')) - \varphi(M(z, z')),
\]
where

\[ M(z, z') = \max \left\{ p(z, z'), p(z, z), p(z', z'), \frac{1}{2} [p(z, z') + p(z, z')] \right\} \]

\[ = p(z, z'), \text{ from } (p_2). \]

Thus

\[ \psi(p(z, z')) \leq \phi(p(z, z')) - \varphi(p(z, z')) < \psi(p(z, z')), \text{ from } (A) \]

which is a contradiction. Hence \( z = z' \). Thus \( f, g, S \) and \( T \) have a unique common fixed point.

Similarly we can prove Theorem 3.2.1 when (3.2.1.6)(b) holds.

Now we give an example to support Theorem 3.2.1.

**Example 3.2.2.** Let \( X = \mathbb{R}^+ \), \( p(x, y) = \max\{x, y\}, \forall x, y \in X \) and define \( x \preceq y \) if \( y \leq x \). Define \( f, g, S, T : X \to X \) by \( f x = \frac{x}{2}, g x = \frac{x}{4}, S x = 8x \) and \( T x = 4x \).

Define \( \alpha : X \times X \to \mathbb{R}^+ \) by \( \alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise}. \end{cases} \)

Define \( \psi, \phi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) by

\[ \psi(t) = 4t, \phi(t) = 7t, \varphi(t) = \frac{7}{2}t, \forall \, t \in \mathbb{R}^+. \]

Clearly \( \psi(t) - \phi(t) + \varphi(t) > 0, \forall \, t > 0. \)

We have \( f x = \frac{x}{2} \leq x \Rightarrow x \preceq f x \) and \( g x = \frac{x}{4} \leq x \Rightarrow x \preceq g x. \)

Also \( f T x = 2x \geq x \Rightarrow f T x \preceq x \) and \( g S x = 2x \geq x \Rightarrow g S x \preceq x. \)

If \( x > \frac{1}{8} \) and \( y \in X \) then \( \alpha(S x, T y) = 0. \)

If \( x \leq \frac{1}{8} \) and \( y > \frac{1}{4} \) then \( \alpha(S x, T y) = 0. \)

In these cases, the condition (2.1.3) is clearly satisfied.

Suppose \( x \leq \frac{1}{8} \) and \( y \in [0, \frac{1}{4}] \) then \( \alpha(S x, T y) = 1. \)
In this case, we have

$$\frac{1}{2} \min \{p(fx, Sx), p(gy, Ty)\} = \frac{1}{2} \min \{8x, 4y\} \leq \max \{8x, 4y\} = p(Sx, Ty) \leq \max \{p(Sx, Ty), p(fx, gy)\}.$$ 

Also

$$\alpha(Sx, Ty)\psi(p(fx, gy)) = (1)4 \max \left\{ \frac{x}{2}, \frac{y}{4} \right\}$$

$$= \max \{2x, y\}.$$ 

$$= \frac{1}{4}p(Sx, Ty)$$

$$\leq \frac{1}{4}M(x, y)$$

$$\leq \phi(M(x, y)) - \varphi(M(x, y))$$

Thus (3.2.1.3) is satisfied.

One can easily verify the remaining conditions of Theorem 3.2.1. Clearly 0 is the unique common fixed point of $f, g, S$ and $T$.

The part of this section was accepted in the following Journal

SECTION 3.3 : COMMON COUPLED FIXED POINTS FOR FOUR
MAPS USING $\alpha$-ADMISSIBLE FUNCTIONS IN
PARTIAL METRIC SPACES

The aim of this section is to introduce $\alpha$-admissible function associated
with four maps of which two maps are $X \to X$ and the other two are $X \times X \to
X$ in partial metric spaces and also to introduce partial (*) compatible condition
to the pair $(F, g)$, where $F : X \times X \to X$ and $g : X \to X$ in partial metric
spaces.

Kaushik et al.[66]introduced the following concept which is a generalization
of the concept introduced by Mursaleen et al.[58].

Definition 3.3.1. ([66]). Let $X$ be a nonempty set and $\alpha : X^2 \times X^2 \to \mathbb{R}^+$ be
a function. Let $F : X \times X \to X$, $S : X \to X$ be mappings. Then $F$ and $S$ are
said to be $\alpha$-admissible if

$$\alpha((Sx, Sy), (Su, Sv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1,$$

$\forall x, y, u, v \in X.$

If $S = I(Identity \ map)$, then the above definition is the concept of
Mursaleen et al.[58].

In this paper, we give a generalization of the above definition.

Definition 3.3.2. Let $X$ be a nonempty set and $\alpha : X^2 \times X^2 \to \mathbb{R}^+$. Let
$F, G : X \times X \to X$ and $S, T : X \to X$ be mappings. Then we say that the pair
$(S, T)$ is $\alpha$-admissible with respect to the pair $(F, G)$ if

(i) $\alpha((Sx, Sy), (Tu, Tv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (G(u, v), G(v, u))) \geq 1,$

(ii) $\alpha((Tx, Ty), (Su, Sv)) \geq 1 \Rightarrow \alpha((G(x, y), G(y, x)), (F(u, v), F(v, u))) \geq 1,$

$\forall x, y, u, v \in X.$

We say that the pair $(S, T)$ is triangular $\alpha$-admissible with respect to the
pair \((F, G)\) if the pair \((S, T)\) is \(\alpha\)-admissible with respect to the pair \((F, G)\) and
\[
\alpha((x_1, y_1), (x_2, y_2)) \geq 1, \alpha((x_2, y_2), (x_3, y_3)) \geq 1 \Rightarrow \alpha((x_1, y_1), (x_3, y_3)) \geq 1,
\]
\(\forall x_1, x_2, x_3, y_1, y_2, y_3 \in X.\)

We extend the Definition 3.1.2. to the maps \(F : X \times X \to X\) and \(S : X \to X.\)

**Definition 3.3.3.** Let \((X, p)\) be a partial metric space and \(F : X \times X \to X\) and \(S : X \to X.\) Then the pair \((F, S)\) is said to be partial \((\ast)\)compatible if
\[
(i) p(z, z) = 0 \Rightarrow p(Sz, Sz) = 0 \text{ whenever } z \in X,
\]
\[
(ii) \lim_{n \to \infty} p(S(F(x_n, y_n)), F(Sx_n, Sy_n)) = 0 \text{ and } \lim_{n \to \infty} p(S(F(y_n, x_n)), F(Sy_n, Sx_n)) = 0 \text{ whenever there exist sequences } \{x_n\} \text{ and } \{y_n\} \text{ in } X \text{ such that } F(x_n, y_n) \to t, Sx_n \to t \text{ and } F(y_n, x_n) \to t', Sy_n \to t'
\]
for some \(t, t' \in X\) with \(p(t, t) = 0\) and \(p(t', t') = 0.\)

**MAIN RESULT :**

We observe that \(\max\{a + b, c + d\} \leq \max\{a, c\} + \max\{b, d\}, \forall a, b, c, d \in \mathbb{R}^+.\)

Throughout this section,let \(\psi, \phi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+\) be such that \(\psi\) is an altering distance function, \(\phi\) is continuous and \(\varphi\) is lower semi continuous, \(\phi(0) = 0 = \varphi(0)\) and
\[
\psi(t) - \phi(t) + \varphi(t) > 0 \text{ for } t > 0 \ldots \ldots (A)
\]

**Theorem 3.3.4.** Let \((X, p)\) be a complete partial metric space and \(\alpha : X^2 \times X^2 \to \mathbb{R}^+\) be a function. Let \(F, G : X \times X \to X\) and \(S, T : X \to X\)
be mappings on \(X\) satisfying
\[
(3.3.4.1) F(X \times X) \subseteq T(X), G(X \times X) \subseteq S(X),
\]
\[
(3.3.4.2) \alpha(Sx, Sy), (Tu, Tv))\psi(p(F(x, y), G(u, v))) \leq \phi(M_{u, v}^{x, y}) - \varphi(M_{u, v}^{x, y}), \\
\forall x, y \in X \text{ where } \psi, \phi, \varphi \text{ are as in above and}
\]

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Then \( F, G, S \) and \( T \) have a common coupled fixed point.

(3.4.4.7) Further if we assume that \( \alpha((z, w), (z', w')) \geq 1 \) and
\[ \alpha((w, z), (w', z')) \geq 1 \] whenever \((z, w)\) and \((z', w')\) are common coupled fixed points of \(F, G, S\) and \(T\) then \(F, G, S\) and \(T\) have a unique common coupled fixed point in \(X \times X\).

**Proof.** Let \(x_1\) and \(y_1\) be in \(X\) satisfying (3.3.4.3). Now define the sequences \(\{z_n\}\) and \(\{w_n\}\) from (3.3.4.1) as follows:

\[
\begin{align*}
  z_{2n+1} & = F(x_{2n+1}, y_{2n+1}) = Tx_{2n+2}, z_{2n+2} = G(x_{2n+2}, y_{2n+2}) = Sx_{2n+3}, \\
  w_{2n+1} & = F(y_{2n+1}, x_{2n+1}) = Ty_{2n+2}, w_{2n+2} = G(y_{2n+2}, x_{2n+2}) = Sy_{2n+3},
\end{align*}
\]

for \(n = 0, 1, 2, 3, \ldots\)

From (3.3.4.3)(a), we have

\[
\alpha((Sx_1, Sy_1), (F(x_1, y_1), F(y_1, x_1))) \geq 1
\]

\[ \Rightarrow \alpha((Sx_1, Sy_1), (Tx_2, Ty_2)) \geq 1, \quad \text{from definition of } \{z_n\} \text{ and } \{w_n\} \]

\[ \Rightarrow \alpha((F(x_1, y_1), F(y_1, x_1)), (G(x_2, y_2), G(y_2, x_2))) \geq 1, \quad \text{from (3.3.4.4)} \]

\[ \Rightarrow \alpha((z_1, w_1), (z_2, w_2)) \geq 1 \quad \text{from definition of } \{z_n\} \text{ and } \{w_n\} \]

\[ \Rightarrow \alpha((Tx_2, Ty_2), (Sx_3, Sy_3)) \geq 1, \quad \text{from definition of } \{z_n\} \text{ and } \{w_n\} \]

\[ \Rightarrow \alpha((G(x_2, y_2), G(y_2, x_2)), (F(x_3, y_3), F(y_3, x_3))) \geq 1, \quad \text{from (3.3.4.4)} \]

\[ \Rightarrow \alpha((z_2, w_2), (z_3, w_3)) \geq 1. \]

Continuing in this way, we have

\[ \alpha((z_n, w_n), (z_{n+1}, w_{n+1})) \geq 1, \quad \forall \ n. \quad (1) \]

Similarly from (3.3.4.3)(c), (3.3.4.3)(b) and (3.3.4.3)(d) we can obtain

\[ \alpha((z_{n+1}, w_{n+1}), (z_n, w_n)) \geq 1. \quad (2) \]

\[ \alpha((w_n, z_n), (w_{n+1}, z_{n+1})) \geq 1. \quad (3) \]

\[ \alpha((w_{n+1}, z_{n+1}), (w_n, z_n)) \geq 1. \quad (4) \]
Let \( R_n = \max \{ p(z_n, z_{n+1}), p(w_n, w_{n+1}) \} \).

**Case (i):** Suppose \( R_{2m} = 0 \) for some \( m \).

Then \( z_{2m} = z_{2m+1} \) and \( w_{2m} = w_{2m+1} \).

Now

\[
\alpha((Sx_{2m+1}, Sy_{2m+1}), (Tx_{2m+2}, Ty_{2m+2})) = \alpha((z_{2m}, w_{2m}), (z_{2m+1}, w_{2m+1})) \geq 1
\]

from (1).

\[
\psi(p(z_{2m+1}, z_{2m+2})) = \psi(p(F(x_{2m+1}, y_{2m+1}), G(x_{2m+2}, y_{2m+2}))) 
\leq \alpha((Sx_{2m+1}, Sy_{2m+1}), (Tx_{2m+2}, Ty_{2m+2}))
\psi(p(F(x_{2m+1}, y_{2m+1}), G(x_{2m+2}, y_{2m+2})))
\leq \phi(M_{x_{2m+1}, y_{2m+1}}^{x_{2m+2}, y_{2m+2}}) - \varphi(M_{x_{2m+1}, y_{2m+1}}^{x_{2m+2}, y_{2m+2}})
\]

where

\[
M_{x_{2m+1}, y_{2m+1}}^{x_{2m+2}, y_{2m+2}} = \max \left\{ \begin{array}{c}
p(z_{2m}, z_{2m+1}), p(w_{2m}, w_{2m+1}), p(z_{2m}, z_{2m+1}), \\
p(w_{2m}, w_{2m+1}), p(z_{2m+1}, z_{2m+2}), p(w_{2m+1}, z_{2m+2}), \\
\frac{1}{2}[p(z_{2m}, z_{2m+2}) + p(z_{2m+1}, z_{2m+1})], \\
\frac{1}{2}[p(w_{2m}, z_{2m+2}) + p(w_{2m+1}, w_{2m+1})]
\end{array} \right\}.
\]

But

\[
\frac{1}{2}[p(z_{2m}, z_{2m+2}) + p(z_{2m+1}, z_{2m+1})] \leq \frac{1}{2}[p(z_{2m}, z_{2m+1}) + p(z_{2m+1}, z_{2m+2})] 
\leq \max \{ p(z_{2m}, z_{2m+1}), p(z_{2m+1}, z_{2m+2}) \}.
\]

Hence \( M_{x_{2m+1}, y_{2m+1}}^{x_{2m+2}, y_{2m+2}} = \max \{ R_{2m}, R_{2m+1} \} = R_{2m+1} \).

Thus \( \psi(p(z_{2m+1}, z_{2m+2})) \leq \phi(R_{2m+1}) - \varphi(R_{2m+1}) \).

Similarly using (3), we can show that

\[
\psi(p(w_{2m+1}, w_{2m+2})) \leq \phi(R_{2m+1}) - \varphi(R_{2m+1}).
\]
Thus
\[
\psi(R_{2m+1}) = \psi \left( \max \{ p(z_{2m+1}, z_{2m+2}), p(w_{2m+1}, w_{2m+2}) \} \right)
\]
\[
= \max \{ \psi(p(z_{2m+1}, z_{2m+2})), \psi(p(w_{2m+1}, w_{2m+2})) \}
\]
\[
\leq \phi(R_{2m+1}) - \varphi(R_{2m+1})
\]
which in turn yields that \(R_{2m+1} = 0\) so that \(z_{2m+1} = z_{2m+2}\) and \(w_{2m+1} = w_{2m+2}\).

Continuing in this way, we get \(z_{2m} = z_{2m+1} = z_{2m+2} = \cdots\) and
\(w_{2m} = w_{2m+1} = w_{2m+2} = \cdots\).

Hence \(\{z_n\}\) and \(\{w_n\}\) are Cauchy sequences in \((X, p)\).

**Case (ii):** Assume that \(R_n \neq 0, \forall n\).

As in Case (i), we have
\[
\psi(R_{2n+1}) \leq \phi(\max\{R_{2n}, R_{2n+1}\}) - \varphi(\max\{R_{2n}, R_{2n+1}\}).
\]
If \(\max\{R_{2n}, R_{2n+1}\} = R_{2n+1}\) then \(\psi(R_{2n+1}) \leq \phi(R_{2n+1}) - \varphi(R_{2n+1})\) which in turn yields that \(R_{2n+1} = 0\). It is a contradiction. Hence
\[
\psi(R_{2n+1}) \leq \phi(R_{2n}) - \varphi(R_{2n})
\]
\[
< \psi(R_{2n}), \text{ from (A).}
\]

Since \(\psi\) is increasing, we have \(R_{2n+1} \leq R_{2n}\).

Using (2) and (4), we can show that \(R_{2n} \leq R_{2n-1}\).

Continuing in this way, we can conclude that \(\{R_n\}\) is a non-increasing sequence of non-negative real numbers and must converge to a real number, say, \(r \geq 0\).

Suppose \(r > 0\).

Letting \(n \to \infty\) in (5), we get
\[
\psi(r) \leq \phi(r) - \varphi(r)
\]
\[
< \psi(r), \text{ from (A).}
\]
Hence \( r = 0 \). Thus

\[
\lim_{n \to \infty} p(z_n, z_{n+1}) = 0 = \lim_{n \to \infty} p(w_n, w_{n+1}) \tag{6}
\]

Hence from \((p_2)\), we have

\[
\lim_{n \to \infty} p(z_n, z_n) = 0 = \lim_{n \to \infty} p(w_n, w_n) \tag{7}
\]

From (6), (7) and from the definition of \( d_p \), we have

\[
\lim_{n \to \infty} d_p(z_n, z_{n+1}) = 0 = \lim_{n \to \infty} d_p(w_n, w_{n+1}) \tag{8}
\]

Now we will show that \( \{z_{2n}\} \) and \( \{w_{2n}\} \) are Cauchy sequences in the metric space \((X, d_p)\). On the contrary, suppose that \( \{z_{2n}\} \) or \( \{w_{2n}\} \) is not Cauchy. This implies that

\[
\max\{d_p(z_{2m}, z_{2n}), d_p(w_{2m}, w_{2n})\} \not\to 0 \quad \text{as} \quad m, n \to \infty
\]

Then there exist an \( \epsilon > 0 \) and monotone increasing sequences of natural numbers \( \{2m_k\} \) and \( \{2n_k\} \) such that \( n_k > m_k > k \),

\[
\max\{d_p(z_{2m_k}, z_{2n_k}), d_p(w_{2m_k}, w_{2n_k})\} \geq \epsilon \tag{9}
\]

and

\[
\max\{d_p(z_{2m_k}, z_{2n_k-2}), d_p(w_{2m_k}, w_{2n_k-2})\} < \epsilon \tag{10}
\]

From (9),

\[
\epsilon \leq \max\{d_p(z_{2m_k}, z_{2n_k}), d_p(w_{2m_k}, w_{2n_k})\}
\]

\[
\leq \max\left\{ d_p(z_{2m_k}, z_{2n_k-2}) + d_p(z_{2n_k-2}, z_{2n_k-1}) + d_p(z_{2n_k-1}, z_{2n_k}),
\right.
\]

\[
\left. d_p(w_{2m_k}, w_{2n_k-2}) + d_p(w_{2n_k-2}, w_{2n_k-1}) + d_p(w_{2n_k-1}, w_{2n_k}) \right\}
\]

\[
\leq \max\{d_p(z_{2m_k}, z_{2n_k-2}), d_p(w_{2m_k}, w_{2n_k-2})\}
\]

\[
+ \max\left\{ d_p(z_{2n_k-2}, z_{2n_k-1}) + d_p(z_{2n_k-1}, z_{2n_k}),
\right.
\]

\[
\left. d_p(w_{2n_k-2}, w_{2n_k-1}) + d_p(w_{2n_k-1}, w_{2n_k}) \right\}
\]
Letting $k \to \infty$ and using (10) and (8), we get

$$\lim_{k \to \infty} \max\{d_p(z_{2m_k}, z_{2n_k}), d_p(w_{2m_k}, w_{2n_k}) = \epsilon$$

(11)

Using definition of $d_p$ and (7), we get

$$\lim_{k \to \infty} \max\{p(z_{2m_k}, z_{2n_k}), p(w_{2m_k}, w_{2n_k}) = \epsilon$$

(12)

Letting $k \to \infty$ and then using (11), (8) in

$$\left| \max\{d_p(z_{2m_k+1}, z_{2n_k}), d_p(w_{2m_k+1}, w_{2n_k}) - \max\{d_p(z_{2m_k}, z_{2n_k}), d_p(w_{2m_k}, w_{2n_k})\} \right|$$

we get

$$\lim_{k \to \infty} \max\{d_p(z_{2m_k+1}, z_{2n_k}), d_p(w_{2m_k+1}, w_{2n_k}) = \epsilon$$

so that

$$\lim_{k \to \infty} \max\{p(z_{2m_k+1}, z_{2n_k}), p(w_{2m_k+1}, w_{2n_k}) = \frac{\epsilon}{2}$$

(13)

Letting $k \to \infty$ and then using (11), (8) in

$$\left| \max\{d_p(z_{2m_k+1}, z_{2n_k-1}), d_p(w_{2m_k+1}, w_{2n_k-1}) - \max\{d_p(z_{2m_k}, z_{2n_k}), d_p(w_{2m_k}, w_{2n_k})\} \right|$$

we get

$$\lim_{k \to \infty} \max\{d_p(z_{2m_k+1}, z_{2n_k}), d_p(w_{2m_k+1}, w_{2n_k-1}) = \epsilon$$

so that

$$\lim_{k \to \infty} \max\{p(z_{2m_k+1}, z_{2n_k-1}), p(w_{2m_k+1}, w_{2n_k-1}) = \frac{\epsilon}{2}$$

(14)
Letting $k \to \infty$ and then using (11), (8) in

\[
\max \{d_p(z_{2m_k}, z_{2n_k-1}), d_p(w_{2m_k}, w_{2n_k-1}) - \max \{d_p(z_{2m_k}, z_{2n_k}), d_p(w_{2m_k}, w_{2n_k})\}
\]

\[
\leq \max \{d_p(z_{2n_k}, z_{2n_k-1}), d_p(w_{2n_k}, w_{2n_k-1})\}
\]

we get

\[
\lim_{k \to \infty} \max \{d_p(z_{2m_k}, z_{2n_k-1}), d_p(w_{2m_k}, w_{2n_k-1}) = \epsilon
\]

so that

\[
\lim_{k \to \infty} \max \{p(z_{2m_k}, z_{2n_k-1}), p(w_{2m_k}, w_{2n_k-1}) = \frac{\epsilon}{2}
\]

(15)

Now by using (1) and triangular property of $\alpha$, we have

\[
\alpha((Sx_{2m_k+1}, Sy_{2m_k+1}), (Tx_{2n_k}, Ty_{2n_k})) = \alpha((z_{2m_k}, w_{2m_k}), (z_{2n_k-1}, w_{2n_k-1})) \geq 1.
\]

Consider

\[
\psi(p(z_{2m_k+1}, z_{2n_k}))
\]

\[
= \psi(p(F(x_{2m_k+1}, y_{2m_k+1}), G(x_{2n_k}, y_{2n_k})))
\]

\[
\leq \alpha((Sx_{2m_k+1}, Sy_{2m_k+1}), (Tx_{2n_k}, Ty_{2n_k}))\psi(p(F(x_{2m_k+1}, y_{2m_k+1}), G(x_{2n_k}, y_{2n_k})))
\]

\[
\leq \phi(M_{x_{2m_k+1}, y_{2m_k+1}}^{x_{2n_k}, y_{2n_k}}) - \varphi(M_{x_{2m_k+1}, y_{2m_k+1}}^{x_{2n_k}, y_{2n_k}})
\]

where

\[
M_{x_{2m_k+1}, y_{2m_k+1}}^{x_{2n_k}, y_{2n_k}} = \max \left\{ \begin{array}{c}
p(z_{2m_k}, z_{2n_k-1}), p(w_{2m_k}, w_{2n_k-1}), p(z_{2m_k}, z_{2m_k+1}), \\
p(w_{2m_k}, w_{2m_k+1}), p(z_{2n_k-1}, z_{2n_k}), p(w_{2n_k-1}, z_{2n_k}), \\
\frac{1}{2}[p(z_{2m_k}, z_{2n_k}) + p(z_{2n_k-1}, z_{2m_k+1})], \\
\frac{1}{2}[p(w_{2m_k}, z_{2n_k}) + p(w_{2n_k-1}, w_{2m_k+1})]
\end{array} \right\}
\]

\[
\to \frac{\epsilon}{2} \text{ from (15), (6), (12), (14).}
\]

Thus

\[
\lim_{k \to \infty} \psi(p(z_{2m_k+1}, z_{2n_k})) < \phi(\frac{\epsilon}{2}) - \varphi(\frac{\epsilon}{2}).
\]

Similarly using (3) and (3.3.4.2) we can show that

\[
\lim_{k \to \infty} \psi(p(w_{2m_k+1}, w_{2n_k})) < \phi(\frac{\epsilon}{2}) - \varphi(\frac{\epsilon}{2}).
\]

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Since $\psi$ is continuous, from (13), we have

$$\psi\left(\frac{\epsilon}{2}\right) = \lim_{k \to \infty} \psi(\max\{p(z_{2m_k+1}, z_{2n_k}), p(w_{2m_k+1}, w_{2n_k})\})$$

$$= \max\left\{ \lim_{k \to \infty} \psi(p(z_{2m_k+1}, z_{2n_k})), \lim_{k \to \infty} \psi(p(w_{2m_k+1}, w_{2n_k})) \right\}$$

$$\leq \phi\left(\frac{\epsilon}{2}\right) - \varphi\left(\frac{\epsilon}{2}\right) < \psi\left(\frac{\epsilon}{2}\right), \text{ from (A).}$$

It is a contradiction. Hence $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences in the metric space $(X, d_p)$.

Letting $n, m \to \infty$ and using (11), (8) in

$$| \max\{d_p(z_{2n+1}, z_{2m+1}), d_p(w_{2n+1}, w_{2m+1}) - \max\{d_p(z_{2n}, z_{2m}), d_p(w_{2n}, w_{2m})\} |$$

$$\leq \max \left\{ d_p(z_{2n+1}, z_{2n}) + d_p(z_{2m}, z_{2m+1}), d_p(w_{2n+1}, w_{2n}) + d_p(w_{2m}, w_{2m+1}) \right\}$$

we get

$$\lim_{n, m \to \infty} d_p(z_{2n+1}, z_{2m+1}) = 0 = \lim_{n, m \to \infty} d_p(w_{2n+1}, w_{2m+1}).$$

Thus $\{z_{2m+1}\}$ and $\{w_{2m+1}\}$ are Cauchy in $(X, d_p)$. Hence $\{z_n\}$ and $\{w_n\}$ are Cauchy in $(X, d_p)$. Hence

$$\lim_{n, m \to \infty} d_p(z_n, z_m) = 0 = \lim_{n, m \to \infty} d_p(w_n, w_m)$$

From the definition of $d_p$ and (7), it follows that

$$\lim_{n, m \to \infty} p(z_n, z_m) = 0 = \lim_{n, m \to \infty} p(w_n, w_m)$$

(16)

Thus $\{z_n\}$ and $\{w_n\}$ are Cauchy in $(X, p)$. Since $(X, p)$ is a complete partial metric space, there exist $z, w \in X$ such that

$$p(z, z) = \lim_{n \to \infty} p(z_n, z) = \lim_{n, m \to \infty} p(z_n, z_m)$$

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and
\[ p(w, w) = \lim_{n \to \infty} p(w_n, w) = \lim_{n, m \to \infty} p(w_n, w_m) \]

From (16), we have
\[ p(z, z) = o = p(w, w) \tag{17} \]

Hence
\[ p(z, z) = \lim_{n \to \infty} p(F(x_{2n+1}, y_{2n+1}), z) = \lim_{n \to \infty} p(G(x_{2n+2}, y_{2n+2}), z) \]
\[ = \lim_{n \to \infty} p(Tx_{2n}, z) = \lim_{n \to \infty} p(Sx_{2n+1}, z) = 0 \tag{18} \]

and
\[ p(w, w) = \lim_{n \to \infty} p(F(y_{2n+1}, x_{2n+1}), w) = \lim_{n \to \infty} p(G(y_{2n+2}, x_{2n+2}), w) \]
\[ = \lim_{n \to \infty} p(Ty_{2n}, w) = \lim_{n \to \infty} p(Sy_{2n+1}, w) = 0 \tag{19} \]

Since the pair \((F, S)\) is partial\(^(*)\) compatible, from (17), we have
\[ p(Sz, Sz) = 0 = p(Sw, Sw) \]

and
\[ \lim_{n \to \infty} p(S(F(x_{2n+1}, y_{2n+1})), F(Sx_{2n+1}, Sy_{2n+1})) = 0 \tag{20} \]

and
\[ \lim_{n \to \infty} p(S(F(y_{2n+1}, x_{2n+1})), F(Sy_{2n+1}, Sx_{2n+1})) = 0 \tag{21} \]

Since \(S\) is continuous at \(z\) and \(w\) we have
\[ \lim_{n \to \infty} p(SSx_{2n+1}, Sz) = p(Sz, Sz) = 0 = \lim_{n \to \infty} p(S(F(x_{2n+1}, y_{2n+1})), Sz) \tag{22} \]
\[ \lim_{n \to \infty} p(SSy_{2n+1}, Sw) = p(Sw, Sw) = 0 = \lim_{n \to \infty} p(S(F(y_{2n+1}, x_{2n+1})), Sw) \tag{23} \]

Now from (3.3.4.6)(i) we have
\[ \alpha((SSx_{2n+1}, SSy_{2n+1}), (Tx_{2n+2}, Ty_{2n+2})) \]
\[ = \alpha((Sz_{2n}, Sw_{2n}), (z_{2n+1}, w_{2n+1})) \geq 1 \]
Letting \( n \to \infty \) and using (22) and (18) in
\[
| p(SSx_{2n+1}, Tx_{2n+2}) - p(Sz, z) | \leq p(SSx_{2n+1}, Sz) + p(z, Tx_{2n+2})
\]
we get
\[
\lim_{n \to \infty} p(SSx_{2n+1}, Tx_{2n+2}) = p(Sz, z) \tag{25}
\]

Letting \( n \to \infty \) and using (23) and (19) in
\[
| p(SSy_{2n+1}, Ty_{2n+2}) - p(Sw, w) | \leq p(SSy_{2n+1}, Sw) + p(w, Ty_{2n+2})
\]
we get
\[
\lim_{n \to \infty} p(SSy_{2n+1}, Ty_{2n+2}) = p(Sw, w) \tag{26}
\]

\[
p(SSx_{2n+1}, F(Sx_{2n+1}, Sy_{2n+1})) \leq p(SSx_{2n+1}, Sz) + p(Sz, S(F(x_{2n+1}, y_{2n+1})))
\]
\[
+ p(S(F(x_{2n+1}, y_{2n+1})), F(Sx_{2n+1}, y_{2n+1}))
\]
\[
\to 0 \text{ as } n \to \infty \text{ from (22) and (20)} \tag{27}
\]

\[
p(SSy_{2n+1}, F(Sy_{2n+1}, Sx_{2n+1})) \leq p(SSy_{2n+1}, Sw) + p(Sw, S(F(y_{2n+1}, x_{2n+1})))
\]
\[
+ p(S(F(y_{2n+1}, x_{2n+1})), F(Sy_{2n+1}, x_{2n+1}))
\]
\[
\to 0 \text{ as } n \to \infty \text{ from (23) and (21)} \tag{28}
\]

Letting \( n \to \infty \) and using (22) and (18) in
\[
| p(SSx_{2n+1}, G(x_{2n+2}, y_{2n+2})) - p(Sz, z) | \leq p(SSx_{2n+1}, Sz) + p(z, G(x_{2n+2}, y_{2n+2}))
\]
we get
\[
\lim_{n \to \infty} p(SSx_{2n+1}, G(x_{2n+2}, y_{2n+2})) = p(Sz, z) \tag{29}
\]

Letting \( n \to \infty \) and using (18), (22), (20) in
\[
| p(Tx_{2n+2}, F(Sx_{2n+1}, Sy_{2n+1})) - p(Sz, z) | \leq p(Tx_{2n+2}, sz) + p(Sz, S(F(x_{2n+1}, y_{2n+1})))
\]
\[
+ p(S(F(x_{2n+1}, y_{2n+1})), F(Sx_{2n+1}, Sy_{2n+1}))
\]
we get
\[
\lim_{n \to \infty} p(Tx_{2n+2}, F(Sx_{2n+1}, Sy_{2n+1})) = p(Sz, z) \tag{30}
\]

Letting \(n \to \infty\) and using (23) and (19) in
\[
| p(SSy_{2n+1}, G(y_{2n+2}, x_{2n+2})) - p(Sw, w)| \leq p(SSy_{2n+1}, Sw) + p(w, G(y_{2n+2}, x_{2n+2}))
\]
we get
\[
\lim_{n \to \infty} p(SSy_{2n+1}, G(y_{2n+2}, x_{2n+2})) = p(Sw, w) \tag{31}
\]

Letting \(n \to \infty\) and using (19), (23) and (21) in
\[
| p(Ty_{2n+2}, F(Sy_{2n+1}, Sx_{2n+1})) - p(Sw, w)| \leq p(Ty_{2n+2}, w) + p(Sw, S(F(y_{2n+1}, x_{2n+1}))
+ p(S(F(y_{2n+1}, x_{2n+1}), F(Sy_{2n+1}, Sx_{2n+1}))
\]
we get
\[
\lim_{n \to \infty} p(Ty_{2n+2}, F(Sy_{2n+1}, Sx_{2n+1})) = p(Sw, w) \tag{32}
\]

Since \(\psi\) is monotonically increasing we have
\[
\psi(p(Sz, z)) \leq \psi \left( p(Sz, S(F(x_{2n+1}, y_{2n+1}))) + p(S(F(x_{2n+1}, y_{2n+1}), F(Sx_{2n+1}, Sy_{2n+1}))
+ p(F(Sx_{2n+1}, Sy_{2n+1}), G(x_{2n+2}, y_{2n+2})) + p(G(x_{2n+2}, y_{2n+2}), z) \right)
\]

Letting \(n \to \infty\) and using continuity of \(\psi\) and (3.3.4.2), we get
\[
\psi(p(Sz, z)) \leq \lim_{n \to \infty} \psi(p(F(Sx_{2n+1}, Sy_{2n+1}), G(x_{2n+2}, y_{2n+2}))
\leq \lim_{n \to \infty} \alpha((SSx_{2n+1}, SSy_{2n+1}), (Tx_{2n+2}, Ty_{2n+2}))
\psi(p(F(Sx_{2n+1}, Sy_{2n+1}), G(x_{2n+2}, y_{2n+2}))), from (24)
\leq \lim_{n \to \infty} \left[ \phi(M^x_{2n+1, y_{2n+2}}) - \varphi(M^y_{2n+1, x_{2n+1}}, Sx_{2n+1}, Sy_{2n+1}) \right]
where
\[
M_{Sx^{2n+1}, S'y^{2n+1}}^{x^{2n+2}, y^{2n+2}} = \max \left\{ \begin{array}{l}
p(SSx_{2n+1}, Tx_{2n+2}), p(SSy_{2n+1}, Ty_{2n+2}), \\
p(SSx_{2n+1}, F(Sx_{2n+1}, Sy_{2n+1})), p(SSy_{2n+1}, F(Sy_{2n+1}, Sx_{2n+1})), \\
p(Tx_{2n+2}, G(x_{2n+2}, y_{2n+2})), p(Ty_{2n+2}, G(y_{2n+2}, x_{2n+2})), \\
\frac{1}{2}[p(SSx_{2n+1}, G(x_{2n+2}, y_{2n+2})) + p(Tx_{2n+2}, F(Sx_{2n+1}, Sy_{2n+1}))], \\
\frac{1}{2}[p(SSy_{2n+1}, G(y_{2n+2}, x_{2n+2})) + p(Ty_{2n+2}, F(Sy_{2n+1}, Sx_{2n+1}))]
\end{array} \right\}
\]

\rightarrow \max\{p(Sz, z), p(Sw, w)\} \text{ from (25) – (32) and (6).}

Thus
\[
\psi(p(Sz, z)) \leq \phi(\max\{p(Sz, z), p(Sw, w)\}) - \varphi(\max\{p(Sz, z), p(Sw, w)\}).
\]

Similarly, using (3.3.4.6)(i) and proceeding as above, we can show that
\[
\psi(p(Sw, w)) \leq \phi(\max\{p(Sz, z), p(Sw, w)\}) - \varphi(\max\{p(Sz, z), p(Sw, w)\}).
\]

Thus
\[
\psi(\max\{p(Sz, z), p(Sw, w)\}) \leq \phi(\max\{p(Sz, z), p(Sw, w)\}) - \varphi(\max\{p(Sz, z), p(Sw, w)\}),
\]

which from (A) gives that \(p(Sz, z) = 0 = p(Sw, w)\) so that \(Sz = z\) and \(Sw = w\). Since the pair \((G, T)\) is partial\(^*\) compatible and \(T\) is continuous, we can show that \(Tz = z\) and \(Tz = w\) by using (3.3.4.6)(ii).

Now from (3.3.4.6)(iii) we have
\[
\alpha((Sz, Sw), (Tx_{2n+2}, Ty_{2n+2})) = \alpha((z, w), (z_{2n+1}, w_{2n+1})) \geq 1.
\]
Since $\psi$ is continuous and by Lemma 1.2.5, we have

$$\psi(p(F(z, w), z)) = \lim_{n \to \infty} \psi(p(F(z, w), G(x_{2n+2}, y_{2n+2})))$$

$$\leq \lim_{n \to \infty} \alpha((S_z, S_w), (T_{x_{2n+2}}, T_{y_{2n+2}}))$$

$$\psi(p(F(z, w), G(x_{2n+2}, y_{2n+2})))$$

$$\leq \lim_{n \to \infty} \left[ \phi \left( M_{z, w}^{x_{2n+2}, y_{2n+2}} \right) - \phi \left( M_{z, w}^{y_{2n+2}} \right) \right]$$

where

$$M_{z, w}^{x_{2n+2}, y_{2n+2}} = \max \left\{ \begin{array}{l}
p(z, z_{2n+1}), p(w, w_{2n+1}), \\
p(z, F(z, w)), p(w, F(w, z)), \\
p(z_{2n+1}, z_{2n+2}), p(w_{2n+1}, w_{2n+2}), \\
\frac{1}{2} [p(z, z_{2n+2}) + p(z_{2n+1}, F(z, w))], \\
\frac{1}{2} [p(w, w_{2n+2}) + p(w_{2n+1}, F(w, z))]
\end{array} \right\}$$

$$\to \max \{ p(z, F(z, w)), p(w, F(w, z)) \}$$

from (18), (19) and (6) and from Lemma 1.2.5.

Thus

$$\psi(p(F(z, w), z)) \leq \phi \left( \max \left\{ \begin{array}{l}
p(z, F(z, w)), \\
p(w, F(w, z))
\end{array} \right\} \right) - \phi \left( \max \left\{ \begin{array}{l}
p(z, F(z, w)), \\
p(w, F(w, z))
\end{array} \right\} \right).$$

Similarly by using (3.3.4.6)(iii), we can show that

$$\psi(p(F(z, w), z)) \leq \phi \left( \max \left\{ \begin{array}{l}
p(z, F(z, w)), \\
p(w, F(w, z))
\end{array} \right\} \right) - \phi \left( \max \left\{ \begin{array}{l}
p(z, F(z, w)), \\
p(w, F(w, z))
\end{array} \right\} \right).$$

Hence

$$\psi \left( \max \left\{ \begin{array}{l}
p(z, F(z, w)), \\
p(w, F(w, z))
\end{array} \right\} \right) \leq \phi \left( \max \left\{ \begin{array}{l}
p(z, F(z, w)), \\
p(w, F(w, z))
\end{array} \right\} \right) - \phi \left( \max \left\{ \begin{array}{l}
p(z, F(z, w)), \\
p(w, F(w, z))
\end{array} \right\} \right).$$
which from (A) gives that $F(z, w) = z$ and $F(w, z) = w$.

Similarly by using (3.3.4.6)(iv), we will prove that $G(z, w) = z$ and $G(w, z) = w$.

Thus $S_z = T_z = z = F(z, w) = G(z, w)$ and $S_w = T_w = w = F(w, z) = G(w, z)$.

Hence $(z, w)$ is a common fixed point of $F, G, S$ and $T$.

Suppose $(z', w')$ is another common fixed point of $F, G, S$ and $T$.

\[
\psi(p(z, z')) = \psi(p(F(z, w), G(z', w')))
\leq \alpha((S_z, S_w), (T_z', T_w'))\psi(p(F(z, w), G(z', w'))), \text{ from (3.3.4.7)}
\leq \phi(M^{z', w'}_{z, w}) - \varphi(M^{z', w'}_{z, w})
= \phi\left(\max\left\{\begin{array}{l} p(z, z') \\ p(w, w') \end{array}\right\}\right) - \varphi\left(\max\left\{\begin{array}{l} p(z, z') \\ p(w, w') \end{array}\right\}\right).
\]

Similarly from (3.3.4.7), we have

\[
\psi(p(w, w')) \leq \phi\left(\max\left\{\begin{array}{l} p(z, z') \\ p(w, w') \end{array}\right\}\right) - \varphi\left(\max\left\{\begin{array}{l} p(z, z') \\ p(w, w') \end{array}\right\}\right).
\]

Thus

\[
\psi\left(\max\left\{\begin{array}{l} p(z, z') \\ p(w, w') \end{array}\right\}\right) \leq \phi\left(\max\left\{\begin{array}{l} p(z, z') \\ p(w, w') \end{array}\right\}\right) - \varphi\left(\max\left\{\begin{array}{l} p(z, z') \\ p(w, w') \end{array}\right\}\right)
\]

which from (A) gives that $w = z$ and $w' = z'$.

Thus $(z, w)$ is the unique common coupled fixed point of $F, G, S$ and $T$.

Now we give an example to illustrate Theorem 3.3.4.

**Example 3.3.5.** Let $X = [0, 2]$ and $p(x, y) = \max\{x, y\}$, $\forall x, y \in X$.

Define $F, G : X \times X \to X$ by $F(x, y) = \frac{x^2 + y^2}{24}$ and $G(x, y) = \frac{x^2 + y^2}{72}$ and $S, T : X \to X$ by $Sx = \frac{x^2}{2}$ and $Tx = \frac{x^2}{3}$.  

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Define $\alpha : X^2 \times X^2 \to \mathbb{R}^+$ by $\alpha((x, y), (u, v)) = \begin{cases} 1, & \text{if } x, y, u, v \in [0, \sqrt{3}], \\ 0, & \text{otherwise} \end{cases}$

Let $\psi, \phi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\psi(t) = 4t$, $\phi(t) = 7t$ and $\varphi(t) = \frac{7}{2}t$.

Clearly $F(X \times X) \subseteq T(X)$ and $G(X \times X) \subseteq S(X)$.

To verify (3.3.4.2) we consider the following two cases.

**Case (a):** Suppose $x, y, u, v \in [0, \sqrt{3}]$.

Then $\alpha((Sx, Sy), (Tu, Tv)) = \alpha((x^2, \frac{y^2}{2}), (u^2, \frac{v^2}{3})) = 1$.

$$\alpha((Sx, Sy), (Tu, Tv))\psi(p(F(x, y), G(x, y))) = 4 \max \left\{\frac{x^2 + y^2}{24}, \frac{u^2 + v^2}{72}\right\}$$

$$= \max \left\{\frac{x^2 + y^2}{6}, \frac{u^2 + v^2}{18}\right\}$$

$$\leq \max \left\{\frac{x^2 + y^2}{6}, \frac{u^2 + v^2}{9}\right\}$$

$$\leq \max \left\{\frac{x^2}{6}, \frac{u^2}{9}\right\} + \max \left\{\frac{y^2}{6}, \frac{v^2}{9}\right\}$$

$$= \frac{1}{3}p(Sx, Tu) + \frac{1}{3}p(Sy, Tv)$$

$$\leq \frac{2}{3} \max\{p(Sx, Tu), p(Sy, Tv)\}$$

$$\leq \frac{2}{3}M_{x,y}u,v$$

$$\leq \frac{7}{2}M_{x,y}u,v$$

$$= \phi(M_{x,y}u,v) - \varphi(M_{x,y}u,v)$$

**Case (b):** Atleast one of $x, y, u, v \notin [0, \sqrt{3}]$.

Then $\alpha((Sx, Sy), (Tu, Tv)) = \alpha((\frac{x^2}{2}, \frac{y^2}{2}), (\frac{u^2}{3}, \frac{v^2}{3}))$.

**Sub case:** If $\frac{x^2}{2}, \frac{y^2}{2}, \frac{u^2}{3}, \frac{v^2}{3} \in (1, \sqrt{3}]$ then $\alpha((Sx, Sy), (Tu, Tv)) = 1$.

The inequality (3.3.4.2) holds as in case (a).

**Sub case:** If at least one of $\frac{x^2}{2}, \frac{y^2}{2}, \frac{u^2}{3}, \frac{v^2}{3} \in (\sqrt{3}, 2]$ then

$\alpha((Sx, Sy), (Tu, Tv)) = 0$. Hence (3.3.4.2) holds, since $0 \leq \frac{7}{2}M_{x,y}u,v$.

By definition of $\alpha$, the condition (3.3.4.3) with $x_1 = 0 = y_1$ and (3.3.4.4) are satisfied clearly.
To verify the partial\(^*\) compatibility of the pair\((F, S)\), let us consider the sequences \(\{x_n\}, \{y_n\}\) in \(X\) such that \(F(x_n, y_n) \to t, Sx_n \to t, F(y_n, x_n) \to t'\) and \(Sy_n \to t'\) for some \(t, t' \in X\) with \(p(t, t) = 0\) and \(p(t', t') = 0\). Then \(t = 0\) and \(t' = 0\).

Clearly \(F(x_n, y_n) \to t\) and \(Sx_n \to t\) implies

\[
\lim_{n \to \infty} p(F(x_n, y_n), t) = p(t, t) = 0 \text{ and } \lim_{n \to \infty} p(F(x_n, y_n), t) = 0.
\]

Thus \(\lim_{n \to \infty} \frac{x_n^2 + y_n^2}{24} = 0 = \lim_{n \to \infty} \frac{x_n^2}{2}\)

which in turn yields that \(x_n^2 \to 0\) and \(y_n^2 \to 0\).

Consider

\[
p(S(F(x_n, y_n)), F(Sx_n, Sy_n)) = \max \left\{ \frac{1}{2}(\frac{x_n^2 + y_n^2}{24}), \frac{1}{24}(\frac{x_n^4}{4} + \frac{y_n^4}{4}) \right\}
\]

\[
\to 0 \text{ as } n \to \infty.
\]

Similarly we can show that

\[
p(S(F(y_n, x_n)), F(Sy_n, Sx_n)) \to 0.\]

Thus the pair \((F, S)\) is partial\(^*\) compatible. Similarly we can show that the pair \((G, T)\) is partial\(^*\) compatible.

Clearly \(S\) and \(T\) are continuous on \(X\). Thus (3.3.4.5) is satisfied.

By definition of \(\alpha\), one can easily verify the conditions (3.3.4.6) and (3.3.4.7).

Thus all the conditions of Theorem 3.3.4 are satisfied and \((0,0)\) is the unique common fixed point of \(F, G, S\) and \(T\).