CHAPTER 1
CHAPTER-1
INTRODUCTION AND PRELIMINARIES

In recent times the study of fixed point theory has been gained an important role because of its wide applications in proving the existence and uniqueness of solutions of differential, integral, integro - differential and impulsive differential equations and in obtaining solutions of optimization problems, in Approximation theory and Non-linear Analysis. Further many fixed point theorems are used not only in various mathematical investigations but also problems in economics, game and computer theory.

In this chapter, we mention some known definitions, propositions and some main theorems in fixed point theory that are relevant to the content of this thesis.

Throughout this thesis, we denote \( \mathbb{R} \) as the set of all real numbers, \( \mathbb{R}^+ \) as the set of all non-negative real numbers, \( \mathbb{N} \) as the set of all natural numbers and \( \mathbb{C} \) as the set of all complex numbers.

Suppose that \( X \) is a non-empty set and \( T : X \rightarrow X \) is a self map on \( X \). If there is an element \( x \in X \) such that \( Tx = x \) then \( x \) is called a fixed point of \( T \) in \( X \).

Section 1.1 : BANACH FIXED POINT THEOREM
FOR SELF AND MULTI MAPS

The fundamental work in fixed point theory is due to Banach (1922), which is famous as “Banach Contraction Principle”.

**Theorem 1.1.1.** (Banach Contraction Principle [74]). Let \((X, d)\) be a complete metric space and \( T \) be a self map on \( X \) and \( 0 \leq k < 1 \) such that
\[
d(Tx, Ty) \leq kd(x, y), \quad \forall \ x, y \in X.
\]
Then \( T \) has a unique fixed point in \( X \).
Further for any \( x_0 \in X \), the sequence of iterates \( \{T^n x_0\} \) is Cauchy and its limit is the unique fixed point of \( T \).

**Definition 1.1.2.** Let \( X \) be a non-empty set and \( T_1, T_2 : X \to X \) be given self maps on \( X \).

1. If \( w = T_1x = T_2x \) for some \( x \in X \) then \( x \) is called a coincidence point of \( T_1 \) and \( T_2 \) and \( w \) is called a point of coincidence of \( T_1 \) and \( T_2 \).

2. If \( x = T_1x = T_2x \) for some \( x \in X \) then \( x \) is called a common fixed point of \( T_1 \) and \( T_2 \).

3. (Jungck and Rhoades[23]). If \( T_1T_2x = T_2T_1x \) whenever \( T_1x = T_2x \), \( x \in X \) then the pair \( (T_1, T_2) \) is said to be weakly compatible.

4. (Jungck[22]). The pair \( (T_1, T_2) \) is said to be compatible if

\[
\lim_{n \to \infty} d(T_1T_2x_n, T_2T_1x_n) = 0 \quad \text{whenever there exists a sequence } \{x_n\} \text{ in } X
\]

such that \( \lim_{n \to \infty} T_1x_n = \lim_{n \to \infty} T_2x_n = t \) for some \( t \in X \).

The study of fixed points for multi-valued contraction mappings using the Hausdorff metric was initiated by Nadler [75].

Let \( (X, d) \) be a metric space. We denote \( CB(X) \) as the family of all non-empty closed and bounded subsets of \( X \) and \( CL(X) \) as the set of all non-empty closed subsets of \( X \). For \( A, B \in CB(X) \) and \( x \in X \), we denote \( D(x, A) = \inf\{d(x, a) : a \in A\} \). Let \( H \) be the Hausdorff metric induced by the metric \( d \) on \( X \), that is,

\[
H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}
\]

for every \( A, B \in CB(X) \).

It is clear that for \( A, B \in CB(X) \) and \( a \in A \) we have \( d(a, B) \leq H(A, B) \).
Lemma 1.1.3. ([75]). Let $A, B \in CB(X)$ and $\epsilon > 0$. Then the for every $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.

Definition 1.1.4. ([75]). An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \rightarrow CB(X)$ if and only if $x \in Tx$.

In 1969, Nadler [75] extended the famous Banach Contraction Principle [74] from single-valued mapping to multi-valued mapping and proved the following fixed point theorem.

Theorem 1.1.5. ([75]). Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $CB(X)$. Assume that there exists $c \in [0, 1)$ such that

$$H(Tx, Ty) \leq c \ d(x, y), \ \forall \ x, y \in X.$$  

Then $T$ has a fixed point.

Now we give the basic definition of a partially ordered set as follows:

Definition 1.1.6. A partially ordered set is a set $X$ and a binary relation $\preceq$ denoted by $(X, \preceq)$ such that, $\forall \ a, b, c \in X$

1. $a \preceq a$ (reflexivity),

2. $a \preceq b$ and $b \preceq a$ implies $a = b$ (anti-symmetry) and

3. $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ (transitivity).

Definition 1.1.7. Let $(X, \preceq)$ be a partially ordered set and $x, y \in X$. We say that $x$ is comparable to $y$ if either $x \preceq y$ or $y \preceq x$.

Recently Abbas et al. [48] introduced the new concepts in a partially ordered set as follows.

Definition 1.1.8. ([48]). Let $(X, \preceq)$ be a partially ordered set and $f, g : X \rightarrow X$.

(i) $f$ is said to be a dominating map if $x \preceq fx$ for every $x \in X$.

(ii) $f$ is said to be a weak annihilator of $g$ if $fgx \preceq x$ for every $x \in X$. 
SUZUKI TYPE FIXED POINT THEOREM:

There are a lot of generalizations of Banach fixed point theorem. Suzuki [98] proved generalized versions of Banach’s and Edelstein’s basic results. The importance of Suzuki contraction theorem is that the contractive condition required to be satisfied not for all points of the domain of mapping involved in it. First we give the following theorem of Suzuki.

**Theorem 1.1.9.** ([98]). Let $(X,d)$ be a complete metric space and let $T$ be a mapping on $X$, define a non-increasing function $\theta$ from $[0,1)$ into $(\frac{1}{2}, 1]$ by

$$
\theta(r) = \begin{cases} 
1, & 0 \leq r \leq \frac{\sqrt{5} - 1}{2} \\
\frac{1 - r}{r^2}, & \frac{\sqrt{5} - 1}{2} < r \leq \frac{1}{\sqrt{2}} \\
\frac{1}{1 + r}, & \frac{1}{\sqrt{2}} < r \leq 1
\end{cases}
$$

Assume that $r \in [0,1)$ such that

$$
\theta(r)d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq r d(x, y),
$$

$\forall \ x, y \in X$. Then there exists a unique fixed point $z$ of $T$. Moreover,

$$
\lim_{n \to \infty} T^n x = z, \forall \ x \in X.
$$

Later an interesting and rich Suzuki-type fixed point theorems were developed. The existence of fixed points for various Suzuki-type fixed point theorems has been studied by many authors under different conditions. For details, we refer [12,18,31,85,99] etc.

**b-METRIC SPACES:**

Some problems, particularly the problem of the convergence of measurable functions with respect to a measure, lead to a generalization of notion of a metric. Using this idea, Czerwik [77] presented a generalization of the well known Banach fixed point theorem [74] in so called b-metric spaces. Consis-
tent with [77,79], we use the following notations and definitions.

**Definition 1.1.10.** ([77,79]). Let $X$ be a non-empty set and $s \geq 1$ a given real number. A function $d : X \times X \rightarrow \mathbb{R}^+$ is called a $b$-metric provided that, \( \forall x, y, z \in X \),

(i) \( d(x, y) = 0 \) if and only if $x = y$,

(ii) \( d(x, y) = d(y, x) \),

(iii) \( d(x, y) \leq s[d(x, z) + d(z, y)] \).

Note that a metric space is evidently a $b$-metric space. Czerwik [77,79] has shown that a $b$–metric on $X$ need not be a metric on $X$ (see also [53,54,78,83].

We cite the following lemmas from Czerwik [77, 78, 79] and Singh et.al.[83].

**Lemma 1.1.11.** Let $(X, d)$ be a $b$-metric space. For any $A, B, C \in CB(X)$ and any $x, y \in X$, we have the following:

(i) \( d(x, B) \leq d(x, b) \) for any $b \in B$,

(ii) \( \delta(A, B) \leq H(A, B) \),

(iii) \( d(x, B) \leq H(A, B) \) for any $x \in A$,

(iv) \( H(A, A) = 0 \),

(v) \( H(A, B) = H(B, A) \),

(vi) \( H(A, C) \leq s(H(A, B) + H(B, C)) \),

(vii) \( d(x, A) \leq s(d(x, y) + d(y, A)) \).

**Lemma 1.1.12.** Let $(X, d)$ be a $b$-metric space. Let $A$ and $B$ be in $CB(X)$. Then for each $\alpha > 0$ and for all $b \in B$ there exists $a \in A$ such that
$d(a, b) \leq H(A, B) + \alpha$.

**Lemma 1.1.13.** Let $(X, d)$ be a $b$-metric space. For $A \in CB(X)$ and $x \in X$, we have $d(x, A) = 0 \iff x \in \overline{A} = A$.

**Section 1.2 : PARTIAL METRIC SPACES AND FUZZY METRIC SPACES**

**PARTIAL METRIC SPACES :**

The basic notion of a partial metric space was introduced by S.G. Mathews [80] as a part of the study of denotational semantics of data flow networks. He presented a modified version of the Banach contraction principle. In fact, the partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory.

First we recall some definitions and results in partial metric spaces.

**Definition 1.2.1.** ([80]). A partial metric on a non-empty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that, $\forall x, y, z \in X$,

1. ($p_1$) $x = y \iff p(x, x) = p(x, y) = p(y, y),$
2. ($p_2$) $p(x, x) \leq p(x, y), p(y, y) \leq p(x, y),$
3. ($p_3$) $p(x, y) = p(y, x),$
4. ($p_4$) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$

The pair $(X, p)$ is called a *partial metric space* (PMS).

From the definition of a partial metric, the following are obvious.

(A) If $p(x, y) = 0$ then $x = y$.

(B) If $x \neq y$, then $p(x, y) > 0$.

(C) If $x = y$, $p(x, y)$ may not be 0. If $p$ is a partial metric on $X$, then the function $d_p : X \times X \to \mathbb{R}^+$ given by $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a
metric on $X$.

**Example 1.2.2.** ([80]) Consider $X = \mathbb{R}^+$ with $p(x, y) = \max\{x, y\}$. Then $(X, p)$ is a partial metric space. It is clear that $p$ is not a (usual) metric. Note that in this case $d_p(x, y) = |x - y|$.

We now state some basic topological notions (such as convergence, completeness, continuity) on partial metric spaces (See e.g. [29, 80, 92]).

**Definition 1.2.3.**

1. A sequence $\{x_n\}$ in the PMS $(X, p)$ converges to the limit $x \in X$ if and only if $p(x, x_n) = \lim_{n \to \infty} p(x, x_n)$.

2. A sequence $\{x_n\}$ in the PMS $(X, p)$ is called a Cauchy sequence if $\lim_{n,m \to \infty} p(x_n, x_m)$ exists and is finite.

3. A PMS $(X, p)$ is called complete if every Cauchy sequence $\{x_n\}$ in $X$ converges with respect to $\tau_p$, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)$.

4. A mapping $F : X \to X$ is said to be continuous at $x \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $F(B_p(x, \delta)) \subseteq B_p(Fx, \epsilon)$.

It is clear that if $F$ is continuous at $x \in X$ then $\{Fx_n\}$ converges to $Fx$ whenever the sequence $\{x_n\} \in X$ converges to $x$.

The following lemma is one of the basic result in PMS.

**Lemma 1.2.4.** ([29, 80, 92]).

(a) A sequence $\{x_n\}$ is a Cauchy sequence in the PMS $(X, p)$ if and only if it is a Cauchy sequence in the metric space $(X, d_p)$.

(b) A PMS $(X, p)$ is complete if and only if the metric space $(X, d_p)$ is complete.

Moreover
\[
\lim_{n \to \infty} d_p(x, x_n) = 0 \iff p(x, x) = \lim_{n \to \infty} p(x, x_n) = \lim_{n,m \to \infty} p(x_n, x_m).
\]

Next, we state a simple Lemma given by [92], which will be used further to prove our main results.

**Lemma 1.2.5.** Assume \( x_n \to z \) as \( n \to \infty \) in a PMS \( (X, p) \) such that \( p(z, z) = 0 \). Then \( \lim_{n \to \infty} p(x_n, y) = p(z, y) \) for every \( y \in X \).

In the year 1994, S. G. Matthews [80] proved the following partial contraction mapping theorem.

**Theorem 1.2.6.** (Theorem 5.3, [80]). (The partial metric contraction mapping theorem) Let \( (U, p) \) be a complete partial metric space and \( f : U \to U \) be such that \( p(f(x), f(y)) \leq cp(x, y), \forall x, y \in U \) and \( 0 \leq c < 1 \).

Then there exists a unique \( a \in U \) such that \( a = f(a) \) and \( p(a, a) = 0 \).

**FUZZY METRIC SPACES :**

The concept of fuzzy sets was introduced initially by Zadeh [45] in 1965. George and Veeramani [4] and Kramosil and Michalek [32] have introduced the concept of fuzzy topological spaces induced by fuzzy metric. Many authors, for example, [15,36,56,72,73,87,90,100] have proved fixed and common fixed point theorems in fuzzy metric spaces.

Now, we give the following preliminaries.

**Definition 1.2.7.** ([7]). A binary operation \( * : [0, 1] \times [0, 1] \to [0, 1] \) is a continuous \( t \)-norm if it satisfies the following conditions:

1. * is associative and commutative,

2. * is continuous,

3. \( a * 1 = a, \forall a \in [0, 1] \),

4. \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \), for each \( a, b, c, d \in [0, 1] \).
Two typical examples of a continuous $t$-norm are $a*b = ab$ and $a*b = \min\{a, b\}$.

**Definition 1.2.8.** ([4]). A 3-tuple $(X, M, \ast)$ is called a fuzzy metric space if $X$ is an arbitrary (non-empty) set, $\ast$ is a continuous $t$-norm and $M$ is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

$(M_1)$ $M(x, y, t) > 0$,  
$(M_2)$ $M(x, y, t) = 1$ if and only if $x = y$,  
$(M_3)$ $M(x, y, t) = M(y, x, t)$,  
$(M_4)$ $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$,  
$(M_5)$ $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Let $(X, M, \ast)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$  

If $(X, M, \ast)$ is a fuzzy metric space, let $\tau$ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then $\tau$ is a topology on $X$ (induced by the fuzzy metric $M$). This topology is Hausdorff and first countable.

A sequence $\{x_n\}$ in $X$ converges to $x$ if and only if $M(x_n, x, t) \rightarrow 1$ as $n \rightarrow \infty$, for each $t > 0$. It is called a G-Cauchy sequence in the sense of [4] if $\lim_{n \rightarrow \infty} M(x_n, x_{n+p}, t) = 1$, $\forall t > 0$ and each positive integer $p$. The fuzzy metric space $(X, M, \ast)$ is said to be $G$-complete if every G-Cauchy sequence is convergent. A subset $A$ of $X$ is said to be $F$-bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$, $\forall x, y \in A$.

**Example 1.2.9.** Let $X = \mathbb{R}$. Put $a \ast b = ab$, $\forall a, b \in [0, 1]$. For each $t \in (0, \infty)$, define $M(x, y, t) = \frac{t}{t + |x-y|}$, $\forall x, y \in X$. 

9
Example 1.2.10. Let $X = [0, 1]$ and $a * b = ab$, $\forall \ a, b \in [0, 1]$ and let $M$ be the fuzzy set on $X \times X \times (0, \infty)$ defined by

$$M(x, y, t) = e^{-\frac{|x-y|}{t}}, \forall \ t \geq 0.$$ 

Then $(X, M, *)$ is a fuzzy metric space.

Example 1.2.11. Let $X = [0, 1]$ and $a * b = ab$, $\forall \ a, b \in [0, 1]$ and let $M$ be the fuzzy set on $X \times X \times (0, \infty)$ defined by

$$M(x, y, t) = \left(\frac{t}{t+1}\right)^{|x-y|}, \forall \ t \geq 0.$$ 

Then $(X, M, *)$ is a fuzzy metric space.

Lemma 1.2.12.\textsuperscript{(56)}. Let $(X, M, *)$ be a fuzzy metric space. Then $M$ is non-decreasing with respect to $t$, $\forall \ x, y$ in $X$.

Definition 1.2.13.\textsuperscript{(35)}. Let $(X, M, *)$ be a fuzzy metric space. Then $M$ is said to be continuous on $X^2 \times (0, \infty)$ if $\lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t)$, whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$. i.e. $\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_n, y, t) = 1$ and $\lim_{n \to \infty} M(x, y, t_n) = M(x, y, t)$.

Lemma 1.2.14.\textsuperscript{(35)}. Let $(X, M, *)$ be a fuzzy metric space. Then $M$ is a continuous function on $X^2 \times (0, \infty)$.

Definition 1.2.15.\textsuperscript{(87)}. Let $(x, M, *)$ be a fuzzy metric space and $f, S : X \to X$. The pair $(f, S)$ is said to be compatible if $\lim_{n \to \infty} M(fSx_n, Sfx_n, t) = 1$ for every $t > 0$, whenever there exists a sequence $\{x_n\}$ in $X$ such that $fx_n \to z$ and $Sx_n \to z$ as $n \to \infty$ for some $z \in X$.

Section 1.3 : GERAGHTY FUNCTION, ALTERING DISTANCE FUNCTION AND ADMISSIBLE FUNCTIONS

GERAGHATY FUNCTION:

In 1973, Geraghty [55] introduced an interesting class of auxiliary
functions to refine the Banach contraction mapping principle. Let $\mathcal{F}$ denote all functions $\beta : \mathbb{R}^+ \to [0, 1)$ satisfying the condition

$$\lim_{n \to \infty} \beta(t_n) = 1 \quad \text{implies} \quad \lim_{n \to \infty} t_n = 0.$$  

By using the function $\beta \in \mathcal{F}$, Geraghty [55] proved the following remarkable theorem.

**Theorem 1.3.1.** ([55]). Let $(X, d)$ be a complete metric space and $T : X \to X$ be an operation. If $T$ satisfies $d(Tx, Ty) \leq \beta(d(x, y))$, $\forall x, y \in X$, where $\beta \in \mathcal{F}$. Then $T$ has a unique fixed point in $X$.

**ALTERING DISTANCE FUNCTION** :

**Definition 1.3.2.** Let $\Psi$ denote the set of all altering distance functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions

(a) $\psi$ is non-decreasing and continuous,

(b) $\psi(t) = 0 \Leftrightarrow t = 0$.

**ADMISSIBLE FUNCTIONS** :

Samet et.al [9] introduced the notion of $\alpha$- admissible mappings as follows

**Definition 1.3.3.** ([9]). Let $X$ be a non-empty set, $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$ be mappings. Then $T$ is called $\alpha$- admissible if $\forall x, y \in X$, we have $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Some interesting examples of such mappings are given in [9]. Actually they proved the following

**Theorem 1.3.4.** ([9]). Let $(X, d)$ be a complete metric space. Suppose that $\alpha : X \times X \to \mathbb{R}^+$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, where $\phi$ is non-decreasing and $\sum \phi^n(t) < \infty$ for each $t > 0$. Suppose that $T : X \to X$ satisfies

\begin{align*}
\alpha(x, y) \geq 1 & \quad \Rightarrow 
\alpha(Tx, Ty) \geq 1, \\
\sum \phi^n(t) & < \infty.
\end{align*}
\[\alpha(x, y)d(Tx, Ty) \leq \phi(d(x, y)), \forall \, x, y \in X.\]

Assume the following

(i) \(T\) is \(\alpha\)-admissible,

(ii) there exits \(x_0 \in X\) such that \(\alpha(x_0, Tx_0) \geq 1\),

(iii) either \(T\) is continuous or if \(\{x_n\}\) is a sequence in \(X\) with \(\alpha(x_n, x_{n+1}) \geq 1\),

\[\forall \, n \in \mathbb{N} \text{ and } x_n \to x \text{ as } n \to \infty, \text{ then } \alpha(x_n, x) \geq 1, \forall \, n \in \mathbb{N}.\]

Then \(T\) has a fixed point in \(X\).

Further, if for any \(x, y \in X\), there exists \(z \in X\) such that \(\alpha(x, z) \geq 1\) and \(\alpha(y, z) \geq 1\) then \(T\) has a unique fixed point in \(X\).

Recently, Karapinar et.al [19] defined the notion of triangular \(\alpha\)-admissible mappings as follows

**Definition 1.3.5.** ([19]). Let \(X\) be a non-empty set, \(T : X \to X\) and \(\alpha : X \times X \to \mathbb{R}^+\). Then \(T\) is called triangular \(\alpha\)-admissible if

\[(i) \, x, y \in X, \, \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1\]

\[(ii) \, x, y, z \in X, \, \alpha(x, z) \geq 1 \text{ and } \alpha(z, y) \geq 1 \Rightarrow \alpha(x, y) \geq 1.\]

Later Shahi et.al [68] and Abdeljawad [93] defined the following

**Definition 1.3.6.** ([68]). Let \(X\) be a non-empty set, \(f, g : X \to X\) and \(\alpha : X \times X \to \mathbb{R}^+\). Then \(f\) is said to be \(\alpha\)-admissible with respect to \(g\) if \(\alpha(gx, gy) \geq 1\) implies \(\alpha(fx, fy) \geq 1, \forall \, x, y \in X\).

**Definition 1.3.7.** ([93]). Let \(X\) be a non-empty set, \(f, g : X \to X\) and \(\alpha : X \times X \to \mathbb{R}^+\). Then the pair \((f, g)\) is said to be \(\alpha\)-admissible if \(\alpha(x, y) \geq 1\) implies \(\alpha(fx, gy) \geq 1\) and \(\alpha(gx, fy) \geq 1, \forall \, x, y \in X\).

In 2013, Salimi et al. [67] modified the concept of \(\alpha\)-admissible mappings as follows.
**Definition 1.3.8.** ([67]). Let $T$ be a self mapping on a metric space $(X, d)$ and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. Then $T$ is said to be $\alpha$-admissible mapping with respect to $\eta$ if $\alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty)$, $\forall \ x, y \in X$.

Very recently Babu et al.[24] extended the above definition for Jungck type maps as follows.

**Definition 1.3.9.** ([24]). Let $f$ and $g$ be two self mappings on a metric space $(X, d)$ and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. Then $f$ is said to be $(\alpha, g)$-admissible mapping with respect to $\eta$ if $\alpha(gx, gy) \geq \eta(gx, gy) \Rightarrow \alpha(fx, fy) \geq \eta(fx, fy)$, $\forall \ x, y \in X$.

**Section 1.4 : COMPLEX VALUED METRIC SPACES AND COMPLEX VALUED $b$-METRIC SPACES**

**COMPLEX VALUED METRIC SPACES :**

Azam et al.[2] introduced the notion of a complex valued metric space which is a generalization of the classical metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a rational contractive condition. Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. However, in complex valued metric spaces, one can study improvements of a host of results of analysis involving divisions. Later several authors proved fixed and common fixed point theorems in complex valued metric spaces, for example, refer[10,21,28,44,49,51,57,64,71,76,97,102]. First we refer the following preliminaries.
Let $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ follows:

$z_1 \preceq z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2), \text{Im}(z_1) \leq \text{Im}(z_2)$.

Thus $z_1 \preceq z_2$ if one of the following holds:

1. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
2. $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
3. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$,
4. $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$.

Clearly $z_1 \preceq z_2 \Rightarrow |z_1| \leq |z_2|$.

We will write $z_1 \preceq z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) is satisfied. Also we will write $z_1 \prec z_2$ if only (4) is satisfied.

**Remark 1.4.1.** One can easily check that the following statements:

(i) if $0 \preceq z_1 \preceq z_2$ then $|z_1| < |z_2|$;

(ii) if $z_1 \preceq z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

**Definition 1.4.2.** Let $X$ be a non-empty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on $X$ if $\forall x, y, z \in X$ the following conditions are satisfied:

(i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$.

The pair $(X, d)$ is called a complex valued metric space.

**Remark 1.4.3.** Let $(X, d)$ be a complex valued metric space. Then

(i) $|d(x, y)| < |1 + d(x, y)|, \forall x, y \in X$.

(ii) $|d(x, y)| \text{ or } |d(u, v)| < |1 + d(x, y) + d(u, v)|, \forall x, y, u, v \in X$.

(iii) $|d(x, y)| > 0$ if $x \neq y$.

**Definition 1.4.4.** Let $(X, d)$ be a complex valued metric space.
(i) A point \(x \in X\) is called interior point of a set \(A \subseteq X\) whenever there exists \(0 < r \in \mathbb{C}\) such that \(B(x, r) = \{y \in X : d(x, y) < r\} \subseteq A\).

(ii) A point \(x \in X\) is called a limit point of a set \(A \subseteq X\) whenever there exists \(0 < r \in \mathbb{C}\) such that \(B(x, r) \cap (X - A) \neq \emptyset\).

(iii) A subset \(B \subseteq X\) is called open whenever each point of \(B\) is an interior point of \(B\).

(iv) A subset \(B \subseteq X\) is called closed whenever each limit point of \(B\) is in \(B\).

(v) The family \(F = \{B(x, r) : x \in X \text{ and } 0 < r\}\) is a sub basis for a topology on \(X\). We denote this complex topology by \(\tau_c\). Indeed, the topology\(\tau_c\) is Hausdorff.

Let \(\{x_n\}\) be a sequence in \(X\) and \(x \in X\). If for every \(c \in \mathbb{C}\) with \(0 \leq c\) there is \(n_0 \in \mathbb{N}\) such that, \(\forall n > n_0, d(x_n, x) < c\), then \(\{x_n\}\) is said to be convergent to \(x\) and \(x\) is the limit point of \(\{x_n\}\). We denote this by \(\lim_{n \to \infty} x_n = x\) or \(x_n \to x\) as \(n \to \infty\). If for every \(c \in \mathbb{C}\) with \(0 < c\) there is \(n_0 \in \mathbb{N}\) such that, \(\forall n > n_0, d(x_n, x_{n+m}) < c\), where \(m \in \mathbb{N}\), then \(\{x_n\}\) is called a Cauchy sequence in \((X, d)\). If every Cauchy sequence is convergent in \((X, d)\) then \((X, d)\) is called a complete complex valued metric space. We require the following lemmas.

**Lemma 1.4.5.** ([2]) Let \((X, d)\) be a complex valued metric space and let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) converges to \(x\) if and only if \(|d(x_n, x)| \to 0\) as \(n \to \infty\).

**Lemma 1.4.6.** ([2]) Let \((X, d)\) be a complex valued metric space and let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is a Cauchy sequence if and only if \(|d(x_n, x_{n+m})|\)
→ 0 as \( n,m \to \infty \).

One can easily prove the following lemma

**Lemma 1.4.7.** Let \( (X,d) \) be a complex valued metric space and let \( \{x_n\} \) and \( \{y_n\} \) be sequences in \( X \) converging to \( x \) and \( y \) respectively. Then \( |d(x_n, y_n)| \to |d(x, y)| \) as \( n \to \infty \).

**COMPLEX VALUED \( b \)-METRIC SPACES :**

We now give the following definitions.

**Definition 1.4.8.** Let \( z_1 = (\alpha, \beta) \) and \( z_2 = (\gamma, \delta) \) be two complex numbers. Then \( \max\{z_1, z_2\} = (\max\{\alpha, \gamma\}, \max\{\beta, \delta\}) \).

Now, we introduce the notion of complex valued \( b \)-metric spaces and related definitions.

**Definition 1.4.9.** Let \( X \) be a non-empty set and \( s \geq 1 \) a given real number. A function \( d : X \times X \to \mathbb{C} \) is called a complex valued \( b \)-metric if it satisfies the following

(i) \( 0 \preccurlyeq d(x, y) \) and \( d(x, y) = 0 \) if and only if \( x = y, \forall \ x, y \in X \);

(ii) \( d(x, y) = d(y, x), \forall \ x, y \in X \);

(iii) \( d(x, y) \preccurlyeq s[d(x, z) + d(z, y)], \forall \ x, y, z \in X \).

The pair \( (X, d) \) is called a complex valued \( b \)-metric space.

**Remark 1.4.10.** Let \( (X, d) \) be a complex valued \( b \)-metric space. Then

(i) \( |d(x, y)| \) or \( |d(u, v)| < |1 + d(x, y) + d(u, v)|, \forall x, y, u, v \in X \).

(ii) If \( x \neq y \) then \( |d(x, y)| > 0 \).

(iii) For \( 0 \leq k < 1 \) and \( z, w \in \mathbb{C} \), if \( |z| \leq k |w| \) and \( |w| \leq k |z| \) then \( z = w = 0 \).

**Definition 1.4.11.** Let \( (X, d) \) be a complex valued \( b \)-metric space.

(i) A point \( x \in X \) is called interior point of a set \( A \subseteq X \) whenever there exists \( 0 \prec r \in \mathbb{C} \) such that \( B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A \).
(ii) A point \( x \in X \) is called limit point of a set \( A \subseteq X \) whenever for every \( 0 < r \in \mathbb{C} \) such that \( B(x, r) \cap (X - A) \neq \emptyset \).

(iii) A subset \( B \subseteq X \) is called open whenever each element of \( B \) is an interior point of \( B \).

(iv) A subset \( B \subseteq X \) is called closed whenever each limit point of \( B \) belongs to \( B \).

(v) The family \( F = \{B(x, r) : x \in X, \text{and } 0 < r\} \) is a sub basis for a topology on \( X \). We denote this complex topology by \( \tau_c \). Indeed, the topology \( \tau_c \) is Hausdorff.

**Definition 1.4.12.** Let \((X, d)\) be a complex valued b-metric space, and let \( \{x_n\} \) be a sequence in \( X \) and \( x \in X \).

(i) If for every \( c \in X \) with \( 0 < c \) there is \( n_0 \in \mathbb{N} \) such that, \( \forall n > n_0 \),

\[
d(x_n, x) < c,
\]

then \( \{x_n\} \) is convergent, \( \{x_n\} \) converges to \( x \) and \( x \) is limit point of \( \{x_n\} \). We denote this by \( x_n \to x \) as \( n \to \infty \) or \( \lim_{n \to \infty} x_n = x \).

(ii) If for every \( c \in X \) with \( 0 < c \) there is \( n_0 \in \mathbb{N} \) such that, \( \forall n > n_0 \),

\[
d(x_n, x_m) < c,
\]

where \( m \in \mathbb{N} \), then \( \{x_n\} \) is said to be Cauchy sequence.

(iii) If every Cauchy sequence is convergent in \((X, d)\), then \((X, d)\) is called a complete complex valued b-metric space.

One can prove the following lemmas in similar lines as in [2].

**Lemma 1.4.13.** Let \((X, d)\) be a complex valued b-metric space, and let \( \{x_n\} \) be a sequence in \( X \). Then, \( \{x_n\} \) converges to \( x \) if and only if \( |d(x_n, x)| \to 0 \) as \( n \to \infty \).

**Lemma 1.4.14.** Let \((X, d)\) be a complex valued b-metric space, and let \( \{x_n\} \) be a sequence in \( X \). Then, \( \{x_n\} \) is a Cauchy sequence if and only if \( |d(x_n, x_{n+m})| \to 0 \) as \( n, m \to \infty \).
One can easily prove the following lemma.

**Lemma 1.4.15.** Let \((X,d)\) be a complex valued \(b\)-metric space and let \(\{x_n\}\) and \(\{y_n\}\) be sequences in \(X\) converging to \(x\) and \(y\) respectively. Then

\[
\begin{align*}
(i) \quad & \frac{1}{s} |d(x, z)| \leq \lim_{n \to \infty} |d(x_n, z)| \leq s |d(x, z)|, \quad \forall \ z \in X, \\
(ii) \quad & \frac{1}{s^2} |d(x, y)| \leq \lim_{n \to \infty} |d(x_n, y_n)| \leq s^2 |d(x, y)|.
\end{align*}
\]

**Section 1.5: COUPLED FIXED POINTS**

Bhaskar and Lakshmikantham [94] introduced the concept of coupled fixed points and Lakshmikantham and Ciric [101] defined the common coupled fixed points. Later several authors proved coupled and common coupled fixed point theorems in various spaces. Now, we mention the following definitions which are needed for further discussions.

**Definition 1.5.1.** Let \(F : X \times X \to X\) and \(f : X \to X\) be mappings.

\(\star\)\(([101])\) An element \((x, y) \in X \times X\) is called a coupled coincidence point of \(F\) and \(f\) if \(fx = F(x, y)\) and \(fy = F(y, x)\).

\(\star\)\(([101])\) An element \((x, y) \in X \times X\) is called a common coupled fixed point of \(F\) and \(f\) if \(x = fx = F(x, y)\) and \(y = fy = F(y, x)\).

\(\star\)\(([47])\) The pair \((F, f)\) is called \(w\)-compatible if \(f(F(x, y)) = F(fx, fy)\)
whenever \(x, y \in X\) such that \(fx = F(x, y)\) and \(fy = F(y, x)\).

**Definition 1.5.2.**([26]). Let \(F, G : X \times X \to X\) be mappings. An element \((x, y) \in X \times X\) is called

\(\star\) coupled coincidence point of \(F\) and \(G\) if \(F(x, y) = G(x, y)\) and 
\[F(y, x) = G(y, x).\]

\(\star\) common coupled fixed point of \(F\) and \(G\) if \(x = F(x, y) = G(x, y)\) and 
\[y = F(y, x) = G(y, x).\]

\(\star\) The pair \((F, G)\) is called \(\tilde{w}\)-compatible if \(x, y \in X\) such that
\begin{align*}
F(x, y) &= G(x, y) \text{ and } F(y, x) = G(y, x) \Rightarrow F(G(x, y),\ G(y, x)).
\end{align*}

**Section 1.6 : DISLOCATED METRIC SPACES AND DISLOCATED QUASI \(D^*-\)METRIC SPACES**

**DISLOCATED METRIC SPACES :**


Now we give some known definitions in dq-metric spaces.

**Definition 1.6.1.** ([20]). Let \(X\) be a non-empty set and let \(d : X \times X \rightarrow \mathbb{R}^+\) be a function satisfying following conditions:

(i) \(d(x, y) = d(y, x) = 0\) implies \(x = y\),

(ii) \(d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X.\)

Then \(d\) is called a dislocated quasi-metric on \(X\).

If \(d\) satisfies \(d(x, x) = 0, \forall x \in X\) then the dislocated quasi-metric is called a quasi-metric on \(X\).

If \(d\) satisfies \(d(x, y) = d(y, x), \forall x, y \in X\) then the dislocated quasi-metric is called a dislocated metric on \(X\).

**Definition 1.6.2.** ([20]). A sequence \(\{x_n\}\) in dq-metric space (dislocated quasi-metric space) \((X, d)\) is called a Cauchy sequence if for given \(\epsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that, \(\forall m, n \geq n_0\) implies \(d(x_m, x_n) < \epsilon\) or \(d(x_n, x_m) < \epsilon\).

**Definition 1.6.3.** ([20]). A sequence \(\{x_n\}\) is said to be dislocated quasi-converges to \(x\) if \(\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0\). In this case \(x\) is called a dq-limit of \(\{x_n\}\) and we write \(x_n \rightarrow x.\)

**Lemma 1.6.4.** ([20]). dq-limit in a dq-metric space is unique.
Definition 1.6.5. ([20]). A $dq$-metric space $(X, d)$ is called complete if every Cauchy sequence in it is $dq$-convergent.

Definition 1.6.6. ([20]). Let $(X, d_1)$ and $(Y, d_2)$ be $dq$-metric spaces and let $f : X \to Y$ be a function. Then $f$ is said to be continuous at $x_0 \in X$, if the sequence $\{f(x_n)\}$ is $d_2q$- convergent to $f(x_0) \in Y$ whenever the sequence $\{x_n\}$ in $X$ is $d_1q$- convergent to $x_0$.

Definition 1.6.7. ([20]). Let $(X, d)$ be a $dq$-metric space. A map $T : X \to X$ is called contraction if there exists $0 \leq \lambda < 1$ such that $d(Tx, Ty) \leq \lambda d(x, y)$, $\forall \ x, y \in X$.

Theorem 1.6.8. ([20]). Let $(X, d)$ be a $dq$-metric space and let $T : X \to X$ be a continuous contraction mapping. Then $T$ has unique fixed point.

DISLOCATED QUASI $D^*$-METRIC SPACES:

Dhage [6] introduced the concept of $D$ - metric spaces and proved several fixed point theorems in it. Unfortunately almost all theorems are not valid (Refer [91]).

Recently Sedghi et. al. [89] introduced the concept of $D^*$ - metric spaces and proved some common fixed point theorems. Using $D^*$- metric concept, we introduce the dislocated quasi $D^*$ - metric on $X$ as follows.

Definition 1.6.9. Let $X$ be a non-empty set and $D^* : X \times X \times X \to \mathbb{R}^+$ be a function satisfying

$(D^*_1) : D^*(x, y, z) = 0$ implies $x = y = z$,

$(D^*_2) : D^*(x, y, z) \leq D^*(a, y, z) + D^*(x, a, a)$, $\forall \ x, y, z, a \in X$,

$(D^*_3) : D^*(x, y, y) = D^*(y, x, x)$, $\forall \ x, y \in X$.

Then $D^*$ is called a dislocated quasi $D^*$ - metric on $X$.

If further, $D^*$ satisfies
\((D^*_4)\) : \(D^*(x,y,z) = D^*(y,z,x) = \ldots \ldots \) (symmetry in all variables)

Then \(D^*\) is called a dislocated \(D^*\) - metric on \(X\).

**Definition 1.6.10.** A sequence \(\{x_n\}\) in dislocated quasi \(D^*\) - metric space \((X,D^*)\) is called Cauchy if for given \(\epsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(n,m \geq n_0\) implies \(D^*(x_m,x_n,x_n) < \epsilon\) or \(D^*(x_n,x_m,x_m) < \epsilon\).

**Definition 1.6.11.** A sequence \(\{x_n\}\) in dislocated quasi \(D^*\) - metric space \((X,D^*)\) converges to \(x \in X\) if

\[
\lim_{n \to \infty} D^*(x_n,x,x) = 0 \quad \text{or} \quad \lim_{n \to \infty} D^*(x,x_n,x) = 0 \quad \text{or} \quad \lim_{n \to \infty} D^*(x,x,x_n) = 0.
\]

In this case, we say that \(x\) is a dislocated quasi - limit of \(\{x_n\}\).

**Lemma 1.6.12.** In dislocated quasi \(D^*\) - metric space \((X,D^*)\), the dislocated quasi - limit of a sequence is unique.

**Proof :** Suppose \(x\) and \(y\) are dislocated quasi - limits of \(\{x_n\}\) in \(X\).

Now

\[
0 \leq D^*(y,x,x) \leq D^*(x_n,x,x) + D^*(y,x_n,x_n) \quad \text{from} \quad (D^*_2)
\]

\[
= D^*(x_n,x,x) + D^*(x_n,y,y) \quad \text{from} \quad (D^*_3)
\]

\[
\rightarrow 0 \quad \text{as} \quad n \to \infty.
\]

Hence \(D^*(y,x,x) = 0\) which implies that \(x = y\).

**SYNOPSIS OF THE THESIS**

This thesis is divided into nine chapters.

Chapter 1: *Introduction and Preliminaries*

In this Chapter we present some known basic notions like Fixed, Common fixed and Common coupled fixed points concepts regarding to self and multi-valued mappings and notions of partial metric spaces, complex valued metric spaces and fuzzy metric spaces and Suzuki type fixed point theorems and as well as the contents of the thesis.
Chapter 2: Common Fixed Point Theorems in Metric Spaces

Chapter 2 is divided into two sections, namely, Section 2.1 and Section 2.2.

In Section 2.1, we introduced $\alpha$-admissible condition associated with four maps and obtained unique common fixed point theorems for four maps satisfying Geraghty contraction condition in metric spaces. Our main theorem generalize the results of Samet et al.[9], Shahi et al.[68] and Cho et al.[81].

In Section 2.2, we introduced $(\alpha, \eta)$-admissible condition associated with four maps and obtained unique common fixed point theorems for four maps satisfying Geraghty contraction condition in metric spaces. Our main theorem generalize the results of Hussain et al.[61] and Hussain et al.[62].

Chapter 3: Common Fixed Points, Common Coupled Fixed Points in Partial Metric Spaces using Admissible Functions

Chapter 3 is divided into three sections, namely, Section 3.1, Section 3.2 and Section 3.3.

In Section 3.1, we introduced $(\alpha, \eta)$-admissible condition associated with four maps. We also modified the definition of partial compatible pair of maps as partial$(^{(*)})$ compatible pair of maps in partial metric spaces. We obtained two common fixed point theorems for four maps in partial metric spaces.

In Section 3.2, we obtained a unique common fixed point theorem of Suzuki type for four maps in ordered partial metric spaces. It generalizes and unifies some results in partial metric spaces.

In Section 3.3, we introduced $\alpha$-admissible condition associated with $F, G : X \times X \to X$ and $S, T : X \to X$ and extended the definition of partial$(^{(*)})$ compatible maps to $F : X \times X \to X$ and $S : X \to X$. We obtained a unique common coupled fixed point theorem for maps satisfying $(\psi, \phi, \varphi)$-contraction...
condition in partial metric spaces.

Chapter 4: Common Fixed Points for Four Maps in Ordered Fuzzy Metric Spaces using $(\psi, \phi, \varphi)$-contractions with Admissible Functions

In this Chapter, we obtained a unique common fixed point theorem for two pairs of mappings satisfying $(\psi, \phi, \varphi)$-contraction condition using $\alpha$-admissible condition in ordered fuzzy metric spaces. Our result generalizes and improves the results of Gregori and Sapena[100] and Gopal and Vetro[13]. We also given an example to support our theorem.

Chapter 5: Suzuki type Common Fixed Point Theorem for Four Maps using $\alpha$-Admissible Functions in Partial Ordered Complex Valued Metric Spaces

In this Chapter we extended the definition of compatible pair of maps in metric spaces to complex valued metric spaces and obtained a Suzuki type unique common fixed point theorem for four self maps using $\alpha$-admissible functions in ordered complex valued metric spaces. An example was given to support our main theorem.

Chapter 6: Common Fixed and Coupled Fixed Point Theorems in Complex Valued b-Metric Spaces

This Chapter is divided into three sections, namely Section 6.1, Section 6.2 and Section 6.3.

In Section 6.1, we obtain a unique common fixed point theorem for two weakly compatible pairs of maps satisfying a quasi-contraction condition in complex valued b-metric spaces.

In Section 6.2, we extended the definition of compatible pair of maps in metric spaces to complex valued metric spaces for maps $F : X \times X \to X$ and
$S : X \to X$, we obtained a unique common coupled fixed point theorem for maps $F, G : X \times X \to X$ and $S, T : X \to X$ in complex valued b-metric spaces, using $\alpha$-admissible functions. Also we have given an example to support our main theorem.

In Section 6.3, we obtained a unique common coupled fixed point theorem for Jungck type maps $F, G : X \times X \to X$ satisfying a rational inequality in complex valued b-metric spaces. An example was given to illustrate our main theorem.

Chapter 7: Common Coupled Fixed Point Theorems for a pair of Hybrid Maps in Complex Valued Metric Spaces.

In this Chapter we obtained a common coupled fixed point theorem for a pair of hybrid maps in complex valued metric spaces. An example was given to illustrate our main theorem.

Chapter 8: On Fixed and Coincidence Points in Dislocated Metric Spaces.
This Chapter is divided into two sections, namely Section 8.1 and Section 8.2.

In Section 8.1, we proved a common coincidence point theorem for four maps and a unique common fixed point theorem for a pair of maps in complete dislocated metric spaces. Also we obtained a unique fixed point theorem for a single map in complete dq-metric spaces.

In Section 8.2, we introduced dislocated $D^*$-quasi metric spaces and dislocated $D^*$-metric spaces and obtained fixed and common fixed point theorems.

Chapter 9: A Coincidence Point Theorem for Two Hybrid Pairs with Quasi-Contraction in $b$-Metric Spaces.

In this Chapter we obtained a coincidence point theorem for two hybrid pairs of maps satisfying a q-set valued quasi contraction in b-metric spaces.
After Chapter 9, we have given a list of references used for the preparation of this thesis.